Spin Hall effect and spin transfer in a disordered Rashba model

D. N. Sheng,¹ L. Sheng,² Z. Y. Weng,³ and F. D. M. Haldane⁴

¹*Department of Physics and Astronomy, California State University, Northridge, California 91330, USA*

²*Department of Physics and Texas Center for Superconductivity, University of Houston, Houston, Texas 77204, USA*

³*Center for Advanced Study, Tsinghua University, Beijing 100084, China*

⁴*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*

(Received 19 July 2005; revised manuscript received 29 August 2005; published 18 October 2005)

Based on numerical study of the Rashba model, we show that the spin Hall conductance remains finite in the presence of disorder up to a characteristic length scale, beyond which it vanishes exponentially with the system size. We further perform a Laughlin's gauge experiment numerically and find that all energy levels cannot cross each other during an adiabatic insertion of the flux in accordance with the general level repulsion rule. It results in zero spin transfer between two edges of the sample as each state always evolves back after the insertion of one flux quantum in contrast to the quantum Hall effect. It implies that the intrinsic spin Hall effect vanishes with the turn-on of disorder.

DOI: [10.1103/PhysRevB.72.153307](https://doi.org/10.1103/PhysRevB.72.153307)

PACS number(s): 72.10.-d, 72.25.Dc

The two dimensional (2D) electron systems exhibit integer and fractional quantum Hall effects (IQHE/FQHE) in the presence of strong magnetic field.¹ The precise connections between the exact quantization of Hall conductance, the topological property of the wavefunctions, and the current carrying edge states in open systems have been well established.¹⁻⁶ As the hallmark of a quantum Hall effect (QHE), the quantized plateaus¹ appear in the presence of disorder where the longitudinal conductance vanishes with the opening of a mobility gap, which ensures a robust dissipationless transport regime. In an open system, the edge states are the chiral current carrying states free from backward scattering, which leads to a precise quantized charge transfer between edges of the sample as demonstrated through Laughlin's gauge argument.^{3,4,6}

The recent proposals of the intrinsic and dissipationless spin Hall effect (SHE) in three-dimensional *p*-doped semiconductors⁷ and 2D electron systems with Rashba spin-orbital coupling (SOC)⁸ have stimulated a great deal of interest.⁹⁻¹⁹ Such intrinsic SHE may provide a way to manipulate electron spins in nonmagnetic semiconductors without the application of magnetic fields. It was also suggested that the recent experimentally observed spin polarization or accumulation¹⁶ in electron systems might be related to this effect.^{17,18}

However, there exists a strong debate regarding the fate of SHE in the presence of disorder. A challenge to SHE in the Rashba model comes from analytical perturbative calculations involving vertex corrections, where SHE is found to vanish in the presence of any weak disorder.^{11,12} On the other hand, numerical calculations based on the Kubo formula¹⁴ using the continuum model in momentum space found that spin Hall conductance (SHC) is finite for a finite-size system with weak disorder while its proper thermodynamic limit is not clear. A consistent picture for the SHE is still absent. Especially the topological property of the SHE has not been well addressed, while all conventional quantum Hall systems are considered to have nontrivial topological origin,^{5,6} including the Hall effect without a magnetic field.^{20,21}

In this paper, we present a numerical Kubo formula cal-

culation of SHC based on the Rashba lattice model. We show that the finite SHC persists in the presence of random disorder scattering for finite-size systems up to a characteristic length scale beyond which SHC decreases and vanishes exponentially. The scaling behavior of the SHC follows a one-parameter scaling law in a fashion similar to the longitudinal conductance in a localized system.²² Furthermore, we perform numerically a Laughlin's gauge experiment^{3,4} by adiabatically inserting flux to directly probe the experimentally measurable spin transfer and accumulation associated with SHE. We find that energy levels do not cross each other during the adiabatic insertion of the flux once the disorder is turned on in accordance with the level-repulsion rule. It results in zero spin transfer between the edges and zero spin accumulation at the boundary as each state always evolves back to itself after the insertion of one flux quantum. This is in contrast to the quantized charge transfer in the IQHE,^{3,4} which is associated with the nontrivial topological property of quantum Hall systems.^{5,6} We thus conclude that the topological SHE vanishes in the disordered 2D Rashba model. An example of a topological SHE in a 2D electron system^{21,23} is also briefly discussed.

We start with a tight-binding lattice model of noninteracting electrons with the Rashba SOC. The Hamiltonian is given as follows:^{24,25}

$$H = -t \sum_{\langle ij \rangle} \hat{c}_i^\dagger \hat{c}_j + \sum_i w_i \hat{c}_i^\dagger \hat{c}_i + V_{so} \left(i \sum_i \hat{c}_i^\dagger \sigma_x \hat{c}_{i+\hat{y}} - i \sum_i \hat{c}_i^\dagger \sigma_y \hat{c}_{i+\hat{x}} + H.c. \right). \quad (1)$$

Here, $\hat{c}_i^\dagger = (\hat{c}_{i\uparrow}^\dagger, \hat{c}_{i\downarrow}^\dagger)$ are electron creation operators with t as the nearest-neighbor hopping integral, V_{so} the Rashba SOC strength, and w_i is an on-site random nonmagnetic potential uniformly distributed in the interval $(-W/2, W/2)$.

We numerically study square samples of side L by using an exact diagonalization method under a twisted boundary condition: $\hat{\Psi}(i+L\hat{x}) = e^{i\theta_x} \hat{\Psi}(i)$ and $\hat{\Psi}(i+L\hat{y}) = e^{i\theta_y} \hat{\Psi}(i)$, where the electron wave function,

FIG. 1. The spin Hall conductance σ_{SH} (in units of $e/4\pi$) as a function of Fermi energy E_f (in units of t), for system size $N=24 \times 24$ at $W=0.4t$ and $V_{so}=0.5t$. The dashed line with open triangles is SHC at a periodic boundary condition averaged over 200 disorder configurations. The solid line with filled triangles is SHC averaged over 10 disorder configurations and 16×16 different boundary phases.

$$\hat{\Psi}(i) = \begin{pmatrix} \Psi_{\uparrow}(i) \\ \Psi_{\downarrow}(i) \end{pmatrix},$$

has two spin components. The linear response SHC at zero temperature can be calculated by using the Kubo formula

$$\sigma_{SH} = \frac{-e\hbar}{N} \sum_{E_m < E_f < E_n} \frac{\text{Im}(\langle \hat{\Psi}_m | J_x^{spin} | \hat{\Psi}_n \rangle \langle \hat{\Psi}_n | v_y | \hat{\Psi}_m \rangle)}{(E_n - E_m)^2}, \quad (2)$$

where E_f is the electron Fermi energy, and $\hat{\Psi}_m$ represents the m th eigenstate with energy E_m . We use the standard velocity operator $\mathbf{v} = i/\hbar[H, \mathbf{r}]$ ($\mathbf{r}$ is the position operator of electron) and the spin current operator $J_x^{spin} = \hbar/4\{v_x, \sigma_z\}$, which measures spin current flowing along the x direction with the spin polarization along the z direction.

The calculated σ_{SH} at a weak disorder strength $W=0.4t$ with a 200 disorder configuration average, is shown in Fig. 1 (dashed line with open triangles) as a function of the Fermi energy E_f with the system size $N=24 \times 24$ under a periodic boundary condition (PBC) with $\theta_x = \theta_y = 0$. Clearly the fluctuations of the SHC are very strong; much larger than $e/4\pi$, in contrast to IQHE where the fluctuations in Hall conductance only appear between plateaus²⁶ with an amplitude smaller than e^2/h (when converting to the units of spin conductance, e^2/h is equal to $e/4\pi$).

The SHC is also very sensitive to the boundary phases that we impose at the edges of the finite-size system. For example, antiperiodic boundary condition ($\theta_x = \theta_y = \pi$) will move the positive and negative peaks in Fig. 1 to different E_f 's. Remarkably, as we change $\theta = (\theta_x, \theta_y)$ in the $2\pi \times 2\pi$ phase space, we find that the abnormal positive and negative fluctuations of the SHC at different θ can largely cancel each other. Thus $\langle \sigma_{SH} \rangle$ averaged over 16×16 different θ 's becomes a smooth function of E_f with a value close to $0.3e/4\pi$, which is also plotted in Fig. 1 as the line with the

FIG. 2. (a) The disorder and boundary phase averaged SHC $\langle \sigma_{SH} \rangle$ (in units of $e/4\pi$) at $E_f = -3.75t$ as a function of the disorder strength W (in units of t) for sizes $N=16 \times 16$, 24×24 , 32×32 , and 48×48 . (b) $\langle \sigma_{SH} \rangle$ for W between $0.05-2.0t$ as a function of the sample length $L=16, 24, 32, 40$, and 48 .

filled triangles. A completely similar behavior of σ_{SH} upon the boundary phase average has been observed for all W 's and V_{so} 's.

We note that the eigenstates of a finite-size system become boundary-phase independent only in the thermodynamic limit. However, when we apply an electric field through a time-dependent vector potential, it acts as a generalized boundary phase evolving with time. Thus the SHC averaged over time is equivalent to the boundary phase averaged SHC. Similar boundary phase averaged charge Hall conductance gives rise to a topological invariant Chern number.^{2,25} In Fig. 2(a), we show $\langle \sigma_{SH} \rangle$ as a function of disorder strength W (in units of t) for system sizes $N=16 \times 16$, 24×24 , 32×32 , 48×48 with $V_{so}=0.5t$ at a fixed Fermi energy $E_f = -3.75t$ near the band bottom, where the spectrum of the lattice model approaches the continuum one. Each data point is averaged over random disorder configurations (about 200 for $L=48$ and 500–1000 for smaller sizes) and boundary phases (16–576 different θ for each disorder configuration). At weak W limit, $\langle \sigma_{SH} \rangle$ reaches the value of $0.48(e/4\pi)$ for all sizes, in agreement with the predicted intrinsic value for the pure system.⁸

The SHC decreases continuously with the increase of W as shown in Fig. 2(a). At very small $W=0.05t$, the SHC is essentially a constant close to the intrinsic value with a neg-

FIG. 3. The energy levels of the 2D Rashba model (in units of t) with an open boundary along y axis as a function of the twisted phase θ_x for a system of $V_{so}=0.1t$, $E_f=-3.75t$, and $N=32 \times 32$. The solid line is for $W=0.1t$ and the dotted line is for $W=0$. The region indicated by the arrow is enlarged in Fig. 4(a).

ligible variation with L . However, the lack of a flat region in $\langle \sigma_{SH} \rangle$ at $W \rightarrow 0$ limit suggests that the effect of disorder is intrinsically important. Very similar results are obtained for other Fermi energies and V_{so} 's. Furthermore, we have also seen a monotonic decrease of $\langle \sigma_{SH} \rangle$ with increasing L for $W \geq 0.1t$. To quantitatively understand such a behavior, we plot $\langle \sigma_{SH} \rangle$ in a logarithmic scale as a function of L in Fig. 2(b) at different W 's with the typical error bar shown. In fact all the data can be nicely fitted into straight lines, as shown in Fig. 2(b), which indicates an exponential decay law

$$\langle \sigma_{SH} \rangle = c_0 \exp(-L/\xi_S), \quad (3)$$

where the constant c_0 is insensitive to W (for the whole range of W 's we find $c_0=0.46 \pm 0.04$ except for the bottom curve with $W=2t$). Therefore, the scaling behavior of SHC can be fitted into the general one-parameter scaling theory:²² $\sigma_{SH} = f(L/\xi_S)$. Here, ξ_S is a spin-related characteristic length scale determined by fitting the data in Fig. 2(b) into Eq. (3), which approximately follows a power-law behavior $\xi_S = 13.8/(W/t)^{1.5}$. It suggests a finite length scale for any weak disorder. Thus SHC remains finite with the sample size up to ξ_S (it can be as large as, say, a few hundreds of the lattice constant at weak disorder strength $W=0.1t$), and is extrapolated to zero exponentially in the thermodynamic limit.

To further probe the spin transfer and accumulation associated with SHE, we perform numerically a Laughlin's gauge experiment^{3,4} by adiabatically inserting flux to a system which is open along the y direction and periodic along the x direction. This is the same geometry considered by Laughlin for IQHE and the adiabatic insertion of one flux quantum is equivalent to varying the twisted boundary phase θ_x by 2π (note that electric field can also be related to the time dependent θ_x). By diagonalizing the Hamiltonian (1) under the open boundary condition along the y axis, at 2000 different θ_x 's, the resulting energies, E_m , around $E_f=-3.75t$ ($N=32 \times 32$, $V_{so}=0.1t$, and $W=0.1t$) form solid lines shown in Fig. 3. Note that the energy spectrum is symmetric at about $\theta_x = \pi$, thus only the half of the spectrum with $\theta_x \leq \pi$ is shown.

FIG. 4. (a) The enlarged energy levels from Fig. 3 show the avoided level crossing. (b) The robust level crossing in IQHE leads to the adiabatic transfer of edge states. This is calculated for the same parameter used in Fig. 3 with an additional perpendicular magnetic field.

A careful examination of these lines reveals that each E_m goes up and down, making several large angle turns due to backward scatterings. These energy levels never cross with each other, except for $\theta_x=0$ and π , where two levels become exactly degenerate (Kramers degeneracy). Nontrivial topological property^{20,25} associated with Kramers degeneracy will give rise to 2D delocalization.²⁵ The region pointed to by an arrow near the left bottom corner of Fig. 3 is enlarged in Fig. 4(a), which clearly demonstrates the noncrossing feature. This is a general observation for all systems that we have checked with different V_{so} 's, W 's and N 's up to $N=200 \times 200$, in accordance with the level repulsion rule of the disordered system. Then if one follows any Kramers degenerated pair of states starting from $\theta_x=0$ to $\theta_x=2\pi$, one will always go back to exactly the same pair of states. Namely, after the adiabatic insertion of one flux quantum, all states evolve exactly back to the starting states. Thus it will result in no spin pumping between two edges and zero spin accumulation at the open boundary. By contrast, in the absence of disorder, all energy levels will simply evolve following the apparent straight lines (dotted lines in Fig. 3) and cross each other with the increase of θ_x . Then after the insertion of one flux quantum, each energy level evolves into a new state which leads to spin transfer across the sample. But this is the trivial case of level crossing due to the absence of disorder scattering. Therefore, the topology of the states becomes completely different with the turn on of a weak disorder, although the energies themselves only shift by very small amounts as shown in Fig. 3.

For comparison, we show a nontrivial case of level crossing in the IQHE for the same Hamiltonian (1) in the presence of a perpendicular magnetic field/flux ($\Phi_B=2\pi/32$ per plaquette).²⁵ As shown in Fig. 4(b), the energy levels do show simple crossing. This is due to the chiral symmetry of the edge states, which suppresses any backward scattering. Through inserting one flux quantum, the two occupied states below E_f are pumped onto states above E_f indicated by the arrow in Fig. 4(b), which leads to two electrons transferred across from one edge to another, corresponding to the IQHE with $\sigma_H=2e^2/h$. We further note the existence of a topologi-

cal SHE in an electron system of 2D graphene with Haldane's model²¹ with SOC,²³ where quantized edge transfers are also detected.

To summarize, we have shown that the SHC persists in the presence of random disorder scattering for finite size systems up to a characteristic length scale beyond which it decreases and vanishes exponentially in agreement with the analytic calculations involving vertex corrections.^{11,12} Thus the length scale ξ_s should also be identified as the spin-relaxation length defined in Ref. 12. Furthermore, by performing numerically a Laughlin's gauge experiment,^{3,4} we have found that all energy levels cannot cross each other and thus must result in zero spin transfer between the edges and zero spin accumulation at the boundary after the insertion of one flux quantum. This is in contrast to the quantized charge transfer in the QHE,^{3,4} which is associated with the nontrivial topological property of quantum Hall systems.^{5,6,27}

Therefore we establish the vanishing of the topological SHE in the disordered 2D Rashba model beyond a linear response theory with the nonconserved electron spin. The backward scattering which reverses electron momentum and spin leading to zero spin transfer is not properly taken into

account in the linear response theory for SHC as it breaks the current continuity condition. While finite SHC in a four terminal setup is obtained from an LB formula¹⁵ by measuring spin current inside the ideal leads, it is related to the boundary spin current in mesoscopic length scale which cannot be interpreted as bulk SHE due to the nonconservation of the spin current.

Note added: Recently, we became aware of a numerical work in which the vanishing SHC in the thermodynamic limit for an electron Rashba model was also concluded.²⁸

The authors would like to thank L. Balents, J. Hu, A. H. MacDonald, B. K. Nikolic, J. Sinova, M. Wu, and S. C. Zhang for stimulating discussions. This work is supported by ACS-PRF 41752-AC10, RC Grant CC5643, the National Science Foundation Grant/DMR-0307170 (D.N.S.) and NSF MRSEC Grant/DMR-0213706 at the Princeton Center for Complex Materials (F.D.M.H.), a grant from the Robert A. Welch Foundation (L.S.), and National Science Foundation of China Grants No. 10374058 and No. 90403016 (Z.Y.W.). D.N.S. and F.D.M.H. wish to thank the KITP for support from NSF PHY99-07949.

¹For a review see, *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, Berlin, 1990).

²D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982); Q. Niu, D. J. Thouless, and Y. S. Wu, Phys. Rev. B **31**, 3372 (1985).

³R. B. Laughlin, Phys. Rev. B **23**, R5632 (1981).

⁴B. I. Halperin, Phys. Rev. B **25**, 2185 (1982).

⁵D. P. Arovas, R. N. Bhatt, F. D. M. Haldane, P. B. Littlewood, and R. Rammal, Phys. Rev. Lett. **60**, 619 (1988).

⁶S. Ryu and Y. Hatsugai, Phys. Rev. Lett. **89**, 077002 (2002); Y. Hatsugai, *ibid.* **71**, 3697 (1993).

⁷S. Murakami, N. Nagaosa, and S. C. Zhang, Science **301**, 1348 (2003).

⁸J. Sinova, D. Culcer, Q. Niu, N. A. Sinitsyn, T. Jungwirth, and A. H. MacDonald, Phys. Rev. Lett. **92**, 126603 (2004); D. Culcer, J. Sinova, N. A. Sinitsyn, T. Jungwirth, A. H. MacDonald, and Q. Niu, *ibid.* **93**, 046602 (2004).

⁹J. P. Hu, B. A. Bernevig, and C. J. Wu, Int. J. Mod. Phys. B **17**, 5991 (2003); S.-Q. Shen, Phys. Rev. B **70**, 081311(R) (2004).

¹⁰J. Schliemann and D. Loss, Phys. Rev. B **69**, 165315 (2004).

¹¹J. I. Inoue, G. E. W. Bauer, and L. W. Molenkamp, Phys. Rev. B **70**, 041303(R) (2004).

¹²E. G. Mishchenko, A. V. Shytov, and B. I. Halperin, Phys. Rev. Lett. **93**, 226602 (2004).

¹³E. I. Rashba, Phys. Rev. B **70**, 201309(R) (2004); **68**, 241315(R) (2003).

¹⁴K. Nomura, J. Sinova, T. Jungwirth, Q. Niu, and A. H. MacDonald, Phys. Rev. B **71**, 041304(R) (2005).

¹⁵L. Sheng, D. N. Sheng, and C. S. Ting, Phys. Rev. Lett. **94**,

016602 (2005); B. K. Nikolić, L. P. Zárbo, and S. Souma, Phys. Rev. B **72**, 075361 (2005); E. M. Hankiewicz, L. W. Molenkamp, T. Jungwirth, and J. Sinova, Phys. Rev. B **70**, 241301(R) (2004).

¹⁶Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom, Science **306**, 1910 (2004); J. Wunderlich, B. Kaestner, J. Sinova, and T. Jungwirth, Phys. Rev. Lett. **94**, 047204 (2005).

¹⁷B. A. Bernevig and S.-C. Zhang, cond-mat/0412550 (unpublished); cond-mat/0411457 (unpublished).

¹⁸B. K. Nikolic, S. Souma, L. P. Zarbo, and J. Sinova, Phys. Rev. Lett. **95**, 046601 (2005).

¹⁹W. Q. Chen, Z. Y. Weng, and D. N. Sheng, Phys. Rev. Lett. **95**, 086605 (2005).

²⁰F. D. M. Haldane, Phys. Rev. Lett. **93**, 206602 (2004).

²¹F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).

²²For a review, see P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

²³C. L. Kane and E. J. Mele, cond-mat/0411737 (unpublished); L. Sheng, D. N. Sheng, C. S. Ting, and F. D. M. Haldane, Phys. Rev. Lett. **95**, 136602 (2005).

²⁴T. Ando, Phys. Rev. B **40**, 5325 (1989).

²⁵D. N. Sheng and Z. Y. Weng, Phys. Rev. B **54**, R11070 (1996); Phys. Rev. Lett. **78**, 318 (1997).

²⁶S. Cho and M. P. A. Fisher, Phys. Rev. B **55**, 1637 (1997).

²⁷D. N. Sheng, X. Wan, E. H. Rezayi, K. Yang, R. N. Bhatt, and F. D. M. Haldane, Phys. Rev. Lett. **90**, 256802 (2003); D. N. Sheng, L. Balents, and Z. Wang, *ibid.* **91**, 116802 (2003).

²⁸K. Nomura, J. Sinova, N. A. Sinitsyn, and A. H. MacDonald, cond-mat/0506189 (unpublished).