The Global Dynamics of Isothermal Chemical Systems with Critical Nonlinearity

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Abstract

In this paper, we study the Cauchy problem of a cubic autocatalytic chemical reaction system

$$u_{1,t} = u_{1,xx} - u_1^\alpha u_2^\beta, u_{2,t} = du_{2,xx} + u_1^\alpha u_2^\beta$$

with non-negative initial data, where the exponents α, β satisfy $1 < \alpha, \beta < 2$, $\alpha + \beta = 3$ and the constant $d > 0$ is the Lewis number. Our purpose is to study the global dynamics of solutions under mild decay of initial data as $|x| \rightarrow \infty$. We show the exact large time behaviour of solutions which is universal.

1 Introduction

In this paper, we study the Cauchy problem of the chemical reaction system

$$u_{1,t} = u_{1,xx} - u_1^\alpha u_2^\beta \tag{1}$$

$$u_{2,t} = du_{2,xx} + u_1^\alpha u_2^\beta \tag{2}$$

with non-negative initial data

$$u_1(x, 0) = a_1(x), \quad u_2(x, 0) = a_2(x), \quad a_1(x), a_2(x) \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}) \tag{3}$$

of arbitrary size, where the exponents α, β satisfy $1 < \alpha, \beta < 2$, $\alpha + \beta = 3$ and the constant $d > 0$ is the Lewis number. Our purpose is to study the global dynamics of solutions under mild decay of initial data as $|x| \rightarrow \infty$.

The system (1)-(2) arises as a model for cubic autocatalytic chemical reaction of the type

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with isothermal reaction rate proportional to $u_1^\alpha u_2^\beta$. Here u_1 is the concentration of reactant A , u_2 is the concentration of auto-catalyst B . The simple power rate is obtained by the usual assumption of the exponential dependence on the temperature rise in flame systems. Such system is also used to model thermal-diffusive combustion problems, see [7]. For recent works on similar systems with application in mathematical biology, see [8].

The system (1)-(2) on a bounded domain is well-studied in the literature, in particular for the case of $\alpha = 1$, $\beta = 2$, see Alikakos [1], Hollis, Martin and Pierre [10], Martin and Pierre [12] and Masuda [13]. Among other things, the authors established boundedness, global existence and large time behavior of solutions. For homogeneous Dirichlet or Neumann boundary conditions, the large time behaviour is that (u_1, u_2) converges to a constant vector (c_1, c_2) such that $c_1 \cdot c_2 = 0$, see Masuda [13].

More recently, Billingham and Needham [4], [5] showed the existence of traveling front, again for the case of $\alpha = 1$, $\beta = 2$ using shooting argument and phase plane methods. In addition, the large time behaviour of solutions are also studied by formal methods and numerical computation by the authors.

Most recently, motivated by thermal-diffusion models with Arrhenius reactions [2], [14], Berlyand and Xin [3] and Bricmont, Kupiainen and Xin [7] considered system (1)-(2) with initial data (3) that have sufficient polynomial decay at infinity for the case of $\alpha = 1$ and $\beta = 2$. The authors of [7], in particular Bricmont and Kupiainen [6], along with Goldenfeld et al [9], pioneered in applying the Renormalization Group to nonlinear parabolic equations. In [3] and [7], upper and lower bounds by self-similar profiles were established and more importantly, the exact large time self-similarity of solutions with initial data decaying sufficiently fast as $|x| \rightarrow \infty$ was proved. We give a detailed description of the main result of [7] in what follows.

Suppose $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$, where $\mathcal{B}$ is the Banach space of continuous functions on $\mathbf{R}$ with the norm

$$\|f\| = \sup_{x \in \mathbf{R}} |f(x)| (1 + |x|)^q, \quad q > 1. \quad (4)$$

Let ϕ_d be the Gaussian

$$\phi_d = \frac{1}{\sqrt{4\pi d}} e^{-x^2/4d}.$$

For $A > 0$, let ψ_A be the normalized ($\int \psi_A^2(x) d\mu(x) = 1$, see below) first eigenfunction of differential operator

$$\mathcal{L}_A = -\frac{d^2}{dx^2} - \frac{1}{2}x \frac{d}{dx} - \frac{1}{2} + A^2 \phi_d^2(x)$$

on $L^2(\mathbf{R}, d\mu)$, with $d\mu(x) = e^{x^2/4} dx$ and E_A , which is positive for $A > 0$, be the corresponding eigenvalue, then the main result of [7] is as follows.

Theorem (Bricmont, Kupiainen and Xin). Suppose $\alpha = 1$, $\beta = 2$, $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$ and $a_i \geq 0$, $a_i \not\equiv 0$, $i = 1, 2$. Let $A = \int_{\mathbf{R}} a_1(x) + a_2(x)dx$, which is conserved in time. Then the system (1)-(2) has a unique global classical solution $(u_1(x, t), u_2(x, t)) \in \mathcal{B} \times \mathcal{B}$ $\forall t \geq 0$. Moreover, there exists a $q(A)$, which is an increasing function of A and tending to ∞ as $A \rightarrow \infty$, such that if $q \geq q(A)$ in (4), there is a positive number B depending continuously on (a_1, a_2) such that

$$\begin{aligned} \|t^{1/2+E_A}u_1(\sqrt{t}, t) - B\psi_A(\cdot)\| &\rightarrow 0 \\ \|t^{1/2}u_2(\sqrt{t}, t) - A\phi_d(\cdot)\| &\rightarrow 0 \end{aligned}$$

as $t \rightarrow \infty$.

Remark. As was pointed out in [7] that the extra decay power E_A in time is due to the critical cubic nonlinearity of the system (1)-(2). In others word, the scaling law which works for non-cubic nonlinearity no longer works for cubic nonlinearity, and thus the appearance of the anomalous exponent E_A .

In this paper, following the work of [7] we continue to investigate the critical cubic nonlinearity (i.e. $\alpha + \beta = 3$) of the system (1)-(2). The main result of the present work is:

Theorem 1 Suppose $1 < \alpha$, $\beta < 2$ and $\alpha + \beta = 3$. Consider initial data $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$ and $a_i \geq 0$, $a_i \not\equiv 0$, $i = 1, 2$, where $q > 1$ is fixed. Let $A = \int_{\mathbf{R}} a_1(x) + a_2(x)dx > 0$ be the total mass, which is conserved in time. Then the system (1)-(2) has a unique global classical solution $(u_1(x, t), u_2(x, t)) \in \mathcal{B} \times \mathcal{B}$ for $\forall t \geq 0$. Furthermore, there exists $B = B(A, d, \alpha, \beta) > 0$ such that

$$\|t^{1/2}(\log t)^{1/(\alpha-1)}u_1(\sqrt{t}, t) - B\phi_1(\cdot)\| \rightarrow 0 \quad (5)$$

$$\|t^{1/2}u_2(\sqrt{t}, t) - A\phi_d(\cdot)\| \rightarrow 0 \quad (6)$$

as $t \rightarrow \infty$.

Remark. Unlike the case of $\alpha = 1$ and $\beta = 2$, where the value of B is not explicitly, we know the precise value of B . The exact expression of the constant B is as follows:

$$B = \left(\frac{4\pi d^{\beta/2}}{(\alpha-1)A^\beta(\alpha+\beta/d)^{1/2}} \right)^{1/(\alpha-1)} \quad (7)$$

Remark. The asymptotic of the solution is in certain sense universal since the large time spatial-temporal profile is independent of the decay rate at $|x| = \infty$ and the size of the initial data except its apparent dependence on the total mass. In addition, the constant B depends explicitly on the known parameters A , d , α and β .

Remark. The $\log t$ term in (5) is not out of a simple scaling argument, rather it is the by-product of the critical cubic nonlinearity and competition between diffusion and mass transfer. The power of $\log t$ is an anomalous exponent.

Remark. In order to understand intuitively why the asymptotic stated in Theorem 1 holds, let us consider a more general case of $\alpha, \beta \geq 1$. If $\alpha + \beta > 3$, it can be shown as in [3] that both u_1 and u_2 go to zero like the solution of pure diffusion as $t \rightarrow \infty$. For, $\alpha + \beta < 3$, one can apply the maximum principle and a simple a priori estimate to bound u_1 by $\bar{u}$, which is a solution of

$$\bar{u}_t = \bar{u}_{xx} - O\left(t^{-(\alpha+\beta-1)/2} \phi_d^\beta\left(\frac{x}{\sqrt{t}}\right)\right) \bar{u}.$$

Then, by applying the Feynmann-Kac formula, one obtains

$$\bar{u} \leq \exp\left(-O\left(t^{3-\alpha-\beta}/2\right)\right) \quad (8)$$

for $|x| \leq O(\sqrt{t})$. For $|x| \gg O(\sqrt{t})$, one gets a diffusive behaviour, depending on the rate of decay, as $|x| \rightarrow \infty$ of the initial data. Next, by inserting the fast decay (8) for u_1 in (2), one shows the nonlinear term is insignificant and that u_2 diffuses to zero. Clearly, the critical case of $\alpha + \beta = 3$ is the most delicate and interesting one. In particular, instead of (8), one gets $\exp(-O(\log \log t))$, which, after detailed analysis, gives rise to (5).

Remark. In comparison of our result with that of Bricmont, Kupiainen and Xin [7], it shows that

- (i) For our case, only minimum requirement on the decay of initial data as $|x| \rightarrow \infty$ is needed, the decay rate is independent of the total mass A , unlike the case in [7] where the decay rate is related to A . We believe our result is optimal. But our method fails for their case.
- (ii) The appearance of $\log t$ indicates the analysis is more involved and subtle. In particular, it is well known in the scientific computation field that a scaling of $\log t$ is hardly detectable in computation.
- (iii) The nonlinear dependence of u_1 , in addition to the cubic nonlinearity, is solely responsible for the arise of $\log t$ term, the power of which tends to infinity as $\alpha \rightarrow 1$. This may explain the appearance of extra decay power of t in the limiting case of $\alpha = 1$.

Remark. To model chemical fronts (flames) propagating down a tube, one has to feed the system at one end of the tube or in an idealized situation at spatial infinity by specifying nonzero value of u_1 or u_2 . But, when the feeding is turned off at a later stage of reaction process and leaves the system to relax freely by itself, the decaying initial condition emerges. Therefore, the dynamics of the present case is vastly different from the case where traveling front solutions develop. In that case, the front solutions in general undergo transitions to chaos as Lewis number d is sufficiently large or the order of autocatalytic reactions is high enough, see Sivashinsky [17] and Metcalf et al [15]. For a more detailed description, see [7]. But, for the critical nonlinearity case studied

in the present work, we know the exact global dynamics of the problem, a rare treat in nonlinear problems. Our result is in spirit similar to the decay of turbulence results in fluid dynamics for the incompressible Navies-Stokes equations ([11], [17]). However, for such problems, only decay rates of solutions in proper Sobolev norms are known.

The plan of the paper is as follows. In section 2, we derive a priori estimates on the solutions of the system (1)-(2) which is similar to the case of $\alpha = 1, \beta = 2$ as in [7] and that of [13] for bounded domain. But, the extra freedom of choice for α, β means the estimates are a bit more involved. These estimates directly imply the solution is global, classical solution. In section 3, we derive key decay estimates using the maximum principle and a careful construction of super-solution. In section 4, we use the renormalization group method to find the exact large time dynamics of solution. In particular, we give a rigorous proof that the RG map converges as is claimed.

Throughout the paper, $\|\cdot\|_p$ stands for the $L^p(\mathbf{R})$ -norm of a function and $\|\cdot\|$ the norm defined in (4). Also, for simplicity of notation, we shall not distinguish generic constant C from line to line.

2 A Priori Estimates

In this section, we show that there is a uniform bound in time of the $L^p(\mathbf{R})$ -norm, $1 \leq p \leq \infty$, for both u_1 and u_2 . Then, it follows from the classical theory that the solutions (u_1, u_2) are smooth and exist global in time.

First we have the simple

Lemma 1 *The solutions (u_1, u_2) satisfy the following L^1 estimates:*

$$\begin{aligned} \left(\int_{\mathbf{R}} (u_1 + u_2) dx \right) (t) &= \int_{\mathbf{R}} (a_1 + a_2) dx, \quad \left(\int_{\mathbf{R}} u_1 dx \right) (t) \leq \int_{\mathbf{R}} a_1 dx \quad \text{for } t \geq 0, \\ \left(\int_{\mathbf{R}} u_2 dx \right) (t) &\geq \int_{\mathbf{R}} a_2 dx \quad \text{for } t \geq 0, \quad \int_0^\infty \int_{\mathbf{R}} u_1^\alpha u_2^\beta dx dt < +\infty. \end{aligned} \tag{9}$$

Proof: By integrating (1)-(2) over $\mathbf{R} \times [0, t]$, we get

$$\int_{\mathbf{R}} u_1(x, t) dx = \int_{\mathbf{R}} a_1(x) dx - \int_0^t \int_{\mathbf{R}} u_1^\alpha u_2^\beta dx d\tau, \tag{10}$$

$$\int_{\mathbf{R}} u_2(x, t) dx = \int_{\mathbf{R}} a_2(x) dx + \int_0^t \int_{\mathbf{R}} u_1^\alpha u_2^\beta dx d\tau. \tag{11}$$

Together, (10) and (11) give the result of Lemma 1.

Q.E.D.

Next, using directly the maximum principle, we have the following result.

Lemma 2

$$0 < u_1(x, t) \leq \|a_1\|_\infty, \quad \forall t > 0;$$

$$0 < \underline{u}_2(x, t) \leq u_2(x, t) \quad \forall t > 0,$$

where $\underline{u}_2$ is a solution of

$$\underline{u}_{2,t} = d\underline{u}_{2,xx}, \quad \underline{u}_2|_{t=0} = a_2(x);$$

$$u_1(x, t) \leq \bar{u}_1(x, t), \quad \forall t > 0,$$

where $\bar{u}_1$ is a solution of

$$\bar{u}_{1,t} = \bar{u}_{1,xx} - \bar{u}_1^\alpha \underline{u}_2^\beta, \quad \bar{u}_1|_{t=0} = a_1(x).$$

The key estimates of this section are stated in the following lemma.

Lemma 3 *The solutions (u_1, u_2) of (1)-(2) are uniformly bounded in time in $L^p(\mathbf{R})$ norm:*

$$\|u_1(\cdot, t)\|_{L^p} + \|u_2(\cdot, t)\|_{L^p} \leq C(a_1, a_2, p) < \infty, \quad 1 \leq p < \infty, \quad (12)$$

where $c(a_1, a_2, p)$ is a constant depending on initial data and p is a positive integer.

Proof: It is clear by Lemmas 1 and 2 that we only need to prove the bound for u_2 . By standard local existence theory, all $L^p(\mathbf{R})$ -norm of (u_1, u_2) and the $L^2(\mathbf{R})$ -norm of $(u_{1,x}, u_{2,x})$ are finite and continuous in time. Therefore, we can freely integrate by parts in what follows. By multiplying (2) by pu_2^{p-1} with $p \geq 2$, integrating over $\mathbf{R} \times R^+$, we get

$$\int_{\mathbf{R}} u_2^p dx = \int_{\mathbf{R}} a_2^p dx - d \int_0^t \int_{\mathbf{R}} p(p-1) u_{2,x}^2 u_2^{p-2} dx d\tau + \int_0^t \int_{\mathbf{R}} pu_1^\alpha u_2^{\beta+p-1} dx d\tau. \quad (13)$$

In addition, with the help of integration by parts, we have the identity

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbf{R}} (u_1^2 + u_1) u_2^p dx \\ &= \int_{\mathbf{R}} (1 + 2u_1)(u_{1,xx} - u_1^\alpha u_2^\beta) u_2^p dx \\ &+ \int_{\mathbf{R}} (u_1^2 + u_1) pu_2^{p-1} (du_{2,xx} + u_1^\alpha u_2^\beta) dx \\ &= -2 \int_{\mathbf{R}} u_{1,x}^2 u_2^p dx - \int_{\mathbf{R}} (1 + 2u_1) u_{1,x} u_{2,x} pu_2^{p-1} - \int_{\mathbf{R}} (1 + 2u_1) u_1^\alpha u_2^\beta u_2^p dx \\ &- d \int_{\mathbf{R}} (1 + 2u_1) u_{1,x} u_{2,x} pu_2^{p-1} - d \int_{\mathbf{R}} (u_1^2 + u_1) p(p-1) u_2^{p-2} u_{2,x}^2 dx \\ &+ \int_{\mathbf{R}} (u_1^2 + u_1) pu_2^{p-1} u_1^\alpha u_2^\beta dx \\ &= I + II + III + IV + V + VI. \end{aligned} \quad (14)$$

It is clear that

$$\begin{aligned}
I + II + IV &\leq (1 + 2\|a_1\|_\infty)(1 + d) \int_{\mathbf{R}} |u_{1,x} u_{2,x}| p u_2^{p-1} - 2 \int_{\mathbf{R}} u_{1,x}^2 u_2^p dx \\
&\leq \int_{\mathbf{R}} u_{1,x}^2 u_2^p dx + C(\|a_1\|_\infty, p) \int_{\mathbf{R}} u_{2,x}^2 u_2^{p-2} dx - 2 \int_{\mathbf{R}} u_{1,x}^2 u_2^p dx \\
&\leq C(\|a_1\|_\infty, p) \int_{\mathbf{R}} u_{2,x}^2 u_2^{p-2} dx - \int_{\mathbf{R}} u_{1,x}^2 u_2^p dx.
\end{aligned}$$

Moreover,

$$III \leq - \int_{\mathbf{R}} u_1^\alpha u_2^\beta u_2^p dx, \quad V \leq 0$$

and

$$VI \leq (\|a_1\|_\infty + \|a_1\|_\infty^2) \int_{\mathbf{R}} p u_1^\alpha u_2^\beta u_2^{p-1} dx.$$

Integrating (14) from 0 to t yields

$$\begin{aligned}
\left(\int_{\mathbf{R}} (u_1^2 + u_1) u_2^p dx \right)(t) &\leq \int_{\mathbf{R}} (a_1^2 + a_1) a_2^p dx + C(p, a_1) \int_0^t \int_{\mathbf{R}} u_{2,x}^2 u_2^{p-2} dx d\tau \\
&\quad + (\|a_1\|_\infty + \|a_1\|_\infty^2) \int_0^t \int_{\mathbf{R}} u_1^\alpha u_2^\beta p u_2^{p-1} dx d\tau \\
&\quad - \int_0^t \int_{\mathbf{R}} u_1^\alpha u_2^\beta u_2^p dx d\tau - \int_0^t \int_{\mathbf{R}} u_{1,x}^2 u_2^p dx d\tau.
\end{aligned} \tag{15}$$

The combination of (13) and (15) then gives

$$\begin{aligned}
&\int_{\mathbf{R}} u_2^p + \int_0^t \int_{\mathbf{R}} (u_{2,x}^2 u_2^{p-2} + u_{1,x}^2 u_2^p + u_1^\alpha u_2^{\beta+p}) dx d\tau \\
&\leq C(a_1, a_2, p) \left(1 + \int_0^t \int_{\mathbf{R}} u_1^\alpha u_2^{\beta+p-1} dx d\tau \right).
\end{aligned} \tag{16}$$

For $p = 1$, proceed as in Lemma 2.3 of [7], we derive

$$\int_0^t \int_{\mathbf{R}} (u_1^\alpha u_2^{\beta+1} + u_{1,x}^2 u_2) dx d\tau \leq C(a_1, a_2) \left(1 + \int_0^t \int_{\mathbf{R}} u_1^\alpha u_2^\beta dx d\tau \right). \tag{17}$$

Apparently, a simple induction on p with the help of (13), (16), (17) and Lemma 1 give the desired uniform bound for u_2 . This completes the proof of the lemma.

Q.E.D.

Lemma 4 *The following estimates for the derivative of (u_1, u_2) of (1)-(2) hold if $t \geq t_0 > 0$:*

$$\|u_{1,x}(\cdot, t)\|_2 + \|u_{2,x}(\cdot, t)\|_2 \leq C(a_1, a_2) < \infty, \tag{18}$$

where $t_0 > 0$ is arbitrary.

Proof: First, multiplying (1) by u_1 and integrating on $\mathbf{R}$ imply

$$\frac{d}{dt} \int_{\mathbf{R}} u_1^2 dx + \int_{\mathbf{R}} u_{1,x}^2 dx \leq 0.$$

Integrating the above inequality over $[0, t]$ gives

$$\int_{\mathbf{R}} u_1^2 dx + \int_0^t \int_{\mathbf{R}} u_{1,x}^2 dx d\tau \leq \int_{\mathbf{R}} a_1^2 dx.$$

Similarly, we have

$$\int_{\mathbf{R}} u_2^2 dx + \int_0^t \int_{\mathbf{R}} u_{2,x}^2 dx d\tau \leq \int_0^t \int_{\mathbf{R}} u_1^\alpha u_2^{\beta+1} dx d\tau + \int_{\mathbf{R}} a_2^2 dx \leq C(a_1, a_2)$$

by Lemma 3. By Fubini's theorem, there exists $0 \leq t_1 \leq t_0$ such that

$$\left(\int_{\mathbf{R}} (u_{1,x}^2 + u_{2,x}^2) dx \right)(t_1) \leq C(a_1, a_2)/t_0. \quad (19)$$

Next, multiplying (1) by $u_{1,xx}$ and integrating over $\mathbf{R}$ yield

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} \|u_{1,x}\|_2^2 &= \int_{\mathbf{R}} u_{1,xx}^2 dx - \int_{\mathbf{R}} u_1^\alpha u_2^\beta u_{1,xx} dx \\ &= \int_{\mathbf{R}} u_{1,xx}^2 dx + \alpha \int_{\mathbf{R}} u_1^{\alpha-1} u_2^\beta u_{1,x}^2 dx + \beta \int_{\mathbf{R}} u_1^\alpha u_2^{\beta-1} u_{1,x} u_{2,x} dx. \end{aligned} \quad (20)$$

Similarly, we have

$$-\frac{1}{2} \frac{d}{dt} \|u_{2,x}\|_2^2 = d \int_{\mathbf{R}} u_{2,xx}^2 dx - \beta \int_{\mathbf{R}} u_1^\alpha u_2^{\beta-1} u_{2,x}^2 dx - \alpha \int_{\mathbf{R}} u_1^{\alpha-1} u_2^\beta u_{1,x} u_{2,x} dx. \quad (21)$$

Add (20) and (21), and integrate from t_1 to t then gives

$$\begin{aligned} (\|u_{1,x}\|_2^2 + \|u_{2,x}\|_2^2)(t) &\leq (\|u_{1,x}\|_2^2 + \|u_{2,x}\|_2^2)(t_1) + \\ &\alpha \int_{t_1}^t \int_{\mathbf{R}} u_1^{\alpha-1} u_2^\beta u_{1,x}^2 dx d\tau + \beta \int_{t_1}^t \int_{\mathbf{R}} u_1^\alpha u_2^{\beta-1} |u_{1,x} u_{2,x}| dx d\tau \\ &+ \beta \int_{t_1}^t \int_{\mathbf{R}} u_1^\alpha u_2^{\beta-1} u_{2,x}^2 dx d\tau + \alpha \int_{t_1}^t \int_{\mathbf{R}} u_1^{\alpha-1} u_2^\beta |u_{1,x} u_{2,x}| dx d\tau. \end{aligned} \quad (22)$$

We now derive bounds for each of the last four terms on the right-hand side of (22).

$$\begin{aligned} \int_{t_1}^t \int_{\mathbf{R}} u_1^{\alpha-1} u_2^\beta u_{1,x}^2 dx d\tau &\leq \|a_1\|_\infty^{\alpha-1} \int_{t_1}^t \int_{\mathbf{R}} u_2^\beta u_{1,x}^2 dx d\tau \\ &\leq \|a_1\|_\infty^{\alpha-1} \left(\int_{t_1}^t \int_{\mathbf{R}} u_2^2 u_{2,x}^2 dx d\tau \right)^{\beta-1} \left(\int_{t_1}^t \int_{\mathbf{R}} u_2 u_{2,x}^2 dx d\tau \right)^{2-\beta} \end{aligned} \quad (23)$$

by Hölder's inequality. Similarly,

$$\begin{aligned} \int_{t_1}^t \int_{\mathbf{R}} u_1^\alpha u_2^{\beta-1} |u_{1,x} u_{2,x}| dx d\tau & \\ \leq \|a_1\|_\infty^\alpha \left(\int_{t_1}^t \int_{\mathbf{R}} u_2^{2(\beta-1)} u_{1,x}^2 dx d\tau \int_{t_1}^t \int_{\mathbf{R}} u_{2,x}^2 dx d\tau \right)^{1/2}. \end{aligned} \quad (24)$$

If $0 < 2(\beta - 1) < 1$, then by Hölder's inequality we find

$$\int_{t_1}^t \int_{\mathbf{R}} u_2^{2(\beta-1)} u_{1,x}^2 dx d\tau \leq \left(\int_{t_1}^t \int_{\mathbf{R}} u_2 u_{1,x}^2 dx d\tau \right)^{2(\beta-1)} \left(\int_{t_1}^t \int_{\mathbf{R}} u_{1,x}^2 dx d\tau \right)^{3-2\beta}. \quad (25)$$

If $1 = 2(\beta - 1)$, the right-hand side of (24) is already bounded by $C(a_1, a_2)$ by Lemma 3.

If $1 < 2(\beta - 1) < 2$, again by a simple application of Hölder's inequality, we get

$$\int_{t_1}^t \int_{\mathbf{R}} u_2^{2(\beta-1)} u_{1,x}^2 dx d\tau \leq \left(\int_{t_1}^t \int_{\mathbf{R}} u_2^2 u_{1,x}^2 dx d\tau \right)^{1/p} \left(\int_{t_1}^t \int_{\mathbf{R}} u_2 u_{1,x}^2 dx d\tau \right)^{(p-1)/p}, \quad (26)$$

where $p = 1/(2(\beta - 1) - 1)$. Continuing the procedure,

$$\begin{aligned} \int_{t_1}^t \int_{\mathbf{R}} u_1^\alpha u_2^{\beta-1} u_{2,x}^2 dx d\tau &\leq \|a_1\|_\infty^\alpha \int_{t_1}^t \int_{\mathbf{R}} u_2^{(\beta-1)} u_{2,x}^2 dx d\tau \\ &\leq \|a_1\|_\infty^\alpha \left(\int_{t_1}^t \int_{\mathbf{R}} u_2 u_{2,x}^2 dx d\tau \right)^{\beta-1} \left(\int_{t_1}^t \int_{\mathbf{R}} u_{2,x}^2 dx d\tau \right)^{2-\beta}. \end{aligned} \quad (27)$$

$$\begin{aligned} \int_{t_1}^t \int_{\mathbf{R}} u_1^{\alpha-1} u_2^\beta |u_{1,x} u_{2,x}| dx d\tau &\leq \|a_1\|_\infty^{\alpha-1} \left(\int_{t_1}^t \int_{\mathbf{R}} u_2^\beta u_{1,x}^2 dx d\tau \int_{t_1}^t \int_{\mathbf{R}} u_2^\beta u_{2,x}^2 dx d\tau \right)^{1/2} \\ &\leq \|a_1\|_\infty^{\alpha-1} \left(\int_{t_1}^t \int_{\mathbf{R}} u_2^2 u_{2,x}^2 dx d\tau \right)^{(\beta-1)/2} \left(\int_{t_1}^t \int_{\mathbf{R}} u_2 u_{2,x}^2 dx d\tau \right)^{(2-\beta)/2} \\ &\times \left(\int_{t_1}^t \int_{\mathbf{R}} u_2^2 u_{1,x}^2 dx d\tau \right)^{1/2\sigma} \left(\int_{t_1}^t \int_{\mathbf{R}} u_2 u_{1,x}^2 dx d\tau \right)^{(\sigma-1)/2\sigma}, \end{aligned} \quad (28)$$

where $\sigma = 1/(\beta - 1)$. Combining (22)-(28) and Lemma 3, we obtain

$$(\|u_{1,x}\|_2^2 + \|u_{2,x}\|_2^2)(t) \leq (\|u_{1,x}\|_2^2 + \|u_{2,x}\|_2^2)(t_1) + C(a_1, a_2)$$

Hence, by (19),

$$(\|u_{1,x}\|_2^2 + \|u_{2,x}\|_2^2)(t) \leq C(a_1, a_2)$$

for all $t \geq t_0$.

Q.E.D.

Proposition 1 *The system (1)-(2) has a unique classical solution satisfying, for all $t > 0$,*

$$\|u_1\|_{L^p} + \|u_2\|_{L^p} \leq C(a_1, a_2), \quad 1 \leq p \leq \infty, \quad (29)$$

where the constant depends only on the initial data $(a_1, a_2) \in (L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}))^2$.

Proof: The bound for u_1 follows directly from Lemmas 1 and 2. For u_2 , the classical theory of local existence yields the bound for small t , say $t \leq t_0$, where $t_0 > 0$. For $t > t_0$, we can use Sobolev embedding, (16) for $p = 2$ and Lemma 4 to get

$$\|u_2\|_\infty \leq C(a_1, a_2).$$

This completes the proof of the proposition.

Q.E.D.

3 Sharp Decay Estimates

The purpose of this section is to prove sharp decay estimates for (u_1, u_2) in time. We assume from now on that the initial data $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$ with $q > 1$ being a fixed constant. To put the solutions in the right scale we let

$$\tilde{u}_1(x, t) = \sqrt{t}u_1(\sqrt{t}x, t), \quad \tilde{u}_2(x, t) = \sqrt{t}u_2(\sqrt{t}x, t).$$

Proposition 2 *The solutions (u_1, u_2) of (1)-(2) satisfies the bounds*

$$\|\tilde{u}_1\|(t) \leq C(a_1, a_2)[\log(1+t)]^{-1/(\alpha-1)}, \quad (30)$$

$$\|\tilde{u}_2\|(t) \leq C(a_1, a_2), \quad (31)$$

where $\|\cdot\|$ stands for the norm in (4).

Remark. We note that as a direct consequence of (30) and (31) we have

$$\begin{aligned} \|u_1\|_\infty &\leq C(a_1, a_2)(1+t)^{-1/2}[\log(1+t)]^{-1/(\alpha-1)}, \\ \|u_2\|_\infty &\leq C(a_1, a_2)(1+t)^{-1/2}. \end{aligned} \quad (32)$$

Lemma 5 *The estimate (30) holds.*

Proof: Let

$$\begin{aligned} s &= \log(t+T), \quad y = \frac{x}{\sqrt{t+T}}, \\ u_1(x, t) &= \frac{v_1(y, s)}{e^{s/2}s^{1/(\alpha-1)}}, \quad u_2(x, t) = \frac{v_2(y, s)}{e^{s/2}}. \end{aligned}$$

where $T > 1$ is fixed. Then, it is easy to verify that v_1, v_2 satisfy, for $s \geq s_0 = \log T > 0$,

$$\begin{cases} v_{1,s} = v_{1,yy} + \frac{1}{2}v_1 + \frac{1}{2}yv_{1,y} + \frac{v_1}{(\alpha-1)s} - \frac{v_1^\alpha v_2^\beta}{s}, \\ v_{2,s} = dv_{2,yy} + \frac{1}{2}v_2 + \frac{1}{2}yv_{2,y} + \frac{v_1^\alpha v_2^\beta}{s^{\frac{\alpha}{\alpha-1}}}, \end{cases}$$

Set

$$\mathcal{L}(v) = v_s - v_{yy} - \frac{1}{2}v - \frac{1}{2}yv_y - \frac{v}{(\alpha-1)s}.$$

It is easy to see that if there exists a function v which has the property that $\mathcal{L}(v) \geq 0$ on $\mathbf{R} \times (s_0, \infty)$ and $v(y, s_0) \geq v_1(y, s_0)$ on $\mathbf{R}$, then $v(y, s) \geq v_1(y, s)$ on $\mathbf{R} \times (s_0, \infty)$. In another word, such v is a super-solution of v_1 . To construct such a super-solution, we let

$$v = Ae^{-y^2/4} \left(1 + \frac{f(y)}{s} \right) + B \frac{(1+y^2)^{-k}}{s^2},$$

where $k > 1/2$ is arbitray and $A, B > 0$ are constants to be fixed later and

$$f(y) = \frac{\alpha}{\alpha-1} \int_0^y e^{y_1^2/4} dy_1 \int_{y_1}^\infty e^{-y_2^2/4} dy_2.$$

Note that f has a bound

$$f(y) \leq \frac{\alpha}{\alpha-1} (\pi e + \log y H(y-1)),$$

where H is the Heaviside function. Detailed calculations show that if $k > 1/2$,

$$\begin{aligned} \mathcal{L}(v) = & \frac{Ae^{-y^2/4}}{s} \left(1 - \frac{\alpha f(y)}{(\alpha-1)s} \right) + B(1+y^2)^{-k} \times \\ & \left(\frac{(2k-1)y^4 - 2(4k^2+k+1)y^2 + 4k-1}{2(1+y^2)^2 s^2} - \frac{2\alpha-1}{(\alpha-1)s^3} \right). \end{aligned}$$

Since $(2k-1)y^4 - 2(4k^2+k+1)y^2 + 4k-1 > m(1+y^2)^2$ for any $m < 2k-1$ with y suitably large, we have, if $k > 1/2$, that

$$\mathcal{L}(v) \geq \frac{Ae^{-y^2/4}}{s} \left(1 - \frac{\alpha f(y)}{(\alpha-1)s} \right) + B \left(\frac{2k-1}{4s^2} - \frac{2\alpha-1}{(\alpha-1)s^3} \right) (1+y^2)^{-k}.$$

Clearly, there exist $y_1(k) > 0$ and $s_1(k) > 0$ such that the right hand side is positive if $|y| \geq y_1(k)$ and $s \geq s_1(k) > 0$ if $A = B > 0$. For $|y| < y_1(k)$, we can choose $s_2(k) \gg 1$ such that $\mathcal{L}(v) > 0$ for all $s \geq s_2(k)$. Hence, v is a super-solution for $s \geq \max(s_1(k), s_2(k)) > 0$.

It follows from our assumption on initial data and the above construction that the estimate (30) holds by noting that A, B can be arbitrarily large independent of k .

Q.E.D.

To prove Proposition 2, we only need to show

Lemma 6 *There exists $C = C(a_1, a_2)$ such that*

$$\|u_2\|_\infty \leq C(a_1, a_2)(1+t)^{-1/2}.$$

Proof: Consider the equation for u_2 . Write the equation in integral form:

$$u_2(x, t) = \frac{1}{\sqrt{4\pi dt}} \int_{\mathbf{R}} e^{-y^2/(4dt)} a_2(x - y) dy + \int_0^t \int_{\mathbf{R}} \frac{1}{\sqrt{4\pi ds}} e^{-y^2/(4ds)} (u_1^\alpha u_2^\beta)(x - y, t - s) dy ds.$$

Take L^∞ -norm then yields

$$\|u_2\|_\infty \leq \frac{c\|a_2\|}{(2+t)^{1/2}} + C \int_0^t s^{-1/2} \|u_1^\alpha u_2^\beta\|_1 (t-s) ds. \quad (33)$$

Now,

$$\|u_1^\alpha u_2^\beta\|_1 \leq \|u_1\|_\infty^\alpha \|u_2^\beta\|_1 \leq C(2+t)^{-\alpha/2} \|u_2^{\beta-1}\|_\infty \|u_2\|_1 \leq C(2+t)^{-\alpha/2} \quad (34)$$

and

$$\int_0^t s^{-1/2} (2+t-s)^{-\alpha/2} ds \leq C(2+t)^{-\min(1, (\alpha-1))/2} = C(2+t)^{-(\alpha-1)/2}.$$

Thus,

$$\|u_2\|_\infty \leq C(2+t)^{-(\alpha-1)/2}. \quad (35)$$

We can use (35) to improve (34) into

$$\|u_1^\alpha u_2^\beta\|_1 \leq C(2+t)^{-(2\alpha-1)/2},$$

which, when inserted into (33) gives

$$\|u_2\|_\infty \leq C(2+t)^{-(\alpha-1)}, \quad (36)$$

and a finite number of iterations gives the desired estimate. This completes the proof of the lemma.

Q.E.D.

Proof of Proposition 2: By the bound of u_1 and Lemma 6, $u_2 \leq \bar{u}_2$, where $\bar{u}_2$ is the solution of the linear heat equation

$$\begin{cases} \bar{u}_{2,t} = d\bar{u}_{2,xx} + C(2+t)^{-1} \log^{-\alpha/(\alpha-1)}(2+t) \bar{u}_2, \\ \bar{u}_2|_{t=0} = a_2. \end{cases}$$

The bound in (31) then follows from the classical theory. This ends the proof of Proposition 2.

Q.E.D.

4 The RG Method and Asymptotics

In this section, we prove Theorem 1. The method we use is the renormalization group method. We refer the reader to the pioneering work of Bricmont, Kupiainen and Lin [6] and the reference therein for a thorough and stimulating discussion of its application to a wide range of PDE problems. Our approach here is close in spirit to the one used in [7].

Define for $n \in N$ and for $L > 1$ large

$$u_i^n = L^n u_i(L^n x, L^{2n} t), \quad t \in [1, L^2], \quad i = 1, 2. \quad (37)$$

(u_1^n, u_2^n) satisfy the system (1)-(2) with initial data (when $t = 1$):

$$a_i^n = L^n u_i(L^n x, L^{2n}), \quad i = 1, 2. \quad (38)$$

The decay estimates we get in Section 3 imply

$$\|a_1^n\| \leq C(L)n^{-1/(\alpha-1)} \quad \text{and} \quad \|a_1^n\|^\alpha \|a_2^n\|^\beta \leq C(L)n^{-\alpha/(\alpha-1)}. \quad (39)$$

We shall consider the RG map $(a_1, a_2) \rightarrow (a_1^1, a_2^1)$ defined in $\mathcal{B} \times \mathcal{B}$ with

$$a_i^1 = L u_i(Lx, L^2), \quad i = 1, 2 \quad \text{inductively,} \quad (40)$$

where (u_1, u_2) solve (1)-(2) with initial data (a_1, a_2) . That is,

$$a_1^1 = L[e^{(L^2-1)\Delta} a_1(Lx)] - L n_1(Lx, L^2), \quad (41)$$

$$a_2^1 = L[e^{d(L^2-1)\Delta} a_2(Lx)] + L n_2(Lx, L^2), \quad (42)$$

where Δ represents the differential operator d^2/dx^2 and

$$n_1(x, t) = \int_1^t \int_{\mathbf{R}} G(x-y, t-s) u_1^\alpha(y, s) u_2^\beta(y, s) dy ds, \quad (43)$$

$$n_2(x, t) = \int_1^t \int_{\mathbf{R}} G_d(x-y, t-s) u_1^\alpha(y, s) u_2^\beta(y, s) dy ds \quad (44)$$

with G and G_d being corresponding Gaussians of $u_t = u_{xx}$ and $u_t = du_{xx}$ respectively.

If we denote $\mathcal{B}^+ = \{f \in \mathcal{B} | f \geq 0\}$. The map S_L from $\mathcal{B}^+ \rightarrow \mathcal{B}^+$ defined as

$$(S_L f)(x) = L f(Lx)$$

is a bounded operator. In fact, we have the following result.

Lemma 7 *Suppose $L \geq 1$. Then,*

$$(a) \quad \|S_L\| \leq L,$$

$$(b) \quad \|e^{\mu(t-s)\Delta}\| \leq e^{c(t-s)}, \text{ for } \mu = 1, d, \quad c < \infty.$$

For each step, we write

$$\begin{aligned} a_1(x) &= A_1\phi(x) + b_1, \\ a_2(x) &= A_2\phi_d(x) + b_2. \end{aligned}$$

where $A_i = \int_{\mathbf{R}} a_i(x)dx$, $i = 1, 2$ and $\int_{\mathbf{R}} b_i(x)dx = 0$, $i = 1, 2$, $\phi(x) = \phi_1(x)$.

In order to estimate the change of coefficients A_i and b_i , we make some preliminary analysis. First, we look at the equation for u_2 ,

$$u_2(x, t) = e^{d(t-1)\Delta}a_2 + n_2(x, t) = u_{20}(x, t) + n_2(x, t). \quad (45)$$

Consider the ball

$$B_R = \{u_2 : |||u_2||| \equiv \sup_{t \in [1, L^2]} ||u_2(\cdot, t)|| \leq R||a_2||\}.$$

Suppose $u_2 \in B_R$. First we have

$$||u_{10}(\cdot, s)|| \leq ||e^{(t-1)\Delta}a_1|| \leq C(L)||a_1||$$

and

$$||u_{20}(\cdot, s)|| \leq ||e^{(t-1)\Delta}a_2|| \leq C(L)||a_2||$$

by Lemma 7. Next, by Lemma 2,

$$||u_1(\cdot, s)||_{\infty} \leq ||a_1||_{\infty} \leq C||a_1||$$

and Lemma 7 (b), we find

$$\begin{aligned} |||n_1||| &\leq \sup_{t \in [1, L^2]} \int_1^t ds \int_{\mathbf{R}} G(x - y, t - s) (c||a_1||)^{\alpha} (R||a_2||)^{\beta} dy \\ &\leq C(L)R^{\beta}||a_1||^{\alpha}||a_2||^{\beta} \leq C(L, R)\epsilon||a_2||, \end{aligned}$$

where $\epsilon = ||a_1||^{\alpha}||a_2||^{\beta-1}$. Similarly,

$$|||n_2||| \leq C(L, R)\epsilon||a_2||.$$

In consequence,

$$||Ln_i(L \cdot, L^2)|| = ||S_L n_i(\cdot, L^2)|| \leq C(L, R)\epsilon||a_i||, \quad i = 1, 2.$$

Therefore, the right-hand side of (45) defines a map from B_R to B_R if $R > 0$ is sufficiently large and ϵ sufficiently small. A more close look reveals it is a contraction map. Hence, there is a unique fixed point in B_R .

For the RG map, we have the relations

$$\begin{aligned} A_1^1 &= A_1 - \int_{\mathbf{R}} Ln_1(Lx, L^2)dx, \quad A_2^1 = A_2 + \int_{\mathbf{R}} Ln_2(Lx, L^2)dx, \\ b_1^1 &= e^{(L^2-1)\Delta}b_1 - Ln_1(Lx, L^2) + (A_1 - A_1^1)\phi, \\ b_2^1 &= e^{d(L^2-1)\Delta}b_2 + Ln_2(Lx, L^2) + (A_2 - A_2^1)\phi_d, \end{aligned} \quad (46)$$

where the major terms in n_1, n_2 are given by

$$\begin{aligned} w(y, s) &= u_1^\alpha(y, s)u_2^\beta(y, s) - (A_1\phi(y))^\alpha(A_2\phi_d(y))^\beta \\ &= (u_{10} - n_1)^\alpha(u_{20} + n_2)^\beta - (A_1\phi)^\alpha(A_2\phi_d)^\beta \\ &= (u_{10} - n_1)^\alpha(u_{20} + n_2)^\beta - (u_{10} - n_1)^\alpha(A_2\phi_d)^\beta + (u_{10} - n_1)^\alpha(A_2\phi_d)^\beta - (A_1\phi)^\alpha(A_2\phi_d)^\beta \\ &= (u_{10} - n_1)^\alpha[(A_2\phi_d + e^{d(s-1)\Delta}b_2 + n_2)^\beta - (A_2\phi_d)^\beta] + \\ &\quad (A_2\phi_d)^\beta[(A_1\phi + e^{(s-1)\Delta}b_1 - n_1)^\alpha - (A_1\phi)^\alpha] \\ &= (u_{10} - n_1)^\alpha\beta(A_2\phi_d + \theta_1(e^{d(s-1)\Delta}b_2 + n_2))^{\beta-1}(e^{d(s-1)\Delta}b_2 + n_2) + \\ &\quad (A_2\phi_d)^\beta\alpha(A_1\phi + \theta_2(e^{(s-1)\Delta}b_1 - n_1))^{\alpha-1}(e^{(s-1)\Delta}b_1 - n_1), \end{aligned}$$

where $0 \leq \theta_i \leq 1$, $i = 1, 2$, such that

$$\begin{aligned} ||w(\cdot, s)|| &\leq \beta||u_1||_\infty^\alpha ||(A_2\phi_d + \theta_1(e^{d(s-1)\Delta}b_2 + n_2))^{\beta-1}||_\infty ||(e^{d(s-1)\Delta}b_2 + n_2)|| \\ &\quad + \alpha ||(A_2\phi_d)||^\beta ||(A_1\phi + \theta_2(e^{(s-1)\Delta}b_1 - n_1))^{\alpha-1}||_\infty ||(e^{(s-1)\Delta}b_1 - n_1)|| \\ &\leq C||a_1||^\alpha (A_2 + ||b_2|| + C(L, R)\epsilon||a_2||)^{\beta-1} \cdot (||b_2|| + C(L, R)\epsilon||a_2||) + \\ &\quad CA_2^\beta (A_1 + ||b_1|| + C(L, R)\epsilon||a_2||)^{\alpha-1} \cdot (||b_1|| + C(L, R)\epsilon||a_2||). \end{aligned} \quad (47)$$

Now for any initial data $(a_1, a_2) \in \mathcal{B} \times \mathcal{B}$, estimates in the previous section imply that if we consider the solution (u_1, u_2) after large enough time T , then $a_1 = u_1(\cdot, T)$ will be very small while $a_2 = u_2(\cdot, T)$ is preserved in $\mathcal{B} \times \mathcal{B}$ and the iterated RG map will be contractive for some $R = R(a_1, a_2)$. Therefore for the rest of this section we will fix R and take $\epsilon = ||a_1||^\alpha ||a_2||^{\beta-1}$ be as small as necessary.

Lemma 8 *Suppose $L \geq 1$. There exists $\epsilon_0(L) > 0$ such that if $||a_1||^\alpha ||a_2||^{\beta-1} < \epsilon \leq \epsilon_0(L)$, the following results hold:*

- (a) $|A_i^1 - A_i| \leq C(L)\epsilon||a_2||$, $i = 1, 2$,
- (b) $||b_2^1|| \leq L^{-2\delta}||b_2|| + C(L)\epsilon||a_2||$,
- (c) $|A_1^1 - A_1 + \gamma A_1^\alpha A_2^\beta| \leq \eta$, where

$$\gamma = \left(\frac{(\alpha + \beta/d)^{1/2}}{4\pi d^{\beta/2}} \right)^{1/(\alpha-1)}$$

and

$$\begin{aligned} \eta &= \eta(||a_1||, ||a_2||, ||b_1||, ||b_2||, A_1, A_2, \epsilon) \\ &= C(L) \left(||a_1||^\alpha (A_2 + ||b_2|| + \epsilon||a_2||)^{\beta-1} \cdot (||b_2|| + \epsilon||a_2||) \right) \\ &\quad + C(L) \left(A_2^\beta (A_1 + ||b_1|| + \epsilon||a_2||)^{\alpha-1} \cdot (||b_1|| + \epsilon||a_2||) \right), \end{aligned}$$

$$(d) \quad ||b_1^1|| \leq L^{-2\delta} ||b_1|| + C(L)\epsilon ||a_2||,$$

where $\delta > 0$ is independent of L .

Proof: (a) follows directly from our previous analysis since $|A_i^1 - A_i| = \int_{\mathbf{R}} Ln_i(Lx, L^2) dx$.

For part (b), by our previous analysis on $Ln_2(Lx, L^2)$, (a) on $A_2^1 - A_2$, and the classical theory for heat equation $u_t = u_{xx}$ of bounded initial data $b_2(x)$ with $\int_{\mathbf{R}} b_2(x) dx = 0$, we get (b).

(c) follows from our estimates in (47) and some elementary calculation.

(d) follows from the same argument as for (b).

Q.E.D.

Applying the Lemma inductively, we have

$$\begin{aligned} |A_2^{n+1} - A_2^n| &\leq C(L)\epsilon_n ||a_2^n||, \\ |A_1^{n+1} - A_1^n| &\leq C(L) ||a_1^n||^\alpha \left((A_2^n)^{\beta-1} + ||b_2^n||^{\beta-1} + \epsilon_n^{\beta-1} ||a_2^n||^{\beta-1} \right) (||b_2^n|| + \epsilon_n ||a_2^n||) + \\ &\quad C(L)(A_2^n)^\beta \left((A_1^n)^{\alpha-1} + ||b_1^n||^{\alpha-1} + (\epsilon_n ||a_2^n||)^{\alpha-1} \right) \cdot (||b_1^n|| + \epsilon_n ||a_2^n||), \\ ||b_2^{n+1}|| &\leq L^{-2\delta} ||b_2^n|| + C(L)\epsilon_n ||a_2^n||, \\ ||b_1^{n+1}|| &\leq L^{-2\delta} ||b_1^n|| + C(L)\epsilon_n ||a_2^n||, \quad n = 1, 2, \dots, \end{aligned}$$

where $\epsilon_n = ||a_1^n||^\alpha ||a_2^n||^{\beta-1}$. We show $A_i^n, ||b_i^n||, i = 1, 2$ converge as $n \rightarrow \infty$. Let

$$A^n = A_1^n + A_2^n, \quad a^n = ||a_1^n|| + ||a_2^n||, \quad b^n = ||b_1^n|| + ||b_2^n||, \quad n = 1, 2, \dots$$

Then,

$$\begin{aligned} 0 &\leq A^{n+1} \leq A^n + C(L)\epsilon_n a^n, \\ b^{n+1} &\leq L^{-2\delta} b^n + C(L)\epsilon_n a^n, \quad n = 1, 2, \dots \end{aligned}$$

By using the fact that

$$\epsilon_n = ||a_1^n||^\alpha ||a_2^n||^{\beta-1} \leq C / (\log L^{2n})^{\alpha/(\alpha-1)} \leq C(L)n^{-\alpha/(\alpha-1)}$$

and a^n is uniformly bounded, we deduce that A^n converges as $n \rightarrow \infty$. Since, $\{A_2^n\}$ is an increasing sequence, this means both A_1^n and A_2^n converges as $n \rightarrow \infty$. As a matter of fact, we know from the evolution of L^1 -norm, as deduced from (39) and Lemma 1, of (u_1, u_2) that $A_1^n \rightarrow 0$ and $A_2^n \rightarrow A$ as $n \rightarrow \infty$.

As to b^n , we have

$$\begin{aligned} b^{n+1} &\leq L^{-2\delta} b^n + C(L)n^{-\alpha/(\alpha-1)} \\ &\leq L^{-2\delta} (L^{-2\delta} b^{n-1} + C(L)(n-1)^{-\alpha/(\alpha-1)}) + C(L)n^{-\alpha/(\alpha-1)} \leq \dots \\ &\leq L^{-2n\delta} b^1 + C(L) \sum_{l=0}^{n-1} L^{-2l\delta} (n-l)^{-\alpha/(\alpha-1)} \\ &\leq C(L)n^{-\alpha/(\alpha-1)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

To calculate the exact limit of A_1^n , we first note $\|a_1^n\| \leq C(L)n^{-1/(\alpha-1)}$ and therefore, $A_1^n \leq C(L)n^{-1/(\alpha-1)}$. Then from (c) of Lemma 8 and the above bound for b^n we get

$$\begin{aligned} |A_1^{n+1} - A_1^n + \gamma(A_1^n)^\alpha(A_2^n)^\beta| &\leq C(L)\{n^{-\alpha/(\alpha-1)} \times n^{-\alpha/(\alpha-1)} + ((A_1^n)^{\alpha-1} + n^{-\alpha})n^{-\alpha/(\alpha-1)}\} \\ &\leq C(L)(n^{-2\alpha/(\alpha-1)} + (A_1^n)^{\alpha-1}n^{-\alpha/(\alpha-1)} + n^{-\alpha^2/(\alpha-1)}) \\ &\leq C(L)n^{-(2\alpha-1)/(\alpha-1)}. \end{aligned}$$

Here we use the fact that $1 < \alpha < 2$. Hence, the difference equation which A_1^n satisfies takes the form

$$A_1^{n+1} = A_1^n - \gamma(A_1^n)^\alpha(A_2^n)^\beta + O(n^{-(2\alpha-1)/(\alpha-1)}).$$

Taking into account that $A_2^n \rightarrow A$ as $n \rightarrow \infty$, we can write it as

$$A_1^{n+1} = A_1^n - \gamma(A_1^n)^\alpha(A)^\beta + o((A_1^n)^\alpha) + O(n^{-(2\alpha-1)/(\alpha-1)}).$$

Using the fact that the difference equation of the form with $\Gamma > 0$

$$y^{n+1} = y^n - \Gamma(y^n)^\alpha + O(n^{-(2\alpha-1)/(\alpha-1)})$$

has the maximum principle if the positive sequence $\{y_n\}$ is sufficiently small, and $y_n = cn^{-1/(\alpha-1)}$ is a sub-solution if $\Gamma c^{\alpha-1} < 1/(\alpha-1)$ but a super-solution if $\Gamma c^{\alpha-1} > 1/(\alpha-1)$, we get $y_n n^{1/(\alpha-1)}$ has a positive finite limit c satisfying $\Gamma c^{\alpha-1} = 1/(\alpha-1)$ as $n \rightarrow \infty$. Therefore, it is also true for A_1^n .

$$A_1^n n^{1/(\alpha-1)} \rightarrow B \quad \text{as } n \rightarrow \infty,$$

where

$$B = \left(\frac{1}{(\alpha-1)\gamma A^\beta} \right)^{1/(\alpha-1)}.$$

Proof of Theorem 1: It is clear that Theorem 1 follows directly from the limits

$$A_1^n n^{1/(\alpha-1)} \rightarrow B \quad \text{and} \quad A_2^n \rightarrow A \quad \text{as } n \rightarrow \infty.$$

Q.E.D.

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