3-symmetric and 3-decomposable geometric drawings of K_n

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Abstract

Even the most superficial glance at the vast majority of crossing-minimal geometric drawings of K_n reveals two hard-to-miss features. First, all such drawings appear to be 3-fold symmetric (or simply 3-*symmetric*). And second, they all are 3-*decomposable*, that is, there is a triangle T enclosing the drawing, and a balanced partition A, B, C of the underlying set of points P, such that the orthogonal projections of P onto the sides of T show A between B and C on one side, B between A and C on another side, and C between A and B on the third side. In fact, we conjecture that all optimal drawings are 3-decomposable, and that there are 3-symmetric optimal constructions for all n multiple of 3. In this paper, we show that any 3-decomposable geometric drawing of K_n has at least $0.380029\binom{n}{4} + \Theta(n^3)$ crossings. On the other hand, we produce 3-symmetric and 3-decomposable drawings that improve the *general* upper bound for the rectilinear crossing number of K_n to $0.380488\binom{n}{4} + \Theta(n^3)$. We also give explicit 3-symmetric and 3-decomposable constructions for $n < 100$ that are at least as good as those previously known.

1 Introduction

For a finite set of points P in general position in the plane, let $\overline{\mathrm{cr}}\,(P)$ denote the number of crossings in the complete *geometric graph* with vertex set P, that is, the complete graph whose edges are straight line segments. Since two edges cross each other if and only if the four points that define them form a convex quadrilateral, it follows that $\overline{\mathrm{cr}}(P)$ equals $\square(P)$, the number of convex quadrilaterals defined by points in P. If P has n vertices, the complete geometric graph with vertex set P is also called a *rectilinear* or *geometric drawing* of K_n. The *rectilinear crossing number* of K_n, denoted $\overline{\mathrm{cr}}(K_n)$, is the minimum number of crossings in a rectilinear drawing of K_n. That is,

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$\overline{\text{cr}}(K_n) = \min_{|P|=n} \overline{\text{cr}}(P)$, where the minimum is taken over all n-point sets P in general position in the plane. Determining $\overline{\text{cr}}(K_n)$ is a well-known problem in combinatorial geometry posed by Erdős and Guy [13].

Figure 1(a) shows the point set of an optimal (crossing minimal) rectilinear drawing of K_{18} [6]. This drawing exhibits a natural partition of the 18 vertices into 3 clusters of 6 vertices each, with two prominent features: (i) rotating any cluster angles of $2\pi/3$ and $4\pi/3$ around a suitable point, one obtains point sets highly resembling the other two clusters; and (ii) the orthogonal projections of these clusters on the sides of an enclosing triangle, have each projected cluster separating the other two. A similar structure is observed in *every* known optimal drawing of K_n, for every n multiple of 3, perhaps after an order-type preserving transformation (see [4, 6]). Even the best available examples for $n > 27$, i.e., for those values of n for which the exact value of $\overline{\text{cr}}(K_n)$ is still unknown, share this property [6].

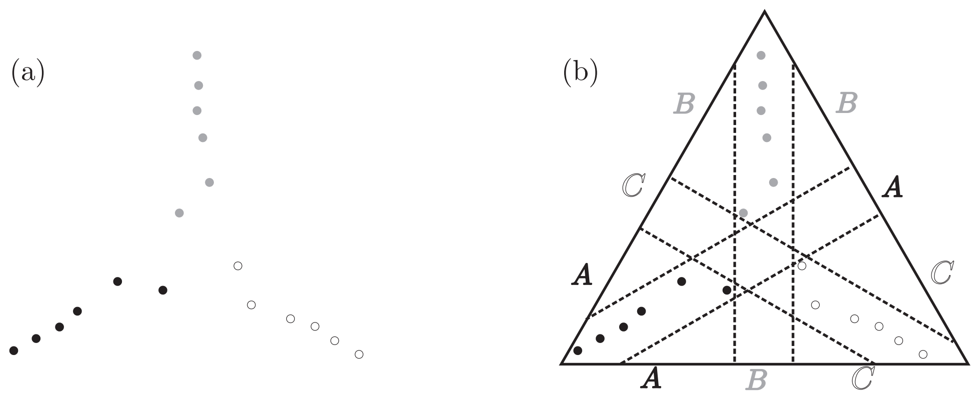

Figure 1: (a) An optimal geometric drawing of K_{18}. (b) The drawing in (a) is 3-decomposable.

To further explore the distinguishing features of these drawings, we introduce the concepts of 3-*symmetry* and 3-*decomposability*. A geometric drawing of K_n is 3-*symmetric* if its underlying point set P is partitioned into three sets (we call them *wings*) of size $n/3$ each, with the property that rotating each wing angles of $2\pi/3$ and $4\pi/3$ around a suitable point generates the other two wings. We also say that P itself is 3-*symmetric*. — A finite point set P is 3-*decomposable* if it can be partitioned into three equal-size sets A, B, and C satisfying the following: there is a triangle T enclosing P such that the orthogonal projections of P onto the — three sides of T show A between B and C on one side, B between A and C on another side, and C between A and B on the third side. We say that a geometric drawing of K_n is 3-*decomposable* if its underlying point set is 3-*decomposable*. We note that whenever we speak of a 3-decomposable or 3-symmetric drawing of K_n, it is implicitly assumed that n is a multiple of 3.

In this paper, we report our recent research on 3-decomposable and 3-symmetric drawings. The rest of the paper is organized as follows.

In Section 2 we recall basic facts about circular sequences, a tool we use throughout the paper.

Then we move on to one of the main results in this paper, namely a lower bound for the number of crossings in 3-decomposable geometric drawings. Following [2] and [15], to bound the number of crossings in a drawing we bound the number of ($\leq k$)-sets in the underlying point set. This is done in Section 3. We then use these bounds in Section 4 to get our improved lower bound for the number of crossings in 3-decomposable geometric drawings.

Our other main result is the construction of 3-symmetric geometric drawings of K_n whose number of crossings is either equal to or smaller than the best previously known (crossing-wise) drawings. The main arguments and techniques are presented in Section 5. The basic idea is to start with a set P of m points, the underlying point set of a geometric drawing of K_m, and to substitute some points of P by clusters of points. We then apply these results in Section 6 to the case in which each point is substituted by two points. In Section 7 we summarize the improved upper bounds we obtained using the arguments and techniques from Sections 5 and 6. An important consequence of the work in Section 7 is given in Section 8, where we establish the best known upper bound for the rectilinear crossing number constant $q_* := \lim_{n\to\infty} \overline{\text{cr}}(K_n)/\binom{n}{4}$.

2 Background: circular sequences

An *allowable sequence* $\mathbf{\Pi}$ is a doubly infinite sequence $\ldots \pi_{-1}, \pi_0, \pi_1, \ldots$ of permutations of n elements, where consecutive permutations differ by a transposition of neighboring elements, and π_i is the reverse permutation of $\pi_{i+\binom{n}{2}}$. Then any subsequence Π of $\binom{n}{2}+1$ consecutive permutations in $\mathbf{\Pi}$ contains all necessary information to reconstruct the entire allowable sequence. Π is called a *halfperiod* of $\mathbf{\Pi}$.

Our interest in allowable sequences derives from the fact that all the combinatorial information of an n-point set P can be encoded by an allowable sequence $\mathbf{\Pi}_P$ on the set P, called the *circular sequence* associated to P. We assume from P that any two lines joining points in P are not parallel (we can assume this without loss of generality, since it can be ensured by sufficiently small perturbations of the points, and this will not affect the number of convex quadrilaterals or the number of ($\leq k$)-sets). A halfperiod Π of $\mathbf{\Pi}_P$ is obtained as follows: Start with a circle C containing P in its interior, and a directed line ℓ tangent to C. By rotating ℓ, while keeping it tangent to C, if necessary, we may guarantee that all points of P have distinct projections over ℓ. Project P orthogonally onto ℓ, and record the order of the points in P on ℓ. This will be the initial permutation π_0 of Π. Now, we continuously rotate ℓ on C (clockwise) and keep projecting P orthogonally onto ℓ. Right after two points overlap in the projection, say p and q, the order of P on ℓ will change. This new order of P on ℓ will be π_1. Note that π_1 is obtained from π_0 by the transposition of pq. Continue doing this, rotating ℓ on C and recording the corresponding permutations of P, until completing half a turn on C. At this time, the order of P on ℓ will be the reverse than the original. Moreover, exactly $\binom{n}{2}$ transpositions have taken place, one per each pair of points.

It is important to note that most allowable sequence are not circular sequences. In fact, allowable sequence are in one-to-one correspondence with generalized configurations of points. We refer the reader to the seminal work by Goodman and Pollack [14] for further details.

Observe that if P is 3-decomposable with partition A, B, and C, then there is a halfperiod $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}})$ of $\mathbf{\Pi}_P$ whose points can be labeled $A = \{a_1, \ldots, a_{n/3}\}$, $B = \{b_1, \ldots, b_{n/3}\}$, and $C = \{c_1, \ldots, c_{n/3}\}$, so that $\pi_0 = (a_1, a_2, ..., a_{n/3}, b_1, b_2, \ldots, b_{n/3}, c_1, c_2, \ldots, c_{n/3})$, and for some indices $0 < s < t \leq \binom{n}{2}$, π_{s+1} shows all the b-elements followed by all the a-elements followed by all the c-elements, and π_{t+1} shows all b-elements followed by all the c-elements followed by all the a-elements. An allowable sequence with a halfperiod satisfying these properties is called 3-*decomposable*, generalizing the definition of 3-decomposability from point-sets to allowable sequences.

We have the following definitions and notation for allowable sequences. A transposition that occurs between elements in sites i and $i+1$ is an i-*transposition*. For $i \leq n/2$, an i-*critical* transposition is either an i-transposition or an $(n-i)$-transposition, and a $(\leq k)$-*critical* transposition is a transposition that is i-critical for some $i \leq k$. If Π is a halfperiod, then $N_{\leq k}(\Pi)$ denotes the number of $(\leq k)$-critical transpositions in Π.

3 Bounding the number of $(\leq k)$-sets in 3-decomposable sets

Throughout this section, P is a set of n points in general position in the plane.

Recall that a $(\leq k)$-*set* of a point set P is a subset of P with at most k elements that can be separated from the rest of P by a straight line. We let $\chi_{\leq k}(P)$ denote the number of $(\leq k)$-sets of P.

If $\mathbf{\Pi} = \mathbf{\Pi}_P$ is the circular sequence associated to P, then $(\leq k)$-critical transpositions in $\mathbf{\Pi}$ correspond to $(\leq k)$-sets of P. Thus, for any halfperiod Π of $\mathbf{\Pi}_P$,

$$\chi_{\leq k}(P) = N_{\leq k}(\Pi). \tag{1}$$

Our main result in this section is the following.

Theorem 1. *Let P be a 3-decomposable set of n points, where n is a multiple of 3, and let $k < n/2$. Then*

$$\chi_{\leq k}(P) \geq B(k,n),$$

where

$$B(k,n) := 3\binom{k+1}{2} + 3\binom{k+1-n/3}{2} + 3\sum_{j=2}^{s-1} j(j+1)\binom{k+1-c_j n}{2}, \tag{2}$$

$c_j := \frac{1}{2} - \frac{1}{3j(j+1)}$, *and $s := s(k,n)$ is the unique integer such that $\binom{s}{2} < \frac{n}{3(n-2k-1)} \leq \binom{s+1}{2}$.*

(In case r is not an integer, we use the formal definition $\binom{r}{2} = \frac{r(r-1)}{2}$. Also, by convention, $\binom{r}{2} = 0$ if $r < 2$.)

In view of Eq. (1), Theorem 1 is a direct consequence of the following (as it happens, more general) version for circular sequences:

Theorem 2. *Let Π be a 3-decomposable halfperiod on n points, and let $k < n/2$. Let $B(k,n)$ be as in Theorem 1. Then*

$$N_{\leq k}(\Pi) \geq B(k,n).$$

Proof. Throughout the proof, $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\binom{n}{2}})$ is a 3-decomposable halfperiod on n points, with initial permutation $\pi_0 = (a_1, \ldots, a_{n/3}, \ldots, a_1, b_1, \ldots, b_{n/3}, c_1, \ldots, c_{n/3})$ and $A = \{a_1, \ldots, a_{n/3}\}$, $B = \{b_1, \ldots, b_{n/3}\}$, and $C = \{c_1, \ldots, c_{n/3}\}$.

In order to lower bound the number of $(\leq k)$-critical transpositions in Π, we distinguish two types of transpositions. A transposition is *monochromatic* if it occurs between two a-elements, between two b-elements, or between two c-elements; otherwise it is called *bichromatic.* We let $N^{mono}_{\leq k}(\Pi)$ (respectively, $N^{bi}_{\leq k}(\Pi)$) denote the number of monochromatic (respectively, bichromatic) $(\leq k)$-critical transpositions in Π, so that

$$N_{\leq k}(\Pi) = N^{mono}_{\leq k}(\Pi) + N^{bi}_{\leq k}(\Pi). \tag{3}$$

We now bound $N^{mono}_{\leq k}(\Pi)$ and $N^{bi}_{\leq k}(\Pi)$ separately.

Claim 1. *Let Π be a 3-decomposable halfperiod on n points, and let $k < n/2$. Then*

$$N^{bi}_{\leq k}(\Pi) = \begin{cases} 3\binom{k+1}{2} & \text{if } k \leq n/3, \\ 3\binom{n/3+1}{2} + (k - n/3)n & \text{if } n/3 < k < n/2. \end{cases}$$

Proof. Each bichromatic transposition is either an ab- or an ac- or a bc-transposition. Since Π is 3-decomposable, A and $B \cup C$ are separated in π_0. Using only this fact, we compute the number of i-critical bichromatic transpositions involving A, that is, the ab- and ac-transpositions together. This number multiplied by $3/2$ is the total number of bichromatic i-critical transpositions of Π. This is because, by definition of 3-decomposable, there is a permutation π_{s+1} of Π where B is separated from $A \cup C$, as well as a permutation π_{t+1} where C is separated from $A \cup B$. Thus, multiplying by 3 counts each i-critical bichromatic transposition twice.

For $x \in \{b, c\}$ each ax-transposition in Π moves the involved a to the right and the involved b or c to the left. Since A occupies the first $n/3$ positions in π_0, then A must occupy the last $n/3$ positions in $\pi_{\binom{n}{2}}$. For each $i \leq n/3$, a bichromatic i-transposition involving A, replaces one a-element occupying one of the first i-positions by a b- or a c-element. This must happen exactly i times in order for A to leave the first i positions. That is, there are exactly i bichromatic i-transpositions involving A. Similarly, for each $i \geq 2n/3$, there are exactly i bichromatic i-transpositions involving A (each of these transpositions replaces one b- or c-element in the last i positions by an a-element). Finally, for $n/3 < i < 2n/3$, there are exactly $n/3$ bichromatic i-transpositions involving A, since all elements of A must leave the region formed by the first i positions. Therefore, the number of $(\leq k)$-critical bichromatic transpositions is exactly $\sum_{i=1}^{k} 3i = 3\binom{k+1}{2}$ if $k \leq n/3$, and $\sum_{i=1}^{n/3} 3i + \sum_{i=n/3}^{k} n = 3\binom{n/3+1}{2} + (k - n/3)n$ if $n/3 < k < n/2$. □

We now move on to establish a lower bound for $N^{mono}_{\leq k}(\Pi)$.

A transposition between elements in positions i and $i+1$ with $k < i < n - k$ is called a $(> k)$-*transposition*. All these transpositions are said to occur in the k-*center* (of Π). Our goal is to give a lower bound (see Claim 3) for $N^{mono}_{\leq k}(\Pi)$. Each monochromatic transposition is an aa- or bb-, or cc-transposition. Our approach is to find an upper bound for the number of $(> k)$-critical aa-, bb-, and cc-transpositions, denoted by $N^{aa}_{>k}(\Pi)$, $N^{bb}_{>k}(\Pi)$, and $N^{cc}_{>k}(\Pi)$, respectively. The lower bound for $N^{mono}_{\leq k}(\Pi)$ follows from the observation that the number of ($\leq k$)-critical aa-transpositions is exactly $\binom{n/3}{2} - N^{aa}_{>k}(\Pi)$, and similarly for bb- and cc-transpositions. Thus

$$N^{mono}_{\leq k}(\Pi) = 3\binom{n/3}{2} - N^{aa}_{>k}(\Pi) - N^{bb}_{>k}(\Pi) - N^{cc}_{>k}(\Pi). \tag{4}$$

Again, we bound $N^{aa}_{>k}(\Pi)$ using only the fact that there is a permutation where A is separated from $B \cup C$, and thus this bound is the same for $N^{bb}_{>k}(\Pi)$ and $N^{cc}_{>k}(\Pi)$.

It is known that for $k \leq n/3$, the bound $N_{\leq k}(\Pi) \geq 3\binom{k+1}{2}$ is tight. Since we have shown that there are $3\binom{k+1}{2}$ bichromatic ($\leq k$)-transposition, we focus on the case $n/3 < k < n/2$. In this case, let D_k be the digraph with vertex set $1, 2, \ldots, n/3$, and such that there is a directed edge from i to j if and only if $i < j$ and the transposition $a_i a_j$ occurs in the k-center. Then the number of edges of D_k is exactly $N^{aa}_{>k}(\Pi)$.

We now bound the number of edges in D_k using the following essential observation. We denote the outdegree and the indegree of a vertex v in a digraph by $[v]^+$ and $[v]^-$, respectively.

Claim 2. *For the graph D_k,*

$$[i]^+ \leq \min\{n - 2k - 1 + [i]^-, n/3 - i\}. \tag{5}$$

Proof. Clearly, $[i]^+ \leq n/3 - i$ because there are only $n/3 - i$ indices $j > i$. To show that $[i]^+ \leq n - 2k - 1 + [i]^-$, note that $n - 2k - 1 + [i]^-$ is the number of ($> k$)-transpositions in which a_i moves right, and only $[i]^+$ of these transpositions involve two a-elements. Indeed, $[i]^-$ is the number of ($> k$)-transpositions involving two a-elements in which a_i moves backward. There are $n - 2k - 1$ forced ($> k$)-transpositions of a_i: since a_i moves from position i to position $n - i + 1$, for each $k < j < n - k$ there is at least one j-transposition in which a_i moves right. Also, each of the $[i]^-$ transpositions in which a_i moves left in the k-center allows an extra transposition in the k-center in which a_i moves right. □

Claim 3. *If Π is a 3-decomposable halfperiod on n points, and $n/3 < k < n/2$, then*

$$N^{mono}_{\leq k}(\Pi) \geq B(k, n) - 3\binom{n/3 + 1}{2} - (k - n/3)n.$$

Proof. We just need to show that D_k has at most $\binom{n/3}{2} - \frac{1}{3}\left(B(k,n) - 3\binom{n/3+1}{2} - (k - n/3)n\right) = \frac{1}{3}(kn - B(k,n))$ edges. We start by giving two definitions. Let $\mathcal{D}_{v,m}$ be the class of all digraphs on v vertices $1, 2, \ldots, v$ satisfying that $[i]^+ \leq m + [i]^-$ for all $1 \leq i \leq v$, and $i < j$ whenever $i \to j$. Let $D_0(v, m)$ be the graph in $\mathcal{D}_{v,m}$ with vertices $1, 2, \ldots, v$ recursively defined by

- $[1]^- = 0$,
- $[i]^+ = \min\left\{[i]^- + m, v - i\right\}$ for each $i \geq 1$, and
- for all $1 \leq i < j \leq v$, $i \to j$ if and only if $i + 1 \leq j \leq i + [i]^+$.

These definitions are equivalent to those in [10] (pages 677 and 683). There, Balogh and Salazar show that the maximum of the function $2\sum_{i=1}^{v}[i]^{-}+\sum_{i=1}^{v}\min\left\{[i]^{-}-[i]^{+}+m,m+1\right\}$ over all digraphs in $\mathcal{D}_{v,m}$ is attained by $D_0\left(v,m\right)$. Their original statement imposes some dependency between v and m, but this is only used to bound the given function applied to $D_0\left(v,m\right)$. And their proof, actually maximizes separately each of the two sums above. In other words, they implicitly show that the maximum number of edges of a graph in $\mathcal{D}_{v,m}$ is attained by $D_0\left(v,m\right)$.

Note that D_k is in $\mathcal{D}_{n/3,n-2k-1}$, and thus its number of edges is bounded above by the number of edges of $D_0\left(n/3,n-2k-1\right)$. Thus, it suffices to bound above the number of edges of $D_0\left(n/3,n-2k-1\right)$.

Claim 4. *$D_0\left(n/3,n-2k-1\right)$ has at most $\frac{1}{3}\left(kn-B\left(k,n\right)\right)$ edges.*

We observe that Claim 4 completes the proof of Claim 3. □

Proof of Claim 4. We first obtain an expression for the exact number of edges in $D_0(n/3,n-2k-1)$, and then we show that this value is upper bounded by the expression in Claim 4. For brevity, in the rest of the section, we use $D_0:=D_0(n/3,n-2k-1)$, $v:=n/3$ and $m:=n-2k-1$.

For positive integers $j\leq i$ define (c.f., Definition 16 in [10]) $S_j(i)$ as the unique nonnegative integer such that

$$\binom{S_j(i)}{2}<\frac{i}{j}\leq\binom{S_j(i)+1}{2}; \text{ and}$$

$T_j(i)$ and $U_j(i)$ as the unique integers satisfying $0\leq T_j(i)\leq j-1$, $0\leq U_j(i)\leq S_j(i)-1$, and

$$i=1+j\binom{S_j(i)}{2}+S_j(i)T_j(i)+U_j(i). \tag{6}$$

The key observation is that we know the indegree of each vertex in D_0. We now find a closed expression for $E(k,n):=\sum_{i=1}^{v}[i]^{-}$, the number of edges in D_0. We break $\sum_{i=1}^{v}[i]^{-}$ into three parts. Let $v_1:=m\binom{S_m(v)}{2}$, $v_2:=S_m(v)T_m(v)$, and set

$$V_1=\sum_{i=1}^{v_1}[i]^{-},\ V_2=\sum_{i=v_1+1}^{v_1+v_2}[i]^{-},\text{ and } V_3=\sum_{i=v_1+v_2+1}^{v}[i]^{-}$$

so that

$$\sum_{i=1}^{v}[i]^{-}=V_1+V_2+V_3. \tag{7}$$

We calculate V_1, V_2, and V_3 separately. If ℓ,j are integers such that $1\leq j\leq S_m(v)-1$ and $0\leq\ell\leq m$, we define $P_j:=\{i:S_m(i)=j\}$ and $Q_{j,\ell}:=\{i\in P_j:T_m(i)=\ell\}$.

We first calculate V_1. Note that $P_1,P_2,\ldots,P_{S_m(v)-1}$ is a partition of $\{1,2,...,v_1\}$ and $Q_{j,0},Q_{j,1},\ldots,Q_{j,m}$ is a partition of P_j, for each $1\leq j\leq S_m(v)-1$. Also, $S_m(v_1+1)=S_m(v)$ and $S_m(i)\leq S_m(v)-1$ for $1\leq i\leq v_1$. Thus V_1 can be rewritten as $\sum_{j=1}^{S_m(v)-1}\sum_{i\in P_j}[i]^{-}$. By

Proposition 17 in [10], for each vertex $1 \leq i \leq v$ of D_0, $[i]^- = m\left(S_m(i)-1\right)+T_m(i)$. Therefore

$$\begin{aligned} V_1 &= \sum_{j=1}^{S_m(v)-1}\left(m\sum_{i\in P_j}\left(S_m(i)-1\right)+\sum_{i\in P_j}T_m(i)\right) \\ &= \sum_{j=1}^{S_m(v)-1}\left(m\sum_{i\in P_j}(j-1)+\sum_{\ell=0}^{m}\sum_{i\in Q_{j,\ell}}\ell\right). \end{aligned}$$

On other hand, by definition $|Q_{j,\ell}| = j$ for $0 \leq \ell \leq m-1$, which implies that $|P_j| = mj$. Therefore

$$\begin{aligned} V_1 &= \sum_{j=1}^{S_m(v)-1}\left(m^2 j\,(j-1)+\sum_{\ell=0}^{m}\ell\,|Q_{j,\ell}|\right) = \sum_{j=1}^{S_m(v)-1}\left(m^2 j\,(j-1)+j\sum_{\ell=1}^{m-1}\ell\right) \\ &= \sum_{j=1}^{S_m(v)-1}\left(2m^2\binom{j}{2}+\binom{m}{2}j\right) = 2m^2\binom{S_m(v)}{3}+\binom{m}{2}\binom{S_m(v)}{2}. \end{aligned} \tag{8}$$

Now, we calculate V_2. Since $S_m(i) = S_m(v)$ for each $v_1+1 \leq i \leq v$, and $[i]^- = m\left(S_m(i)-1\right)+T_m(i)$, then $V_2 = \sum_{i=v_1+1}^{v_1+v_2}[i]^- = \sum_{i=v_1+1}^{v_1+v_2} m\left(S_m(v)-1\right)+T_m(i)$. Therefore

$$V_2 = \sum_{i=v_1+1}^{v_1+v_2} m\left(S_m(v)-1\right)+\sum_{i=v_1+1}^{v_1+v_2}T_m(i) = m\left(S_m(v)-1\right)S_m(v)T_m(v)+\sum_{i=v_1+1}^{v_1+v_2}T_m(i).$$

Again, we have that $\left|Q_{S_m(v),\ell}\right| = S_m(v)$ for every $0 \leq \ell \leq m-1$. Because $0 \leq T_m(i) \leq T_m(v)-1$ for every $v_1+1 \leq i \leq v_1+v_2$, and $T_m(v_1+v_2+1) = T_m(v)$, it follows that $Q_{S_m(v),0}, Q_{S_m(v),1}, \ldots, Q_{S_m(v),T_m(v)-1}$ is a partition of $\{v_1+1, v_1+2, \ldots, v_1+v_2\}$. Thus

$$\sum_{i=v_1+1}^{v_1+v_2}T_m(i) = \sum_{\ell=0}^{T_m(v)-1}\sum_{i\in Q_{S_m(v),\ell}}T_m(i) = \sum_{\ell=0}^{T_m(v)-1}\ell\left|Q_{S_m(v),\ell}\right| = \sum_{\ell=1}^{T_m(v)-1}\ell\cdot S_m(v) = S_m(v)\binom{T_m(v)}{2}.$$

Then

$$V_2 = 2m\binom{S_m(v)}{2}T_m(v)+S_m(v)\binom{T_m(v)}{2}. \tag{9}$$

Finally, we calculate V_3. Since $S_m(i) = S_m(v)$ and $T_m(i) = T_m(v)$ for every $v_1+v_2+1 \leq i \leq v$ and $[i]^- = m\left(S_m(i)-1\right)+T_m(i)$, it follows that

$$\begin{aligned} V_3 = \sum_{i=v_1+v_2+1}^{v}[i]^- &= \sum_{i=v_1+v_2+1}^{v} m\left(S_m(i)-1\right)+T_m(i) = \sum_{i=v_1+v_2+1}^{v} m\left(S_m(v)-1\right)+T_m(v) \\ &= (v-v_1-v_2)\left(m\left(S_m(v)-1\right)+T_m(v)\right) \end{aligned}$$

From (6) it follows that $U_m(v)+1 = v-v_1-v_2$, and so

$$V_3 = \left(U_m(v)+1\right)\left(m\left(S_m(v)-1\right)+T_m(v)\right). \tag{10}$$

Now from (7), (8), (9), and (10), it follows that

$$E(k,n) := 2m^2\binom{S_m(v)}{3} + \binom{m}{2}\binom{S_m(v)}{2} + 2m\cdot T_m(v)\binom{S_m(v)}{2} + \tag{11}$$

$$\binom{T_m(v)}{2}S_m(v) + (U_m(v)+1)\left(m(S_m(v)-1)+T_m(v)\right).$$

Now recall that $v := n/3$ and $m := n-2k-1$. If $k > n/3$, then $v \geq m$. From (6), it follows that

$$T_m(v) = \frac{v-1-m\binom{S_m(v)}{2} - U_m(v)}{S_m(v)}.$$

Note that $s = s(k,n)$ in the definition of $B(k,n)$ is equal to $S_m(v)$. We use this fact, together with the previous identity substituted in the expression of $E(k,n)$ in (11), to obtain the following expression for $E(k,n) + (B(k,n) - kn)/3$. The next identity follows from a long, yet elementary, simplification (which can be efficiently performed in a CAS like Maxima, Mathematica or Maple).

$$\begin{aligned} E(k,n) + \frac{1}{3}\left(B(k,n) - kn\right) &= \frac{4s^2 - s^4 - 3\left(2+2U_m(v)-s\right)^2}{24s} \\ &\leq \frac{s^2\left(4-s^2\right)}{24} \leq 0, \end{aligned}$$

where the last inequality follows from the fact that $s = S_m(v) \geq 2$ whenever $k > n/3$. Thus $E(k,n) \leq \frac{1}{3}\left(kn - B(k,n)\right)$, thus completing the proof of Claim 4. $\square$

We now complete the proof of Theorem 2. By Eq. (5), $N_{\leq k}(\Pi) = N^{mono}_{\leq k}(\Pi) + N^{bi}_{\leq k}(\Pi)$. If $k \leq n/3$, then by Claim 1, $N_{\leq k}(\Pi) \geq N^{bi}_{\leq k}(\Pi) \geq 3\binom{k+1}{2} = B(k,n)$. If $n/3 < k < n/2$, then by Claims 1 and 3, $N_{\leq k}(\Pi) \geq B(k,n)$. $\square$

4 Bounding the number of crossings in 3-decomposable sets

We are now ready to prove a lower bound for the crossing number for 3-decomposable sets.

Theorem 3. *Let P be a 3-decomposable set of n points. Then*

$$\overline{\mathrm{cr}}(P) \geq \frac{2}{27}\left(15-\pi^2\right)\binom{n}{4} + \Theta(n^3) > 0.380029\binom{n}{4} + \Theta(n^3).$$

Proof. Let P be a 3-decomposable set of n points in general position.

First we recall the following relationship between rectilinear crossing numbers and $(\leq k)$-sets, unveiled independently by Ábrego and Fernández-Merchant [2] and by Lovász et al. [15]:

$$\overline{\mathrm{cr}}(P) = \sum_{k=1}^{(n-2)/2} (n-2k-1)\chi_{\leq k}(P) + \Theta(n^3). \tag{12}$$

Now combining Theorem 1 and Eq. (12), and noting that the -1 in the factor $n-2k-1$ only contributes to smaller order terms, we obtain

$$
\begin{aligned}
\overline{\mathrm{cr}}(P) \geq & \sum_{k=1}^{(n-2)/2} (n-2k)\, B(k,n) + \Theta(n^3) \\
= & 36\binom{n}{4}\left(\sum_{k=1}^{(n-2)/2} \frac{1}{n}\left(1-2\left(\frac{k}{n}\right)\right)\left(\frac{k}{n}\right)^2 + \sum_{k=n/3}^{(n-2)/2} \frac{1}{n}\left(1-2\left(\frac{k}{n}\right)\right)\left(\frac{k}{n}-\frac{1}{3}\right)^2 \right.\\
& \left. + \sum_{k=1}^{(n-2)/2}\sum_{j=2}^{s-1} j(j+1)\frac{1}{n}\left(1-2\left(\frac{k}{n}\right)\right)\left(\frac{k}{n}-c_j\right)^2\right) + \Theta(n^3),
\end{aligned}
$$

since $j \leq s(k,n)-1$ if and only if $k > c_j n - 1/2$, then

$$
\begin{aligned}
\overline{\mathrm{cr}}(P) \geq 36\binom{n}{4} & \left(\sum_{k=1}^{(n-2)/2} \frac{1}{n}\left(1-2\left(\frac{k}{n}\right)\right)\left(\frac{k}{n}\right)^2 + \sum_{k=n/3}^{(n-2)/2} \frac{1}{n}\left(1-2\left(\frac{k}{n}\right)\right)\left(\frac{k}{n}-\frac{1}{3}\right)^2 \right.\\
& \left. + \sum_{j=2}^{\infty} j(j+1) \sum_{c_j n - 1/2 < k \leq (n-2)/2} \frac{1}{n}\left(1-2\left(\frac{k}{n}\right)\right)\left(\frac{k}{n}-c_j\right)^2\right) + \Theta(n^3).
\end{aligned}
$$

Each of the inner sums is a Riemann Sum which we estimate using the corresponding integrals. Note that all the error terms are bounded uniformly by $\Theta(n^3)$.

$$
\begin{aligned}
\overline{\mathrm{cr}}(P) \geq 36\binom{n}{4} & \left(\int_0^{1/2}(1-2x)x^2 dx + \int_{1/3}^{1/2}(1-2x)\left(x-\frac{1}{3}\right)^2 dx \right.\\
& \left. + \sum_{j=2}^{\infty} j(j+1)\int_{c_j}^{1/2}(1-2x)(x-c_j)dx\right) + \Theta(n^3) \\
= \binom{n}{4} & \left(\frac{3}{8} + \frac{1}{216} + \frac{2}{27}\sum_{j=2}^{\infty}\frac{1}{j^3(j+1)^3}\right) + \Theta(n^3).
\end{aligned}
$$

Since

$$
\sum_{j=2}^{\infty}\frac{1}{j^3(j+1)^3} = \sum_{j=2}^{\infty}\left(\frac{1}{j^3} - \frac{3}{j^2} + \frac{6}{j} - \frac{1}{(j+1)^3} - \frac{3}{(j+1)^2} - \frac{6}{j+1}\right) = \frac{79}{8} - \pi^2,
$$

then

$$
\overline{\mathrm{cr}}(P) \geq \frac{2}{27}\left(15-\pi^2\right)\binom{n}{4} + \Theta(n^3). \qquad \square
$$

5 Constructing geometric drawings from smaller ones

In this section, we describe a refinement of a method used in [3, 7, 12] to grow a geometric drawing D_m of K_m (the *base* drawing) into a geometric drawing of K_n (the *augmented* drawing) for some $n > m$. The goal is to produce geometric drawings of complete graphs with as few crossings as possible. Our technique refines previous constructions by Brodsky et al. [12], Aichholzer et al. [7], and Ábrego and Fernández-Merchant [3].

The method substitutes each point p_i in the underlying point set of D_m by a *cluster* of points C_i. The cluster C_i is an affine copy of a preset *cluster model* S_i (so that the order types of C_i and S_i are the same) carefully placed near p_i and almost aligned along a line ℓ_i through p_i. More precisely, if $C = \bigcup_{j=1}^{m} C_j$, then ℓ_i divides the set $C \setminus C_i$ into two sets of sizes as equal as possible, and any line spanned by two points in C_i has the same "halving" property as ℓ_i on $C \setminus C_i$. Such a placement helps to minimize the number of convex quadrilaterals that involve two points in C_i and, as a consequence, the total number of crossings in the augmented drawing.

In a nutshell, the difference between our approach and that in [7] is that, for each i, we allow one cluster C_j with $j \neq i$ to be splitted by ℓ_i, and ask that no two clusters split each other. Whereas in [7], each cluster C_j other than C_i is completely contained in a semiplane of ℓ_i. While this step further is more general and powerful, it brings new technical complications that are analyzed and sorted out throughout this section.

Before moving on to describing formally the constructions (starting in Subsection 5.2), we present in the next subsection a fully worked out example, with the intention of both introducing and motivating the main ideas in our constructions.

5.1 An example: drawing a K_{18} from a drawing of K_6

Consider the 6-point set $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ given in Figure 2(a), the underlying point set of a geometric drawing of K_6. Our aim is to substitute each of p_1, p_3, and p_6 with a 2-point set (C_1, C_3, and C_6, respectively), and each of p_2, p_4, and p_5 with a 4-point set (C_2, C_4, and C_5, respectively). At this point it may help the reader to take a sneak peek at Figure 4, the final 18-point set Q.

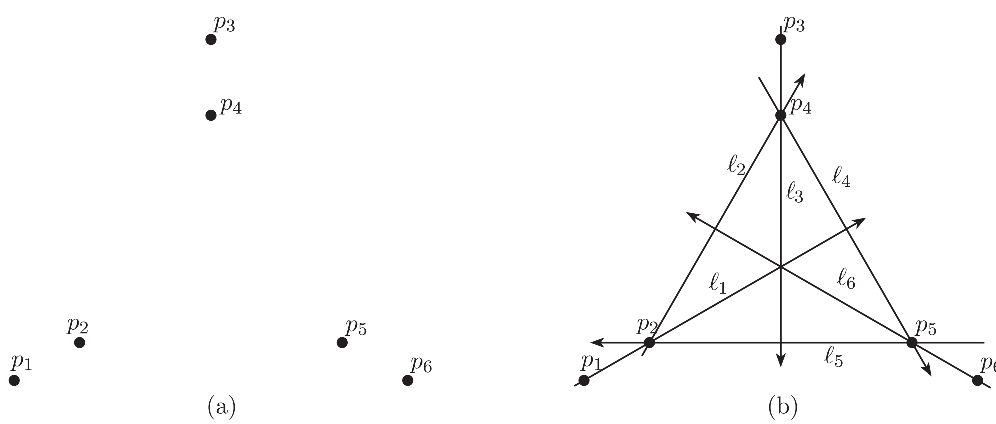

Figure 2: (a) The base 6-point set, (b) A pre-halving set of lines.

The idea is to grow this 6-point set as described so that the final 18-point set is the underlying point set of a K_{18} with as few crossings as possible. Now in order to be able to easily count the final number of crossings, the points in each C_i need to be all placed very closely to each other and to the corresponding p_i. To minimize the final number of crossings, the crucial observations are: (i) the number of crossings defined by points in the same C_i must be as small as possible; and (ii) it is desirable to let the points in each C_i be placed so that the line spanned by any two of them is as close as possible to being a halving line of the final set.

The convenience of accomplishing (i) is evident. To justify (ii) it suffices to take a look at Eq. (12): pairs of points that define a halving line contribute the least to the crossing number.

To satisfy (i) in our current example, it suffices to ensure that the convex hull of each of the 4-point sets of C_2, C_4, and C_5 is a triangle (in general, if some C_i had n points, we would naturally choose the underlying point set of a K_n with as few crossings as possible). Now our strategy to satisfy (ii) is to try to place the points in each C_i very close to a line through p_i that halves $Q \setminus C_i$. Since at this initial stage we have obviously not determined Q, our strategy is, if possible, to find for each p_i a line that is not incident to another p_j and that halves $Q \setminus C_i$. (This is done by taking into account the size of each C_i; even if we have not determined them yet, we know their size in advance.) It is easy to check that this ideal scenario does not occur in our current example: every line through p_1 that does not go through another p_j, does not have one half of the points of $C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6$ in each of its open halfplanes. Thus we adjust the strategy by choosing a line ℓ_1 that spans p_1 and p_2, with the following in mind: the points of p_1 will be placed very close to this line, and we have to be careful to place the correct amount of points of p_2 in each halfplane of this line (so that the line indeed halves $Q \setminus C_1$). Here, the six points of $C_3 \cup C_4$ are on the left halfplane of ℓ_1, and the six points of $C_5 \cup C_6$ are on its right halfplane. Thus (recall that $|C_2| = 2$) one of the points of C_2 will have to be placed on each halfplane of ℓ_1.

We find a line ℓ_i with similar properties for each $i = 2, 3, 4, 5, 6$. For obvious reasons, we call $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$ a *pre-halving* set of lines. The result is illustrated in Figure 2(b).

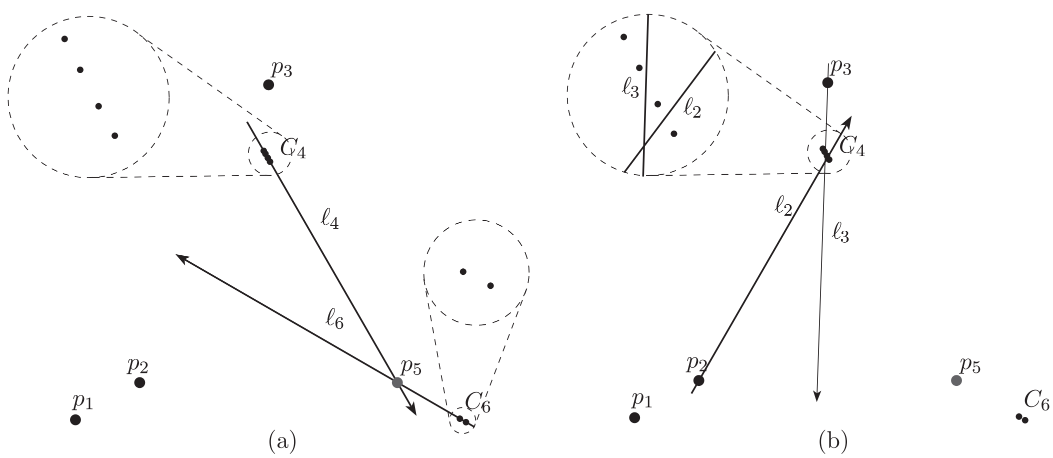

Figure 3: (a) Replacing the points p_i. (b) Ensuring that ℓ_2 and ℓ_3 are halving lines.

It is now the time to start the process of replacing each p_i by its corresponding C_i. It is convenient to do so in two stages: first we replace each p_i by a set U_i of $c_i := |C_i|$ collinear points, placed along ℓ_i. This process of replacing the points p_i is illustrated in Figure 3(a). We note that the points in each U_i must be placed in a very small neighborhood of its corresponding p_i, so that for any three pairwise distinct i, j, k, if we choose one point q_i of U_i, one point q_j of U_j, and one point q_k of U_k, the order type of (q_i, q_j, q_k) is the same as the order type of (p_i, p_j, p_k).

When we replace a point p_i, we must be very careful: if a line ℓ_j spans p_j and p_i, then after the replacement process, we must leave on each halfplane of ℓ_j the same number of points of $Q \setminus U_j$. For instance: ℓ_2 spans p_2 and p_4, and its left (respectively, right) halfplane contains p_1 and p_3 (respectively, p_5 and p_6). Thus, after replacing p_1, p_3, p_5, and p_6, by U_1, U_3, U_5, and U_6, respectively, ℓ_2 will have 4 points to its left (namely $U_1 \cup U_3$) and 6 points to its right (namely $U_5 \cup U_6$). Therefore, when we replace p_4 by U_4, we must ensure that 3 points of U_4 stay to the left of ℓ_2, and the other point of U_4 stays to its right, so that each halfplane of ℓ_2 contains exactly 7 points, see Figure 3(b). Similar care must be taken when we replace each p_i by its corresponding U_i.

Finally, we perturb very slightly each U_i to a set C_i in general position, so that the order type of the whole set remains otherwise unchanged.

In the formal description we shall do in the following subsections, each C_i is taken from a *cluster model* S_i: the ultimate goal is to have C_i with the same order type as S_i, which must be a point set whose crossing number we happen to know exactly.

The final result is illustrated in Figure 4. Admittedly, the points in each C_i, $i = 2, 4, 5$ do not appear to be in general position, but this is a product of the limitation of our drawing and printing technology: a figure in which the four points of C_2 are visibly not collinear would require a much larger size and higher definition.

The geometric drawing of K_{18} resulting from the 18-point set in Figure 4 happens to be quite good in terms of its crossing number: it is not difficult to argue (this is done in its full generality in Subsection 5.5 below) that this geometric drawing of K_{18} has exactly 1035 crossings, as opposed to the 1029 drawings of optimal drawings of K_{18}.

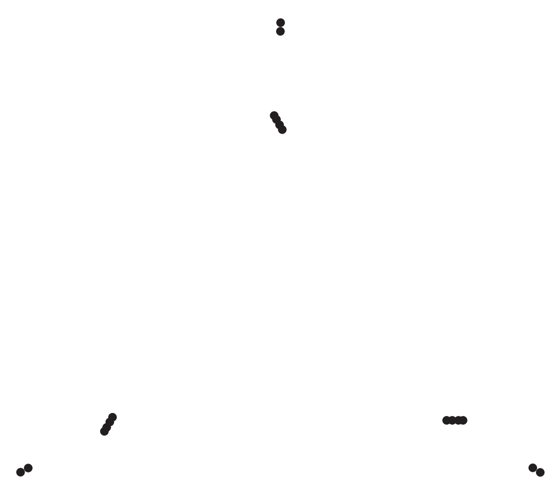

Figure 4: The resulting set, the underlying point set of a geometric drawing of K_{18}.

5.2 Input and output

The primary ingredients of our construction are a base point-set P, sets S_i that serve as models for our clusters, and what we call a *pre-halving* set of lines (Condition 3 below), which is a generalization of the corresponding "halving properties" required in [3, 7].

The input

1. The *base set*: a point set $P = \{p_1, p_2, \ldots, p_m\}$ in general position. This is the underlying set of the *base* geometric drawing of K_m.

2. The *cluster models*: for each $i = 1, 2, \ldots, m$, a nonempty point set S_i in general position. We ask that no two points in a cluster S_i have the same x-coordinate. Let $s_i = |S_i|$ and $I = \{i : s_i > 1\}$.

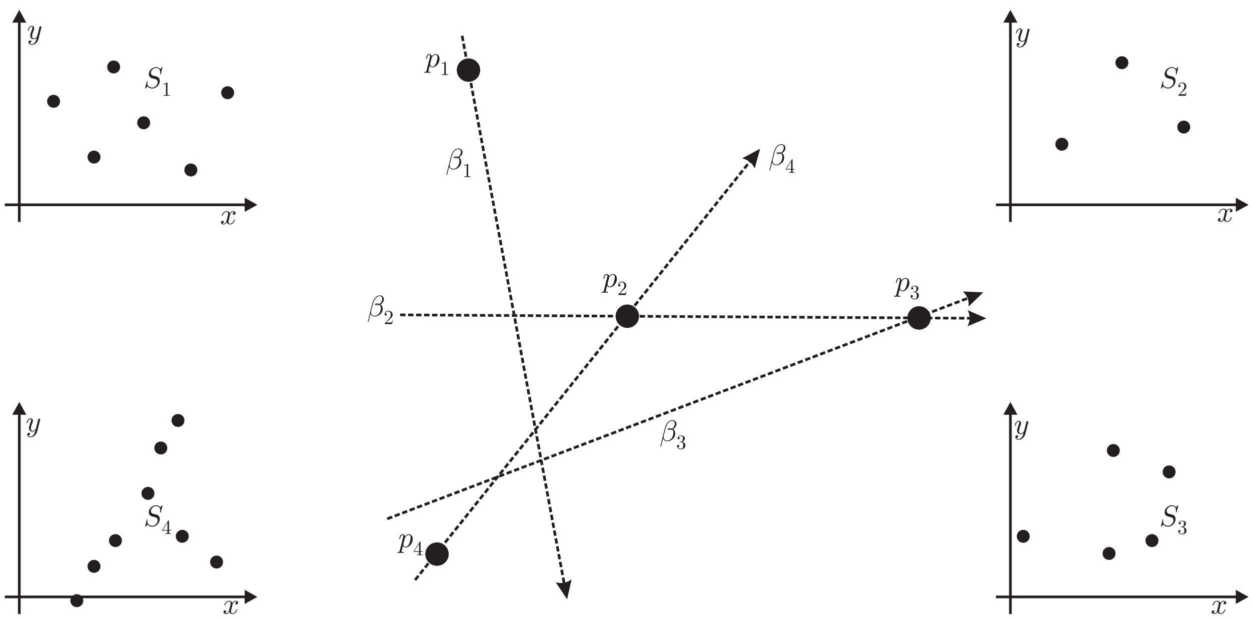

Figure 5: The sets S_1, S_2, S_3, and S_4 are cluster models. We show a pre-halving set of lines $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ for the base point-set $P = \{p_1, p_2, p_3, p_4\}$ and the integers $s_1 = |S_1| = 6, s_2 = |S_2| = 3, s_3 = |S_3| = 5$, and $s_4 = |S_4| = 8$.

3. The pre-halving set of lines: for each $i \in I$, a directed line β_i containing p_i. For each β_i, we let $\mathcal{L}(i)$ (respectively, $\mathcal{R}(i)$) denote the set of those k such that p_k is on the left (respectively, right) semiplane of β_i. If β_i goes through a p_j other than p_i, we say that p_i and β_i are *splitting*. In this case, we say that β_i *splits* p_j, and write $j = \sigma(i)$. Otherwise, p_i and β_i are called *simple*. (Note that $\sigma(i)$ is defined if and only if p_i and β_i are splitting.) The collection of these lines must satisfy the following properties.

   (a) If $i \neq j$, then $\beta_i \neq \beta_j$ and $\beta_i \neq -\beta_j$, the reverse line of β_j.

(b) If β_i is simple, then $0 \leq \sum_{k \in \mathcal{L}(i)} s_k - \sum_{k \in \mathcal{R}(i)} s_k \leq 1$.

(c) If β_i is splitting, then β_i is directed from p_i to $p_{\sigma(i)}$ and $|\sum_{k \in \mathcal{L}(i)} s_k - \sum_{k \in \mathcal{R}(i)} s_k| \leq s_{\sigma(i)} - 1$.

Note that properties (a) to (c) relate only to the point set P and to the integers s_i, and are independent of the order types of the sets S_i.

The output

The construction consists of substituting each p_i, with $i \in I$, by a *cluster* C_i. C_i is a suitable affine copy of S_i whose points are aligned along a line ℓ_i. If $s_i = 1$, then $C_i = \{p_i\}$. The result is a set $C := \bigcup_{i=1}^m C_i$ of $n := |C|$ points in general position, the *augmented* point set. To describe in detail the properties of C_i and ℓ_i, we need a couple of definitions.

A directed line ℓ *halves* a set of points T if the left semiplane of ℓ contains $\lceil |T|/2 \rceil$ points of T, and the right semiplane contains the remaining $\lfloor |T|/2 \rfloor$ points. It follows from the definition that ℓ and T are disjoint. If ℓ is a line that halves a set T, and S is a set of points disjoint from T, then S *halves* T *as* ℓ, if every line ℓ' spanned by two points in S can be directed so that it halves T in exactly the same way as ℓ. That is, the left (respectively, right) semiplane of ℓ' contains the same subset of T as the left (respectively, right) semiplane of ℓ.

With this terminology, the key properties of the sets C_i and of the lines ℓ_i are the following.

(1) **Inherited order type property.** For any three pairwise distinct i, j, k, and $q_i \in C_i, q_j \in C_j, q_k \in C_k$, the order type of the triple $q_i q_j q_k$ is the same as the order type of $p_i p_j p_k$.

(2) **Halving property.** For each $i \in I$, ℓ_i halves $C \setminus C_i$ and C_i halves $C \setminus C_i$ as ℓ_i.

5.3 The construction

Step 1 *Enlarging each point p_i to a very small disc D^i that will contain the cluster C_i.*

For each $i = 1, \ldots, m$, let D^i be a disc of radius r_i centered at p_i, such that the collection D^i satisfies the following. If $q_i \in D^i, q_j \in D^j, q_k \in D^k$ (with i, j, k pairwise distinct), then the order type of the triple $q_i q_j q_k$ is the same as the order type of $p_i p_j p_k$. It is clear that this can be achieved by making the radius of each D^i sufficiently small.

Step 2 *Replacing each p_i with a set U_i contained on $D^i \cap \beta_i$.*

We now construct a first approximation U_i to each cluster C_i. The first simplification is that the each set U_i is collinear, as opposed to C_i, which is in general position. Although, we might certainly describe the construction without using intermediate collinear sets, it is a convenient device that greatly simplifies our work.

For each $i \in I$, consider a similarity transformation that takes the origin to p_i and the x-axis to β_i, such that the image C_i of S_i is contained in the interior of the disc centered at the origin with radius $r_i/2$. Let U_i be the projection of C_i onto β_i, thus U_i lies on β_i. If $s_i = 1$, we make $U_i = C_i = \{p_i\}$. Then U_i is completely contained in D^i for every i. Let $U = \bigcup_{j=1}^m U_j$. See Figure 6.

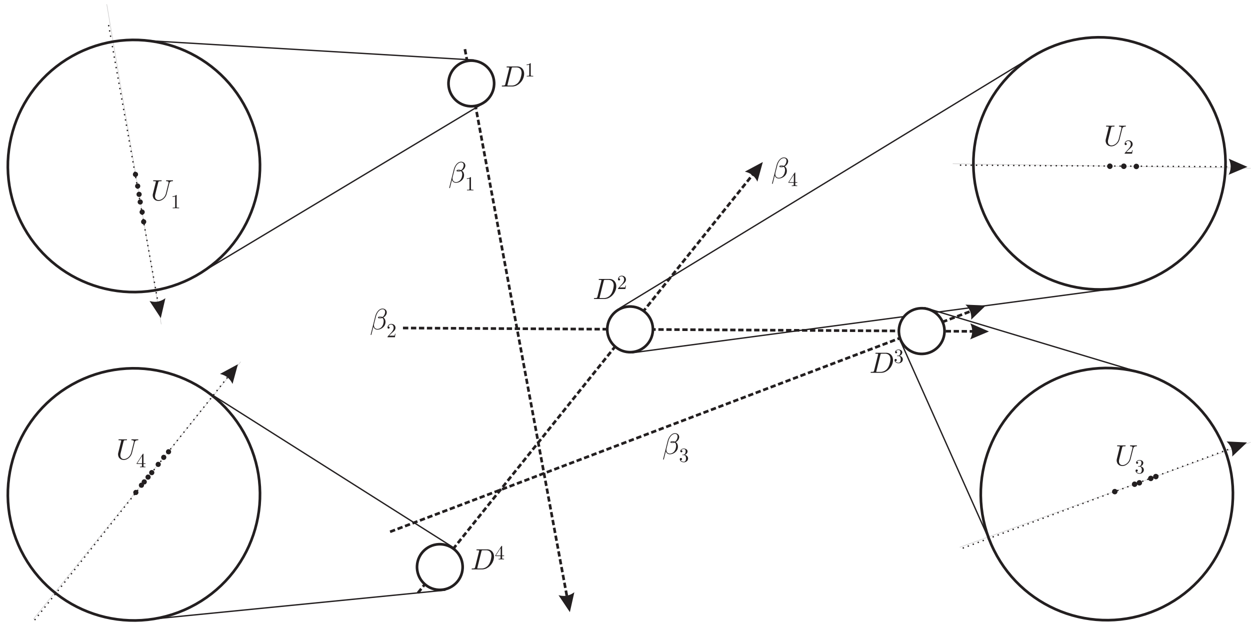

Figure 6: Enlarging each point p_i to a small disc D^i of radius r_i (example from Figure 5) and the sets U_1, U_2, U_3, and U_4 from Step 2. Eeach set U_i lies on β_i and is contained in the disc D^i.

Before moving on to the next step, we observe that each set β_i has a good halving potential. In fact, if β_i is simple, it already halves $U \setminus U_i$. And if β_i is splitting, then the difference between the number of points in $U \setminus U_i$ on each side of β_i is at most $s_{\sigma(i)} - 1$. In this case, β_i does not necessarily halve $U \setminus U_i$, but it intersects $D^{\sigma(i)}$, which contains exactly $s_{\sigma(i)}$ points of $U \setminus U_i$. Thus, a very small rotation of U_i (and β_i) may balance this difference. A preview of Figure 7 may be of help here. Unfortunately, there is a significant gap to be filled: we may certainly perform this rotation to adjust any particular β_i, but whenever the turn comes for $\beta_{\sigma(i)}$ to be adjusted, if we rotate this line we may break the halving property previously achieved by β_i. Taking care of this possible scenario transforms an otherwise intuitive, straightforward procedure into a somewhat technical one. This is the task for the next step.

Step 3 *Moving the sets U_i, so that each U_i lies on a line ℓ_i that halves $U \setminus U_i$.*

Our goal in this step is to slightly move (rotate or translate) each set U_i with $i \in I$, so that the line containing U_i passes through p_i and halves $U \setminus U_i$. In what follows, ℓ_i denotes the line containing U_i. We describe a dynamic process that moves U_i, and accordingly ℓ_i and C_i. Even when we are actually transforming the U_i, ℓ_i, and C_i, we keep their names all the way through. If $s_i = 1$, $U_i = C_i = \{p_i\}$ remains unchanged throughout this process. The central feature of the whole process is the following

Key property *The set U_i is contained in the interior of D^i and lies on ℓ_i (whenever $s_i > 1$) during the entire process. In their final position, ℓ_i goes through p_i and halves $U \setminus U_i$.*

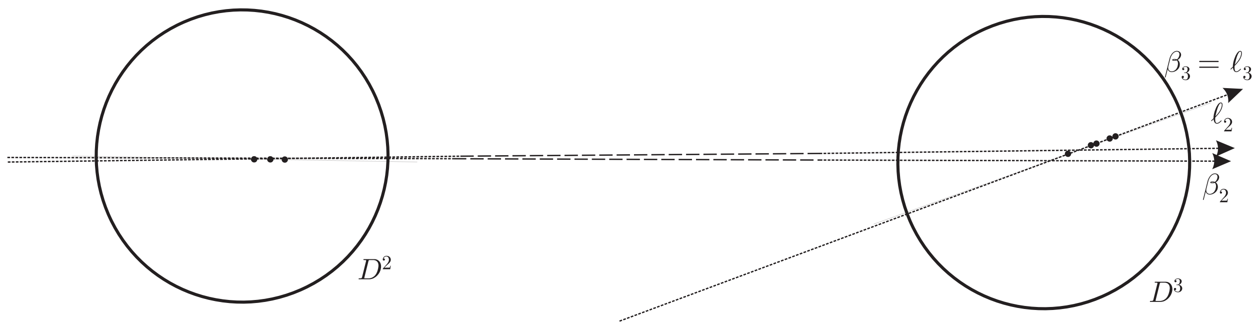

Figure 7: We consider D^2 and D^3 from Figure 6. Line ℓ_2 halves $U \setminus U_2$ and ℓ_3 halves $U \setminus U_3$. Since p_3 is simple, U_3 remains unchanged and $\ell_3 = \beta_3$. p_2 is splitting, with β_2 through p_3. There are 19 points in $U \setminus U_2$, 10 of which must be on the left of ℓ_2. U_1 (6 points) is on the left of β_2 and U_4 (8 points) is on its right. We use U_3 to balance: rotate U_2 around p_2, so that ℓ_2 leaves 4 points of U_3 on its left.

To describe the process, we consider the digraph G with vertex set $P' = \{p_i \in P : i \in I\}$, induced by the set of splitting pre-halving lines, that is, there is an arc from p_i to p_j if and only if $\sigma(i) = j$, see Figure 8. Thus, if p_i is simple, then its outdegree is zero, and if it is splitting, then its outdegree is one. These properties guarantee that each strong component of G is either acyclic, or contains at most one directed cycle. In any case, each strong component must have a vertex, called *root*, that can be *reached* from all other vertices in the component. (That is, for each vertex p in the component, there is a directed path from p to the root.)

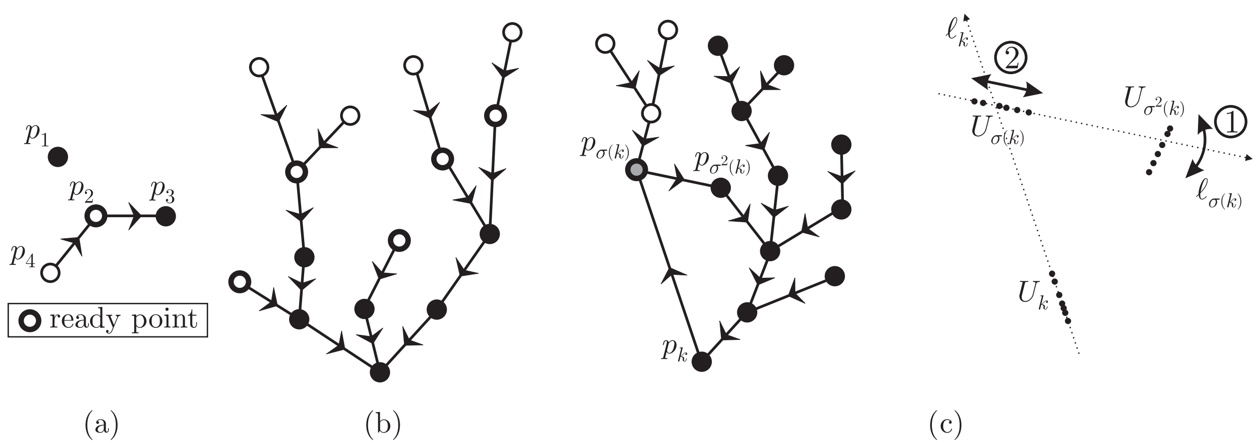

Figure 8: (a) The graph G corresponding to the example in Figure 5. (b) An acyclic component. (c) A component with a cycle at the time (2) is applied in Step 3.

We work on one component at a time. Let $P_c \subseteq P'$ be a strong component of G and p_k its root. Start by coloring all vertices of G white. Coloring a point p_i black means that ℓ_i and U_i have reached their final position. Color p_k black, and if p_k is splitting, then color $p_{\sigma(k)}$ grey. A white or a grey point is said to be *ready* if $p_{\sigma(k)}$ is black. As long as there are ready points, we apply (1) or (2) below.

(1) If possible, arbitrarily choose a white ready point p_i. Slightly rotate U_i around p_i until ℓ_i halves $U \setminus U_i$. This is always possible asking that ℓ_i intersects $D^{\sigma(i)}$ at all times, because $\beta_i \neq \pm\beta_j$, ℓ_i intersects $D^{\sigma(i)}$, $D^{\sigma(i)}$ has $s_{\sigma(i)}$ points, and before rotating U_i, we have an unbalance of at most $s_{\sigma(i)} - 1$. Color p_i black.

(2) If (1) cannot be applied, then work with the grey point $p_{\sigma(k)}$. First, proceed as in (1), that is, rotate $\ell_{\sigma(k)}$ until it halves $U \setminus U_{\sigma(k)}$. Then translate $U_{\sigma(k)}$ along $\ell_{\sigma(k)}$ until ℓ_k (which stays still) halves $U \setminus U_k$. Since $U_{\sigma(k)}$ was originally contained on a disc of radius $r_{\sigma(k)}/2$ centered at $p_{\sigma(k)}$, then $U_{\sigma(k)}$ is still contained in $D^{\sigma(k)}$ during the translation. Color $p_{\sigma(k)}$ black. See Figure 8(c).

Note that (2) is applied at most once, and if we cannot apply (1) or (2), then all points are already black. Since the key property is maintained at all times during the process, then at the end we have achieved our goal: Each U_i lies on ℓ_i and is contained in the interior of D^i. Also, ℓ_i goes through p_i and halves $U \setminus U_i$.

STEP 4 *Flattening C_i towards U_i.*

Finally, for $i \in I$, we affinely flatten each C_i towards U_i to obtain its final position. Again, if $s_i = 1$, then $C_i = \{p_i\}$. For each $0 \leq \epsilon \leq 1$ and each $i \in I$, let $C_i(\epsilon)$ be the set obtained from C_i by orthogonally moving its points towards U_i reducing their distance to ℓ_i by a factor of ϵ. (If $s_i = 1$, then $C_i(\epsilon) = \{p_i\}$. For each i, measure the distances from all points in $\bigcup_{j \neq i} S_j(\epsilon)$ to ℓ_i, making it negative if the point and its corresponding point in U are on different sides of ℓ_i. Let $f(\epsilon)$ be the minimum of these distances for fixed ϵ and over all $i \in I$. Note that the function f is continuous and $f(0) > 0$ as $\bigcup_{i=1}^{m} C_i(0) = U$. Then there must be an $\epsilon' > 0$ such that $f(\epsilon') > 0$. The final position of C_i is $C_i(\epsilon')$. Let $C := \bigcup_{i=1}^{m} C_i$. Since each C_i is contained in D^i, then C satisfies the inherited order type property and the halving property. And because U satisfies the halving property then C also satisfies it. The fact that each C_i is an affine copy of S_i, preserving this way its order type, will allow us to count the number of crossings in C.

5.4 Keeping 3-symmetry and 3-decomposability

Let θ be the counterclockwise rotation of $2\pi/3$ around the origin. We say that the input set $(P, \{\beta_i\}_{i \in I}, \{S_i\}_{i=1}^{m})$ is 3-*symmetric* if: the base point-set P is 3-symmetric, say via the function θ, the pre-halving set of lines $\{\beta_i\}_{i \in I}$ is 3-symmetric under the same function θ, and the collection of cluster models $\{S_i\}_{i=1}^{m}$ is partitioned into orbits of equal clusters according to the function θ. That is, if $p_i = \theta(p_j) = \theta^2(p_k)$, then $\beta_i = \theta(\beta_j) = \theta^2(\beta_k)$ and $S_i = S_j = S_k$.

Similarly, we say that the input set is 3-*decomposable*, if the base point-set P is 3-decomposable, with partition A, B, and C, and if the collection of cluster models satisfies that

$$\sum_{i: p_i \in A} s_i = \sum_{j: p_j \in B} s_j = \sum_{k: p_k \in C} s_k.$$

Note that no assumption is made on the pre-halving set of lines.

The following observations are worth highlighting.

Remark 1. *If the input set is* 3*-symmetric, then the construction can be performed so that the resulting augmented point set* C *is* 3*-symmetric. Similarly, if the input set is* 3*-decomposable, then the construction can be performed so that the resulting augmented point set* C *is* 3*-decomposable.*

5.5 Counting the crossings in the augmented drawing

Now we count the number of crossings in the resulting point set $C = \bigcup_{i=1}^{m} C_i$, equivalently, the number of convex quadrilaterals $\square(C)$. The most important aspect of the calculation is that it only depends on the input set, that is, on the base point set P, the cluster models S_i, and the collection of pre-halving lines. Thus the number of crossings in the augmented drawing can be calculated (perhaps using a computer) without explicitly doing the construction. This is particularly useful in Section 6, where we iterate this construction and, as a consequence, we obtain the currently best general drawings of K_n.

5.5.1 A closer look into how clusters get splitted

Before going into the calculation, we introduce some terminology. If p_i is simple (respectively, splitting), then we say that C_i itself is *simple* (respectively, *splitting*). If C_i is simple, then each C_j with $i \neq j$ is completely contained in a semiplane of ℓ_i. If C_i is splitting, then the same holds except for the cluster $C_{\sigma(i)}$: a nonempty subset L_i of $C_{\sigma(i)}$ is on the left semiplane of ℓ_i, while the also nonempty subset $R_i = C_{\sigma(i)} \setminus L_i$ is on the right semiplane. We remark that L_i and R_i are *not* subsets of C_i, but of $C_{\sigma(i)}$. By convention, if C_i is simple, so that $\sigma(i)$ is not defined, then we let $L_i = R_i = \emptyset$.

Note that the previously defined set $\mathcal{L}(i)$ (respectively, $\mathcal{R}(i)$) coincides with the set of those j such that C_j is completely contained in the left (respectively, right) semiplane of ℓ_i. Thus, if C_i is simple, then $\mathcal{L}(i) \cup \mathcal{R}(i) = \{1, 2, \ldots, m\}$, and if C_i is splitting, then $\mathcal{L}(i) \cup \mathcal{R}(i) = \{1, 2, \ldots, m\} \setminus \{\sigma(i)\}$. We also remark that the sizes of L_i and R_i are fully determined by $\sum_{j \in \mathcal{L}(i)} s_j$ and s_i. Indeed, the left semiplane of ℓ_i contains $\lceil (n - s_i)/2 \rceil$ points of $C \setminus C_i$, $\sum_{j \in \mathcal{L}(i)} s_j$ of which belong to a C_j other than $C_{\sigma(i)}$. Therefore, $|L_i| = \lceil (n - s_i)/2 \rceil - \sum_{j \in \mathcal{L}(i)} s_j$. The size of R_i is analogously calculated.

5.5.2 The calculation of crossings

We now count the number of crossings in D_n, that is, the number $\square(C)$ of convex quadrilaterals defined by points in C. We count separately five different types of convex quadrilaterals contributing to $\square(C)$. Adding the five contributions gives the exact value of $\square(C)$.

Type I *Convex quadrilaterals whose points all belong to different clusters.*

It follows from the inherited order type property that the number of quadrilaterals of Type I is:

$$\sum_{\substack{i<j<k<\ell \\ p_i, p_j, p_k, p_\ell \text{ is a convex quadrilateral}}} s_i s_j s_k s_\ell .$$

Type II *Convex quadrilaterals whose points belong to three distinct clusters.*

Every convex quadrilateral of Type II has two points in a cluster C_i and the other two points in clusters C_j, C_k, with i, j, k pairwise distinct. Now any four such points define a convex quadrilateral if and only if the points in C_j and C_k are on the same semiplane determined by ℓ_i. Recalling that the set of points in $C \backslash C_i$ on the left (respectively, right) halfplane of ℓ_i is $(\bigcup_{j\in\mathcal{L}(i)} C_j) \cup L_i$ (respectively, $(\bigcup_{j\in\mathcal{R}(i)} C_j) \cup R_i$), it follows that the total number of convex quadrilaterals of Type II equals:

$$\sum_{i=1}^{m} \binom{s_i}{2}\Bigg(\sum_{\substack{j,k\in\mathcal{L}(i)\\ j<k}} s_j s_k + \sum_{j\in\mathcal{L}(i)} s_j|L_i| + \sum_{\substack{j,k\in\mathcal{R}(i)\\ j<k}} s_j s_k + \sum_{j\in\mathcal{R}(i)} s_j|R_i| \Bigg).$$

Type III *Convex quadrilaterals whose points belong to two distinct clusters, with two points in each cluster.*

For each fixed C_i, and points p, q in C_i, p and q define a convex quadrilateral of Type III with those pairs of points that are on the same C_j and on the same halfspace of ℓ_i, except when $i = \sigma(j)$ and one of p and q belongs to L_j and the other to R_j. Thus the number of convex quadrilaterals of Type III that involve two points in C_i is $\binom{s_i}{2}\left(\sum_{j\notin\{i,\sigma(i)\}}\binom{s_j}{2} + \binom{|L_i|}{2} + \binom{|R_i|}{2}\right) - \sum_{j:i=\sigma(j)}\binom{s_j}{2}|L_j||R_j|$. When summing over all i, each convex quadrilateral of Type III gets counted exactly twice. Thus the total number of convex quadrilaterals of Type III is:

$$\frac{1}{2}\sum_{i=1}^{m}\left(\binom{s_i}{2}\left(\sum_{j\notin\{i,\sigma(i)\}}\binom{s_j}{2} + \binom{|L_i|}{2} + \binom{|R_i|}{2}\right) - \sum_{\substack{j:\\ i=\sigma(j)}}\binom{s_j}{2}|L_j||R_j|\right).$$

Type IV *Convex quadrilaterals with three points in the same cluster and the other point in a distinct cluster.*

To count these crossings we need to introduce a bit of terminology. If S is a point set in general position in the plane, and $p = (p_x, p_y), q = (q_x, q_y), r = (r_x, r_y) \in S$, with $p_x < q_x < r_x$, then the concatenation of the segments $\overline{pq}$ and $\overline{qr}$ is either concave up or concave down. In the former case, we say that $\{p, q, r\}$ is itself *concave up*, and in the latter case, we say it is *concave down*. We let $\sqcup(S)$ (respectively, $\sqcap(S)$) denote the number of 3-subsets of S that are concave up (respectively, concave down). If no two points in S have the same x-coordinate, then each 3-subset of S is either concave up or concave down, and so in this case $\sqcup(S) + \sqcap(S) = \binom{|S|}{3}$.

Now it follows from the construction of the clusters C_i, that given any 3 points $p, q, r \in C_i$, then a fourth point s in another cluster forms a convex quadrilateral with p, q, and r if and only if either (i) s is in the left semiplane of ℓ_i and $\{p, q, r\}$ is concave up in S_i; or (ii) s is in the right semiplane of ℓ_i and $\{p, q, r\}$ is concave down in S_i.

Since there are $\lceil (n - s_i)/2 \rceil$ points s in $C \setminus C_i$ in the left halfspace of ℓ_i, and $\lfloor (n - s_i)/2 \rfloor$ points s of $C \setminus C_i$ in the right halfspace of ℓ_i, it follows that the total number of quadrilaterals of Type IV equals:

$$\sum_{i=1}^{m} \left(\sqcup(S_i) \cdot \left\lceil \frac{n - s_i}{2} \right\rceil + \sqcap(S_i) \cdot \left\lfloor \frac{n - s_i}{2} \right\rfloor \right).$$

Type V *Convex quadrilaterals with all four points in the same cluster.*

This is simply the sum of the number of convex quadrilaterals in each C_i, or equivalently, in each S_i:

$$\sum_{i=1}^{m} \square(C_i) = \sum_{i=1}^{m} \square(S_i).$$

6 Doubling all points of a set with an odd number of points

There is a case in which the construction from Section 5 is particularly useful: when the cluster models are all equal to each other. This is the approach followed by Aichholzer et al. [7] and by Ábrego and Fernández-Merchant [3].

In [7], the equivalent of our ℓ_is do not split any cluster, and the cluster models are sets in convex position called *lens arrangements*. This is the best possible choice (under the no-splitting assumption) to minimize the number of crossings of the augmented point set.

In [3], clusters of size 2 are used in an iterative process, starting from a base point set with m points, and producing augmented point sets with $2^k m$ points for $k = 0, 1, \ldots$. This has been used to obtain the best upper bounds known for the rectilinear crossing number prior to the present work. The only limitations of the process in [3] are that (i) the base configuration P is assumed to have an even number of points; and (ii) the base configuration P is assumed to have a *halving matching*, that is, an injection from P to the set of halving lines of P, such that each $p \in P$ gets mapped to a line incident with p. The base for this iterative process is the following result.

Lemma 3 in [3]. If P is an m-element set, m even, and P has a halving-line matching, then there is a point set $Q = Q(P)$ in general position, $|Q| = 2m$, Q also has a halving-line matching, and $\square(Q) = 16\square(P) + (m/2)(2m^2 - 7m + 5)$.

As in [3], we now use clusters of size 2, but within the more general framework described in the previous section, we can use a base configuration with an *odd* number of points. This also has the advantage that the existence of a pre-halving set of lines is trivially satisfied. Moreover, after one iteration, we get a set with an even number of points and a halving matching, allowing us to use the iterative construction in [3].

Proposition 1. *Starting from any point set P with $m := |P|$ odd, and duplicating each point (that is, substituting each point by a 2-point cluster), our construction yields a $2m$-point set C in general position with $\square(C) = 16\square(P) + (m/2)(2m^2 - 7m + 5)$. Moreover, C has a halving matching.*

Proof. To apply our construction, we first need to check the existence of a pre-halving set of lines. This is trivial because $s_i = 2$ for every $i = 1, \ldots, m$. That is, it suffices to choose, for each p_i, a line β_i through p_i that leaves $(m-1)/2$ points of P on each side. Moreover, such a line is simple, and thus $L_i = R_i = \emptyset$. Knowing the existence of a pre-halving set of lines, we may proceed to calculate the number of convex quadrilaterals in the augmented $2m$-set C.

- **Type I.** Since $s_i = 2$ for each i, then C has $16\square(P)$ convex quadrilaterals of Type I.
- **Type II.** For each i, the line ℓ_i has exactly $(m-1)/2$ clusters C_j on each side. Thus C has $\sum_{i=1}^{m} \binom{2}{2} \left(\binom{(m-1)/2}{2} \cdot 4 + \binom{(m-1)/2}{2} \cdot 4 \right) = m(m-1)(m-3)$ convex quadrilaterals of Type II.
- **Type III.** For each i, $\sigma(i)$ is undefined and $L_i = R_i = \emptyset$. Thus C has $\frac{1}{2} \sum_{i=1}^{m} \binom{2}{2} \left(\sum_{j \neq i} \binom{2}{2} \right)$ $= \frac{1}{2} m(m-1)$ convex quadrilaterals in C of Type III.
- **Types IV and V.** Since there are no clusters of size 3 or larger, then C has no convex quadrilaterals of Types IV or V.

Summing up the contributions of Types I, II, and III, it follows that $\square(C) = 16\square(P) + (m/2)(2m^2 - 7m + 5)$, as claimed.

Finally, we show that C has a halving matching. If ℓ is a directed line that spans points p and q, then p is *before* q *in* ℓ if as we traverse ℓ, first we find p and then q. Recall that in the last step in the construction we start with all points in each cluster C_i lying on line ℓ_i, and perturb them so that the order type of C_i coincides with that of S_i. Since here all clusters have size 2, there is no need to perturb them: their final position may as well be on ℓ_i. For each $p_i \in P$, we let p_i', p_i'' denote the two points in C into which p_i get splitted, labelled so that p_i' is before p_i'' in ℓ_i. We assume without any loss of generality that all lines ℓ_i are directed so that their angles with the x-axis are between 0 and π.

Now ℓ_i is clearly a halving line for every i. Thus we may associate ℓ_i to one of p_i' and p_i'', and only need to seek a halving line to associate to the other point. We rotate ℓ_i counterclockwise around p_i' until we hit another point in C (say q), and let $\overline{\ell_i'}$ denote the line through p_i' and q, with the direction it naturally inherits from ℓ_i. If q is before p_i' in $\overline{\ell_i'}$, then let $\overline{\overline{\ell_i'}} := \overline{\ell_i'}$. Otherwise, let $\overline{\overline{\ell_i'}}$ denote the line spanning p_i'' and q with the orientation it naturally inherits from ℓ_i, that is, so that q is before p_i'' in $\overline{\overline{\ell_i'}}$. In either case, $\overline{\overline{\ell_i'}}$ is a halving line that goes through one of p_i' or p_i''. We associate this halving line to the point in $\{p_i', p_i''\}$ belonging to it, and to the other point we associate ℓ_i. It is easily checked that if $i \neq j$, then $\overline{\overline{\ell_i'}} \neq \overline{\overline{\ell_j'}}$ (and trivially $\ell_i \neq \ell_j$). Therefore this defines an injection from C to the set of its halving lines. Thus C has a halving matching, as claimed. $\square$

We are now ready to prove the main result in this section.

Theorem 4. *If P is an m-element point set in general position, with m odd, then*

$$\overline{\mathrm{cr}}(K_n) \leq \frac{24\overline{\mathrm{cr}}(P) + 3m^3 - 7m^2 + (30/7)m}{m^4} \binom{n}{4} + \Theta(n^3). \tag{13}$$

Proof. We closely follow the proof of Theorem 2 in [3]. (Note that Lemma 3 in [3], the equivalent to our Proposition 1, may also be derived from the construction in Section 5).

Applying Proposition 1 to $P_{-1} := P$, we obtain an even cardinality point set P_0 with a halving matching. Thus, we can apply iteratively Lemma 3 in [3] with P_0 as the base configuration. Then, for all $k > 0$, if P_k denotes the set obtained from P_{k-1} using Lemma 3 in [3], we have

$$\overline{\text{cr}}(P_k) = 16\overline{\text{cr}}(P) + m^3 8^{k-1}(2^k - 1) - \frac{7}{6}m^2 4^{k-1}(4^k - 1) + \frac{5}{14}m 2^{k-1}(8^k - 1).$$

Now by letting $n := |P_k| = 2^k m$, we get

$$\overline{\text{cr}}(P_k) = \left(\frac{24\overline{\text{cr}}(P) + 3m^3 - 7m^2 + (30/7)m}{24m^4}\right) n^4 - \frac{1}{8}n^3 + \frac{7}{24}n^2 - \frac{5}{28}n. \qquad \square$$

This inequality was previously known (Theorem 2 in [3]) only for drawings with an *even* number of points. The existence of a point set satisfying this halving property, together with this theorem, constitute the best tools available to obtain upper bounds for the rectilinear crossing number constant.

We cannot overemphasize the importance of Theorem 4 and Theorem 2 in [3]: they constitute the best tools available to obtain upper bounds for the rectilinear crossing number constant q_*. As of the time of writing, the best bound known for q_*, namely

$$q_* \leq \frac{83247328}{218791125} < 0.380488,$$

is obtained by applying Theorem 4 to a particular drawing of K_{315}, see Section 7.

7 Symmetric geometric drawings

The most fruitful and comprehensive effort to produce good geometric drawings of K_n is the Rectilinear Crossing Number Project, led by Oswin Aichholzer [6]. Prior to the present work, the drawings in [6] constitute the state-of-the-art in the subject: for every $n \leq 100$, the previously best crossing-wise geometric drawing of K_n can be found in [6]. A detailed look at the information in [6] shows that the *vast majority* of drawings seem close to being 3-symmetric.

We have successfully produced 3-symmetric and 3-decomposable drawings that match or improve the best drawings reported in [6]. Our results are summarized as follows.

(1) For every positive integer $n < 100$, n a multiple of 3, we produced a 3-symmetric and 3-decomposable geometric drawing of K_n whose number of crossings is less than or equal to that in [6]. Some of these drawings were obtained using heuristic methods based on previous drawings, and the rest using our replacing-by-clusters construction in Section 5. For a brief summary of our results, see Table 1.

(2) The best upper bound for the rectilinear crossing number constant $q_* = \lim_{n\to\infty} \overline{\text{cr}}(K_n)/\binom{n}{4}$ is now achieved by 3-symmetric and 3-decomposable drawings. For this we apply Theorem 4 to a 3-symmetric and 3-decomposable drawing of K_{315} with 152210640 crossings, and recall Remark 1.

Trying to produce 3-symmetric geometric drawings of K_n that improve those of Aichholzer is a formidable task, specially for large values of n. Prior to our work, no good crossing-wise 3-symmetric drawings had been reported, other than those for very small values of n. For each positive integer n multiple of 3, we produced 3-symmetric drawings of K_n whose number of crossings is less than or equal to the previous best drawing. Our drawings are optimal for $n \leq 27$ [4], and we conjecture they are optimal for $n = 36$, 39, and 45. The drawings for $n \leq 57$, with the exception of $n = 33$, were obtained independently. A good sample of these drawings is our 3-symmetric drawing of K_{24}, sketched in Figure 9. The precise coordinates of the eight points in one wing W are: $p_1 = (-51, 113); p_2 = (6, 834); p_3 = (16, 989); p_4 = (18, 644); p_5 = (18, 1068); p_6 = (22, 211); p_7 = (-26, 313); p_8 = (17, 1036)$. If θ denotes the counterclockwise rotation of $2\pi/3$ around the origin, then the whole 24-point set is $P = W \cup \theta(W) \cup \theta^2(W)$.

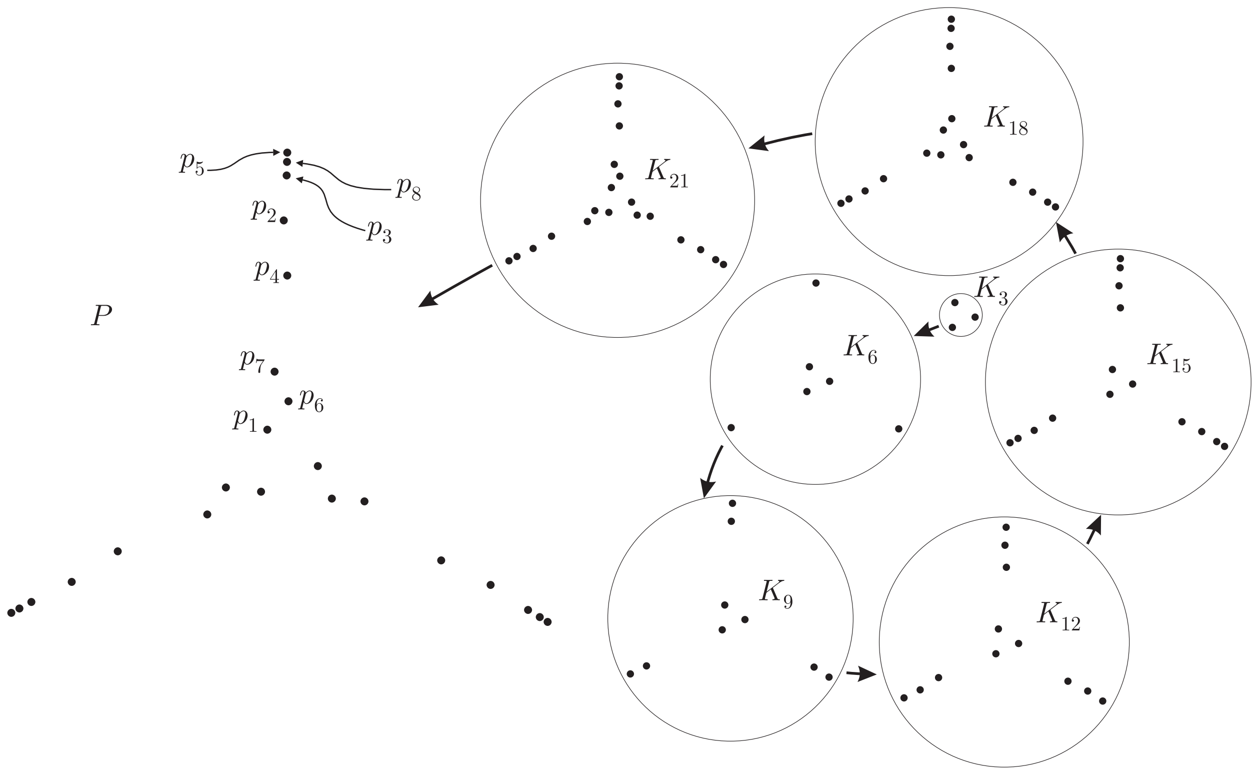

Figure 9: The underlying vertex set of an optimal 3-symmetric geometric drawing of K_{24}. This point set contains optimal nested 3-symmetric drawings of $K_{21}, K_{18}, K_{15}, K_{12}, K_9, K_6$, and K_3.

The geometric drawing induced by this point-set has 3699 crossings, and is thus optimal [4]. A remarkable property of this drawing is that it contains a chain of optimal 3-symmetric subdrawings of $K_{21}, K_{18}, K_{15}, K_{12}, K_9, K_6$, and K_3. Indeed, if $W_i = \{p_1, p_2, \ldots, p_i\}$ then the point-set $W_i \cup \theta(W_i) \cup \theta^2(W_i)$ is an optimal drawing of K_{3i} for $1 \leq i \leq 8$, that is, its number of crossings matches the one known to be optimal (see [4] and [8]).

We also include 3-symmetric drawings of K_{27} and K_{30} (Figure 10), K_{36} and K_{39} (Figure 11),

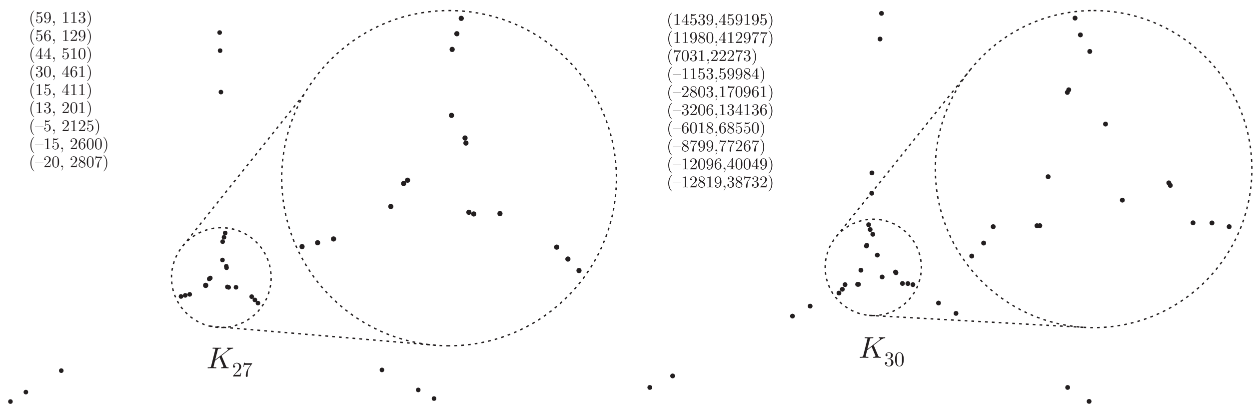

Figure 10: The underlying vertex sets of 3-symmetric geometric drawings of K_{27} (left) and K_{30} (right). In each case, the coordinates given correspond to one third of the points; the other two thirds are obtained by rotating the given set angles of $2\pi/3$ and $4\pi/3$, respectively. The induced drawing of K_{27} is known to be optimal, and we conjecture that the induced drawing of K_{30} is also optimal.

and K_{45} (Figure 12). The drawing of K_{27} is known to be optimal [4]. For reasons that are beyond the scope of this work, we firmly believe that the given drawings of K_{30}, K_{36}, K_{39}, and K_{45} are also optimal.

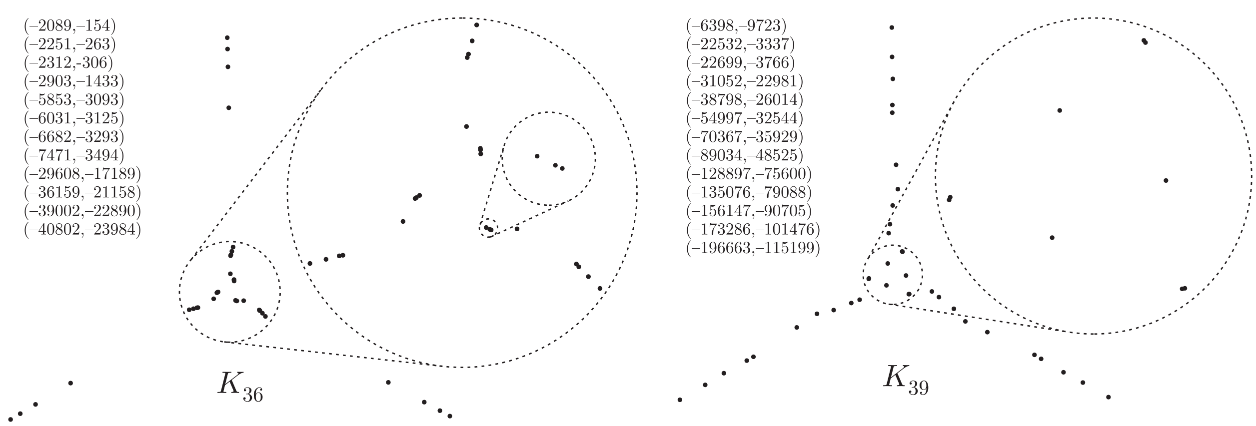

Figure 11: The underlying vertex sets of 3-symmetric geometric drawings of K_{36} (left) and K_{39} (right), both of which we conjecture are optimal. In each case, the coordinates given correspond to one third of the points; the other two thirds are obtained by rotating the given set angles of $2\pi/3$ and $4\pi/3$, respectively.

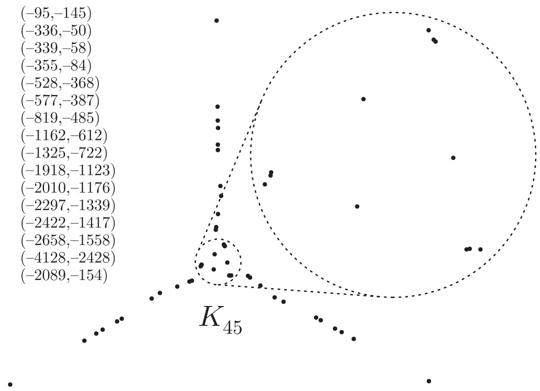

Figure 12: The underlying vertex set of a 3-symmetric geometric drawing of K_{45}, which we conjecture is optimal. The coordinates given correspond to one third of the points; the other two thirds are obtained by rotating the given set angles of $2\pi/3$ and $4\pi/3$, respectively.

Given the evolving nature of our symmetric drawings of K_{42}, K_{48}, K_{51}, K_{54}, K_{57}, and for space reasons, they are hardly worth including in the present work. They are available in an extended version of this paper [1].

To obtain the drawings for $n \geq 60$, and for the special case $n = 33$, we use the construction in Section 5. For each such K_n, it suffices to give the base drawing D_m for some suitable $m < n$, the cluster models S_i, and a pre-halving set of lines $\{\beta_i\}_{i\in I}$ for D_m. This determines the information relevant to calculate the number of crossings of the resulting drawing of K_n: the sizes of the clusters that lie to the left of each line ℓ_i, and the sizes of the sets L_i and R_i of the cluster (if any) that is splitted by ℓ_i. We use a base drawing of K_{30} to obtain drawings for K_{33} and K_{60}, and a base drawing of K_{51} to obtain drawings of K_n with $60 < n < 100$. It makes little sense to include here the details of any such example, not only because of the amount of information required to do so, but also because, by no means, we believe that the drawings found as of the time of the writing are optimal. They are the best current examples and support our conjectures that optimal 3-symmetric and 3-decomposable geometric drawings exist for every N multiple of 3. Instead, we gathered all the information relevant to these drawings in its full detail in [1]. Here, we include Table 1 as a summary of our results.

8 Improved upper bound for the rectilinear crossing number constant

We finally bring together the results from the previous two sections to establish the best known upper bound for the *rectilinear crossing number constant* $q_* := \lim_{n\to\infty} \overline{\mathrm{cr}}(K_n)/\binom{n}{4}$.

Theorem 5. *The rectilinear crossing number constant q_* satisfies $q_* \leq \frac{83247328}{218791125} < 0.380488$.*

Proof. It follows from Theorem 4 and the definition of q_* that if P is a point set, and $|P| = m$ is odd, then

$$q_* \leq \frac{24\overline{\mathrm{cr}}(P) + 3m^3 - 7m^2 + (30/7)m}{m^4}. \tag{14}$$

Now, as we pointed out in Section 7, we have produced a geometric drawing of K_{315} with 152210640 crossings. Plugging in these values into the previous equation yields Theorem 5. $\square$

9 Concluding Remarks

The previously best known general bounds for the rectilinear crossing number of K_n are $0.379972\binom{n}{4} + \Theta(n^3) < \overline{\mathrm{cr}}(K_n) < 0.38054415\binom{n}{4} + \Theta(n^3)$; see [5] for the lower bound, and [3] with a drawing of K_{90} with 951526 crossings by Aichholzer for the upper bound. Thus the general upper bound in Theorem 5, together with the lower bound given by Theorem 3, closes this gap by about 10%, and by 20% under the quite feasible assumption of 3-decomposability. In fact, we strongly believe that:

Conjecture 1. *For each positive integer n multiple of 3, all optimal rectilinear drawings of K_n are 3-decomposable.*

The reasons for this belief go beyond the evidence of all known optimal drawings: the underlying point sets of all the best crossing-wise known drawings of K_n happen to minimize the number of $(\leq k)$-sets for every $k \leq n/3$, and a point set with this property is in turn 3-decomposable (an equivalent form of this statement appears in [9].

Another strong feeling that we have is about the symmetry. We note that *none* of the explicit best known constructions, prior to this paper, is 3-symmetric (except for some very small values of n). Yet, they resemble a 3-symmetric set. This hints to the existence of equally good drawings of K_n that are 3-symmetric (which seems to be a wide spread belief). In this context we believe that:

Conjecture 2. *For each positive integer n multiple of 3, there is an optimal geometric drawing of K_n that is 3-symmetric.*

Our main findings back up Conjectures 1 and 2. Indeed, we have found, for every n multiple of 3, a 3-decomposable and 3-symmetric geometric drawing of K_n with the fewest number of crossings known to date. Thus, in particular, for each n multiple of 3 for which the exact value of $\overline{\mathrm{cr}}(K_n)$ is known (that is, $n \leq 27$), we have found an optimal geometric drawing that is 3-decomposable and 3-symmetric. These drawings are described in Section 7. Some were obtained using heuristic methods based of previously known constructions; the rest were obtained applying our replacing-by-clusters construction from Section 5, with base drawings of K_{30} or K_{51}. In fact, this drawing of K_{315} is obtained from a base drawing of K_{51}, and it is the initial base drawing used to establish Theorem 5.

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n	Number of crossings in previous best drawing [6]	Number of crossings in currently best 3-symmetric drawing	Number of crossings in currently best (nonnecesarily rectilinear) drawing $\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$	How we obtained the drawing reported in the third column
$n \leq 27$, n divisible by 3	Optimal for each n	Optimal for each n	Optimal for $n \leq 12$	Independently
30	9726	9726	9555	Independently
33	14634	14634	14400	From K_{30}
36	21175	21174	20808	Independently
39	29715	29715	29241	Independently
42	40595	40593	39900	Independently
45	54213	54213	53361	Independently
48	71025	71022	69828	Independently
51	91452	91452	90000	Independently
54	115994	115977	114075	Independently
57	145178	145176	142884	Independently
60	179541	179541	176610	From K_{30}
63	219683	219681	216225	From K_{51}
66	266188	266181	261888	From K_{51}
69	319737	319731	314721	From K_{51}
72	380978	380964	374850	From K_{51}
75	450550	450540	443556	From K_{51}
78	529350	529332	520923	From K_{51}
81	618048	618018	608400	From K_{51}
84	717384	717360	706020	From K_{51}
87	828233	828225	815409	From K_{51}
90	951526	951459	936540	From K_{51}
93	1088217	1088055	1071225	From K_{51}
96	1239003	1238646	1219368	From K_{51}
99	1405132	1404552	1382976	From K_{51}
315	–	152210640	149964516	From K_{51}

Table 1: For each $n < 100$, n a multiple of 3, we have found a 3-symmetric and 3-decomposable drawing whose number of crossings is less than or equals to the number of crossings in the previously best geometric drawing of K_n. We also include our current record for K_{315}, the drawing that gives, in combination with Theorem 4, $q_* < 0.380488$. For comparison purposes, the entries in the fourth column show the number of crossings in the best known unrestricted (i.e., nonnecesarily rectilinear) drawing of K_n: for each $n \geq 5$, it is known that K_n can be drawn with $\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ crossings. A long standing conjecture states that this is the crossing number of K_n for all $n \geq 5$.