See discussions, stats, and author profiles for this publication at:
<https://www.researchgate.net/publication/227228456>

Singular perturbations of integral equations with degenerate kernels

Article in *Integral Equations and Operator Theory* · January 1996

DOI: 10.1007/BF01204605

CITATION

1

READS

16

1 author:

Carol Shubin

California State University, Northridge

21 PUBLICATIONS 148 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:

Proceedings of the Conference in Honor of Leonid Pastur, Markov Processes and Related Fields, vol 9 #4, (2003), 829-840. [View project](#)

American Mathematical Society, University Lecture Series, Vol 29 (Editors: I. Laba and C. Shubin), 2003. [View project](#)

All content following this page was uploaded by [Carol Shubin](#) on 26 May 2014.

The user has requested enhancement of the downloaded file.

SINGULAR PERTURBATIONS OF INTEGRAL EQUATIONS WITH DEGENERATE KERNELS

CAROL ANN SHUBIN

Singularly perturbed Fredholm equations of the second kind are investigated. The kernels are allowed to have a jump discontinuity which vanishes at a point along the diagonal. Sufficient conditions for existence and uniqueness of solutions are found and the behavior of the solutions is studied.

Introduction.

This work is concerned with singularly perturbed integral equations on the interval $I = (a, b)$. Let ϵ be a small positive parameter. The perturbed equation is a Fredholm equation of the second kind

$$(1) \quad \epsilon u_\epsilon(x) + \int_a^b K(x, y) u_\epsilon(y) dy = f(x).$$

By setting $\epsilon = 0$, we obtain the unperturbed equation

$$(2) \quad \int_a^b K(x, y) u_0(y) dy = f(x)$$

which is a integral equation of the first kind. We consider kernels $K(x, y)$ defined on $[a, b] \times [a, b]$ which have the form

$$K(x, y) = \begin{cases} K_1(x, y) & y < x \\ K_2(x, y) & x < y \end{cases}.$$

We assume that $K_1(x, y)$ and $K_2(x, y)$ are smooth on the $[a, b] \times [a, b]$. The kernel $K(x, y)$ has a jump discontinuity along the diagonal

$$(3) \quad K_1(x, x^-) - K_2(x, x^+) = a(x), \quad a(x) \in C^\infty[a, b].$$

We study the degenerate case which occurs when $a(x)$ vanishes to order k at one endpoint or at one interior point of the interval.

The connection between the solution to (1) and (2) in the nondegenerate case, when $a(x) \neq 0$ for $x \in [a, b]$, was considered in [LS] and [S]. This case is a subclass of elliptic singular perturbations of elliptic pseudodifferential equations. The regular degeneration of elliptic operators of higher order to elliptic operators was studied extensively by Višik and Ljusternik [VL]. An example of this situation is $\epsilon^2 \Delta^2 - \Delta$. Demidov [D], Eskin [E1], Frank [F], Frank and Wendt [FW1,2,3], and Grubb [G] have studied elliptic pseudodifferential equations with a small parameter. When $a(x)$ vanishes at some point the unperturbed equation ceases to be elliptic and the methods of these works do not apply. Very little is known about the degenerate case but the behavior of the solutions appear to be far more complex than in the nondegenerate case. This work shows that the behavior of the solution to a Fredholm integral equation on an interval (1) which degenerates to an integral equation (2) (which is not a Fredholm equation) is already complicated. In particular the behavior of the solution depends on the sign of $a(x)$ in a neighborhood of the vanishing point. There is an analogy between the class of singularly perturbed integral equations considered here and the study of subelliptic pseudodifferential equations related to the oblique derivative problem, see [E3], [MP], [Sj], [T], and [VG]. These works consider the inversion of the first order differential operator, $L_\epsilon = \frac{d}{dx} + a(x)|\xi|$ where $|\xi|$ is a large parameter.

In §1, 2, and 3 we take $a(x) = x^k b(x)$, k a positive integer, $b(x) \in C^\infty$, and $b(x) > 0$. We show that (1) can be rewritten as

$$L_\epsilon(B + \epsilon T)u_\epsilon(x) = f$$

where L_ϵ is a first order differential operator

$$L_\epsilon = \epsilon \frac{d}{dx} + a(x),$$

B is an associated integral operator of order -1 which does not depend on ϵ and ϵT is a small perturbation of order -1. In particular, B and T map L^2 into H^1 and B is assumed to have a one dimensional cokernel. Under this assumption, (1) is reduced to a regular perturbation problem. In our case, B is shown to be the reduced operator; this is in contrast to the elliptic case when the reduced operator and the unperturbed operator coincide. Sufficient conditions are given for the existence of a unique solution to (1) in L^2 for any $f \in L^2$ and $0 < \epsilon < \epsilon_0$.

§4 examines singularly perturbed integrals when $a(x)$ is replaced by $-a(x)$. We reduce the perturbed problem to regular perturbation problem by assuming that the reduced operator which maps H^{-1} to L^2 has a one dimensional kernel. Sufficient conditions are given for the existence and uniqueness of the solution u_ϵ for any $f \in L^2$.

At the end of sections 1, 2 and 3, the behavior of the solution u_ϵ as $\epsilon \rightarrow 0$ is described. In the nondegenerate case when $a(x) \neq 0$ on $[a, b]$, u_ϵ tends to the solution of the unperturbed equation (2) in some Sobolev norm, see [E1]. This is not true in the degenerate case. The behavior of u_ϵ as $\epsilon \rightarrow 0$ is more complicated. Clearly, the solution to (2) is singular at $x = 0$. Under suitable hypotheses, the solution of the perturbed equation converges to the solution of the reduced equation in $L^2[0, 1]$ as ϵ tends to zero.

I would like to thank my advisor Professor G. Eskin for his valuable comments and interest in this work. Also I gratefully acknowledge NSF support DMS-9401735.

1. Degeneracy at an endpoint.

We consider equations (1) and (2) defined on $[0, 1]$. The most general class of kernels for (2) which the results in this section hold is the following: let $K(x, y) = c(x)K_0(x, y)$ where $c(x) = x^k d(x)$, $k \in \mathbb{Z}^+$, $d(x) > 0$, $d(x) \in C^\infty[0, 1]$ and

$$K_0(x, y) = \begin{cases} K_{01}(x, y) & x > y \\ K_{02}(x, y) & x < y \end{cases}$$

with K_{01} and $K_{02} \in C^\infty([0, 1] \times [0, 1])$ and $K_0(x, x^-) - K_0(x, x^+) = g(x) > 0$. Let $b(x) = g(x)d(x)$ and $a(x) = x^k b(x)$. So we can write $K_0(x, y) = a(x)\Theta(x - y) + Q(x, y)$ where $\Theta(x) = 1$ if $x > 0$ and 0 otherwise. By Taylor's expansion about the diagonal

$$(1.1) \quad Q(x, y) = \sum_{i=1}^n \alpha_i(x)(x - y)_+^i + \sum_{i=1}^n \beta_i(x)(x - y)_-^i + R_n(x, y)$$

where $\alpha_i(x), \beta_i(x) \in C^\infty[0, 1]$, $i = 1, \dots, n$, $x_+^i = x^i$ if $x > 0$ and 0 if $x < 0$, $x_-^i = x^i$ if $x < 0$ and 0 if $x > 0$, and $R_n(x, y) \in C^n([0, 1] \times [0, 1])$.

Let us establish some notation which will be used throughout the paper. c denotes a real positive number which may change from line to line. The H^1 -norm is defined as $\|v\|_{H^1} \leq \|\frac{d}{dx}v\|_{L^2} + c\|v\|_{L^2}$.

Thus the singularly perturbed integral equation (1) can be rewritten as

$$(1.2) \quad \epsilon u_\epsilon(x) + a(x) \left(\int_0^x u_\epsilon(y) dy + \int_0^1 Q(x, y) u_\epsilon(y) dy \right) = f(x)$$

for $x \in (0, 1)$ and ϵ is a small positive parameter. We will find conditions to ensure that there exists a unique solution to (1.2), $u_\epsilon(x) \in L^2(0, 1)$, for each $f \in L^2(0, 1)$. Then we will study the behavior of the solution u_ϵ as ϵ tends to zero.

We associate to (1.2) the first order differential operator

$$(1.3) \quad L_\epsilon = \epsilon \frac{d}{dx} + a(x).$$

The fundamental solution of the initial value problem for L_ϵ , $x > 0$, is given by

$$G_\epsilon(x, y) = \frac{1}{\epsilon} \exp\left(-\int_y^x \frac{a(s)}{\epsilon} ds\right) \Theta(x - y).$$

Now we define the operator R_ϵ :

$$(1.4) \quad R_\epsilon f(x) = \int_0^1 G_\epsilon(x, y) f(y) dy \quad f \in L^2(0, 1)$$

and set

$$(1.5) \quad E_\epsilon(x, y) = \exp\left(-\int_y^x \frac{a(s)}{\epsilon} ds\right).$$

Using integration by parts one can verify the relations:

$$(1.6a) \quad R_\epsilon L_\epsilon w(x) = w(x) - w(0)E_\epsilon(x, 0) \quad w \in H^1[0, 1]$$

$$(1.6b) \quad L_\epsilon R_\epsilon g(x) = g(x) \quad g \in L^2(0, 1).$$

By (1.3), $a = L_\epsilon - \epsilon \frac{d}{dx}$. So for $\frac{f}{a} \in H^1[0, 1]$, $R_\epsilon f$ can be written as $R_\epsilon f = R_\epsilon a(\frac{f}{a}) = R_\epsilon(L_\epsilon - \epsilon \frac{d}{dx})\frac{f}{a}$. Therefore, using (1.6a) we have

$$(1.7) \quad R_\epsilon f = \frac{f}{a} - \frac{f}{a}(0)E_\epsilon(x, 0) - \epsilon R_\epsilon \frac{d}{dx} \frac{f}{a}.$$

We can rewrite (1.2) as

$$(1.8) \quad L_\epsilon B u_\epsilon(x) - \epsilon T u_\epsilon(x) = f(x)$$

where

$$(1.9) \quad B u_\epsilon(x) = \int_0^x u_\epsilon(y) dy + \int_0^1 Q(x, y) u_\epsilon(y) dy,$$

$$(1.10) \quad T u_\epsilon(x) = \int_0^1 \frac{\partial Q}{\partial x}(x, y) u_\epsilon(y) dy.$$

Let's also define a regular perturbation of B as follows:

$$(1.11) \quad B_\epsilon v = Bv - \epsilon R_\epsilon T v.$$

Now apply R_ϵ to (1.8) and use (1.6a) and (1.11) to obtain

$$(1.12) \quad B_\epsilon u_\epsilon(x) = R_\epsilon f + c_\epsilon E_\epsilon(x, 0)$$

where

$$(1.13) \quad c_\epsilon = \int_0^1 Q(0, y) u_\epsilon(y) dy.$$

c_ϵ is an unspecified constant to be determined shortly.

We will modify the usage of term the 'reduced' equation in the following way. By (1.12) and (1.7), we have

$$(1.14) \quad \begin{aligned} B_\epsilon u_\epsilon &= R_\epsilon f + c_\epsilon E_\epsilon(x, 0) \\ &= \frac{f}{a} - \frac{f}{a}(0) E_\epsilon(x, 0) - \epsilon R_\epsilon \frac{d}{dx} \frac{f}{a} + c_\epsilon E_\epsilon(x, 0). \end{aligned}$$

Then we are able to associate a 'reduced' equation to (1.2):

$$(1.15) \quad Bu_0 = g \quad g \in H^1(0, 1).$$

By (1.9), B has index -1 as an operator from $L^2(0, 1)$ into $H^1(0, 1)$.

Lemma 1.1. *Every element of the cokernel of B has the form:*

$$v_0(x) = c_1 \delta(x) + c_2 \delta(x - 1) + w(x)$$

with $w(x) \in C^\infty[0, 1]$.

Proof. First we show that if $u \in H^{-1}$ then $u = c\delta(x) + v$ where $v = \frac{df}{dx}$, with $f \in L^2[0, 1]$. To show this, let u be a bounded linear functional on H^1 . We will show that there exists a $c \in \mathbb{R}$ and $f \in L^2$ such that $u(h) = ch(0) - \int_0^1 h' f dx$.

Let $H_0^1 = \{h \in H^1 : h(0) = 0\}$. Define $T : H_0^1 \rightarrow L^2$ such that $T(h) = h'$. Note that T is one-to-one and onto L^2 . Now $u \circ T^{-1}$ is a linear functional on L^2 so by the Riesz Representation theorem there exists an f such that $u(T^{-1}g) = -\int_0^1 g f dx$. Let $g = Th$. Then $u(h) = -\int_0^1 h' f dx$. Let $c = u(1)$. Take any function $h \in H^1$. Then $h - h(0) \in H_0^1$ and since u is linear, we have $u(h) = u(h - h(0)) + u(h(0)) = -\int_0^1 h' f dx + ch(0)$.

Let B^* be the adjoint of B . B^* maps $H^{-1} \rightarrow L^2$. Next we show that if $u \in \ker B^*$, then $u = c\delta(x) + v$ where $v = \frac{df}{dx}$, $f \in H^k$ for $k = 0, 1, 2, \dots$. The proof is by induction. By the above paragraphs, the case for $k = 0$ holds by the definition of f . Assume true for $k - 1$, $k \geq 1$. We will show it is true for $f \in H^k$.

Since $u \in \ker B^*$, then $u \in H^{-1}$ so $u = c\delta(x) + v$ where $v = \frac{df}{dx}$ with $f \in L^2$. Then we have

$$B^* u = \int_x^1 (c\delta(y) + \frac{df}{dy}) dy + \int_0^1 Q(y, x) (c\delta(y) + \frac{df}{dy}) dy = 0.$$

Then

$$\frac{d}{dx} B^* u = -\frac{df}{dx} + \int_0^1 \frac{\partial}{\partial x} Q(y, x) \frac{df}{dy} dy + c \frac{d}{dx} Q(0, x) = 0.$$

This implies that on $(0, 1)$

$$\frac{df}{dx} = \int_0^1 \frac{\partial}{\partial x} Q(y, x) \frac{df}{dy} dy + c \frac{d}{dx} Q(0, x).$$

By (1.1), $Q(x, y) = \sum_{i=1}^n \alpha_i(x)(x-y)_+^i + \beta_i(x)(x-y)_-^i + R_n(x, y)$ with $\alpha_i(x), \beta_i(x) \in C^\infty[0, 1]$, $i = 1, \dots, n$ and $R_n(x, y) \in C^n([0, 1] \times [0, 1])$. Therefore, $\frac{\partial}{\partial x} Q(y, x)$ has the expansion: $\frac{\partial Q}{\partial x}(y, x) = \rho_0(x)\Theta(x-y) + \gamma_0(x)\Theta(y-x) + \text{smoother terms}$, with $\rho_0, \gamma_0 \in C^\infty$. Also $\frac{\partial}{\partial x} Q(0, x) \in C^\infty[0, 1]$. So if $f \in H^{k-1}$, then $\frac{df}{dx} \in H^{k-2}$ and $\int_0^1 \frac{\partial}{\partial x} Q(y, x) \frac{df}{dy} dy \in H^{k-1}$. Thus $\frac{df}{dx} \in H^{k-1}$ on $(0, 1)$, so $f \in H^k$ for all k . Consequently $f \in C^\infty[0, 1]$. So we can conclude that $\frac{df}{dx} = c_1\delta(x) + c_2\delta(x-1) + w$ where $w \in C^\infty[0, 1]$, and then clearly u has this form also.

We note that Lemma 1.1 is actually a special case of the transmission property - see [E2], chapter 23.

We will make the following assumption, suggested by the fact that B has index -1 .

Assumption 1.1 The cokernel of B is one dimensional, and the constant c_1 of Lemma 1.1 is nonzero.

Let $(\cdot, \cdot)$ denote the duality pairing between $H^1(0, 1)$ and its dual $H^{-1}(0, 1)$. Then $(Bu, v_0) = 0$ for all $u \in L^2(0, 1)$. Assumption 1.1 implies that there exists a bounded operator $R_0 : H^1(0, 1) \rightarrow L^2(0, 1)$ such that

$$(1.16) \quad \begin{aligned} R_0 Bu &= u & u &\in L^2(0, 1), \\ BR_0 g &= g & \text{if } (g, v_0) &= 0 \quad g \in H^1(0, 1). \end{aligned}$$

We apply R_0 to (1.12) to obtain

$$(1.17) \quad u_\epsilon(x) - \epsilon R_0 R_\epsilon \int_0^1 \frac{\partial Q}{\partial x}(x, y) u_\epsilon(y) dy = R_0(R_\epsilon f + c_\epsilon E_\epsilon(x, 0)).$$

Set

$$(1.18) \quad T_\epsilon w(x) = \epsilon R_0 R_\epsilon \int_0^1 \frac{\partial Q}{\partial x}(x, y) w(y) dy, \quad w \in L^2(0, 1).$$

We will prove a series of lemmas which will show that T_ϵ is a bounded operator from $L^2(0, 1)$ into $H^1(0, 1)$ which satisfies the estimate

$$\|T_\epsilon w\|_{L^2} \leq C \epsilon^{\frac{1}{2k+2}} \|w\|_{H^1}.$$

These lemmas and a lemma describing the behavior of c_ϵ as $\epsilon \rightarrow 0$ will imply the main results:

Theorem 1.1. Suppose that Assumption 1.1 holds, then there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, there exists a unique solution $u_\epsilon(x) \in L^2(0, 1)$, which is given by

$$u_\epsilon(x)(x) = (I - T_\epsilon)^{-1} R_0(R_\epsilon f + c_\epsilon E_\epsilon(x, 0)).$$

Theorem 1.2. Suppose Assumption 1.1 holds. If $\frac{f}{a} \in H^1[0, 1]$ and $\frac{f}{a}$ is orthogonal to the cokernel of B , then $u_\epsilon(x) \rightarrow u_0(x) = R_0(\frac{f}{a})$ in $L^2[0, 1]$ as $\epsilon \rightarrow 0$.

Lemma 1.2. The following estimates hold:

$$(1.19) \quad \|R_0 E_\epsilon(x, 0)\|_{L^2} \leq c\epsilon^{-\frac{1}{2k+2}}$$

$$(1.20) \quad \|E_\epsilon(x, 0)\|_{L^2} \leq c\epsilon^{\frac{1}{2k+2}}$$

Proof. Since $R_0 : L^2 \rightarrow H^1$ and $\|R_0 E_\epsilon(x, 0)\|_{L^2}^2 \approx \|\frac{d}{dx} E_\epsilon(x, 0)\|_{L^2}^2 + \|E_\epsilon(x, 0)\|_{L^2}^2$, it suffices to consider $\|\frac{d}{dx} E_\epsilon(x, 0)\|_{L^2}^2$ and $\|E_\epsilon(x, 0)\|_{L^2}^2$. Using that $c_1 x^k < a(x) < c_2 x^k$ we have

$$\begin{aligned} \|\frac{d}{dx} E_\epsilon(x, 0)\|_{L^2}^2 &= \int_0^1 \frac{a(x)^2}{\epsilon^2} \exp(-\int_0^x \frac{2a(s)}{\epsilon} ds) dx \\ &\approx \int_0^1 \frac{x^{2k}}{\epsilon^2} \exp(-cx^{k+1}/\epsilon) dx \\ &\approx \epsilon^{-\frac{1}{k+1}} \int_0^\infty t^{k/k+1} \exp(-ct) dt \\ &\approx c\epsilon^{-\frac{1}{k+1}}. \end{aligned}$$

Hence (1.19).

Now

$$\begin{aligned} \|E_\epsilon(x, 0)\|_{L^2}^2 &= \int_0^1 \exp(-\int_0^x \frac{2a(s)}{\epsilon} ds) dx \\ &\approx \int_0^1 \exp(-cx^{k+1}/\epsilon) dx \\ &\approx \epsilon^{\frac{1}{k+1}} \int_0^\infty t^{k/k+1} \exp(-ct) dt \\ &\approx c\epsilon^{\frac{1}{k+1}}, \end{aligned}$$

so (1.20) follows. $\square$

Lemma 1.3. If $f \in L^2$, then

$$(1.21) \quad \|\epsilon R_\epsilon f\|_{L^2} \leq c\epsilon^{\frac{1}{k+1}} \|f\|_{L^2}.$$

Proof. Let $f \in L^2$. By Cauchy-Schwarz,

$$\begin{aligned}\|\epsilon R_\epsilon f\|^2 &= \int_0^1 \left(\int_0^x E_\epsilon(x, y) f(y) dy \right)^2 dx \\ &\leq \int_0^1 x \int_0^x E_\epsilon^2(x, y) f^2(y) dy dx.\end{aligned}$$

Interchanging the order of integration, we obtain

$$\begin{aligned}\|\epsilon R_\epsilon f\|^2 &\leq \int_0^1 \int_y^1 x E_\epsilon^2(x, y) dx f^2(y) dy \\ &\leq \int_0^1 \int_y^1 x \exp\left(c \frac{y^{k+1} - x^{k+1}}{\epsilon}\right) dx f^2(y) dy.\end{aligned}$$

By the change of variables $t = \frac{x^{k+1} - y^{k+1}}{\epsilon}$,

$$\begin{aligned}\|R_\epsilon f\|^2 &\leq \int_0^1 \int_0^\infty (\epsilon t + y^{k+1})^{\frac{-k+1}{k+1}} \exp(-ct) dt f^2(y) dy \\ &\leq c \epsilon^{\frac{2k}{k+1}} \|f\|^2.\end{aligned}$$

Therefore, the lemma is proved. $\square$

Lemma 1.4. For $v \in H^1[0, 1]$, then

$$(1.22) \quad \left\| \epsilon \frac{d}{dx} R_\epsilon v \right\|_{L^2} \leq c \epsilon^{\frac{1}{2k+2}} \|v\|_{H^1}.$$

Proof. Let $v(x) \in H^1(0, 1)$. Then

$$\epsilon \frac{d}{dx} R_\epsilon v = v(x) - \frac{a(x)}{\epsilon} \int_0^x E_\epsilon(x, y) v(y) dy.$$

Let

$$I_\epsilon(x) = \frac{a(x)}{\epsilon} \int_0^x E_\epsilon(x, y) v(y) dy$$

and

$$I_\epsilon(x) = I_\epsilon^1(x) + I_\epsilon^2(x)$$

where

$$\begin{aligned}I_\epsilon^1(x) &= \int_0^x \frac{a(y)}{\epsilon} E_\epsilon(x, y) v(y) dy, \\ I_\epsilon^2(x) &= \int_0^x \frac{a(x) - a(y)}{\epsilon} E_\epsilon(x, y) v(y) dy.\end{aligned}$$

Clearly,

$$I_\epsilon^1(x) = v(x) - v(0)E_\epsilon(x, 0) - \int_0^x E_\epsilon(x, y)v'(y) dy,$$

By Lemma 1.3, $\|\epsilon R_\epsilon v'\|_{L^2} \leq c\epsilon^{1/k+1}\|v'\|_{L^2}$. By Lemma 1.2, $\|E_\epsilon(x, 0)\|_{L^2} \leq c\epsilon^{1/2k+2}$. So $\|\epsilon \frac{d}{dx} R_\epsilon v\| = \|v - I_\epsilon^1 - I_\epsilon^2\| \leq c\epsilon^{1/2k+2}\|v\|_{H^1} + \|I_\epsilon^2\|$.

We will use the observation that there exists some $c > 0$ such that $|a(x) - a(y)| \leq c|x^k - y^k|$ when estimating the $L^2(0, 1)$ norm of $I_\epsilon^2(x)$.

$$\begin{aligned} I_\epsilon^2(x) &= \int_0^x \frac{a(x) - a(y)}{\epsilon} E_\epsilon(x, y)v(y) dy \\ &\leq c\|v\|_{L^\infty(0,1)} \int_0^x \frac{x^k - y^k}{\epsilon} \exp(c \frac{y^{k+1} - x^{k+1}}{\epsilon}) dy. \end{aligned}$$

Of course $\|v\|_{L^\infty} \leq c\|v\|_{H^1} < c$. By the change of variables $s = \frac{x^k - y^k}{\epsilon}$,

$$I_\epsilon^2(x) \leq c\epsilon \int_0^{\frac{x^k}{\epsilon}} s \exp(-csx)(x^k - \epsilon s)^{\frac{1}{k}-1} ds.$$

We write $I_\epsilon^2(x) = I_\epsilon^3(x) + I_\epsilon^4(x)$ where

$$\begin{aligned} I_\epsilon^3(x) &= \epsilon \int_0^{\frac{x^k}{2\epsilon}} s \exp(-csx)(x^k - \epsilon s)^{\frac{1}{k}-1} ds, \\ I_\epsilon^4(x) &= \epsilon \int_{\frac{x^k}{2\epsilon}}^{\frac{x^k}{\epsilon}} s \exp(-csx)(x^k - \epsilon s)^{\frac{1}{k}-1} ds. \end{aligned}$$

Let us show

$$\|I_\epsilon^3(x)\| \leq c\epsilon^{1/2k+2}.$$

Using the fact that $(x^k - \epsilon s)^{\frac{1}{k}-1}$ is monotonically increasing for $0 < s < \frac{x^k}{2\epsilon}$ and integrating by parts, we find:

$$I_\epsilon^3(x) \leq c_1 x^{k+1} \exp(-\frac{cx^{k+1}}{\epsilon}) + c_2 \epsilon (\exp(-\frac{cx^{k+1}}{2\epsilon}) - 1).$$

By (1.20), the $L^2(0, 1)$ norm of the first term on the righthand side is less than $c\epsilon^{1/2k+2}$. Clearly, the second term on the righthand side is $O(\epsilon)$. Finally, we consider $I_\epsilon^4(x)$. We have:

$$I_\epsilon^4(x) \leq c \frac{x^{k+1}}{\epsilon} \exp(-\frac{cx^{k+1}}{2\epsilon}).$$

Therefore,

$$\|I_\epsilon^4(x)\| \leq c\epsilon^{1/2k+2}.$$

So the lemma is proved. $\square$

Lemma 1.5. T_ϵ is a bounded operator from $L^2[0, 1] \rightarrow H^1[0, 1]$ which satisfies the estimate

$$(1.23) \quad \|T_\epsilon w\|_{L^2} \leq c\epsilon^{1/2k+2} \|w\|_{H^1}.$$

Proof. By definition $T_\epsilon w = \epsilon R_0 R_\epsilon \int_0^1 \frac{\partial}{\partial x} Q(x, y) w(y) dy$. Let $v = \int_0^1 \frac{\partial}{\partial x} Q(x, y) w(y) dy$ then $v \in H^1[0, 1]$. Using Lemma 1.4, we have $\|\epsilon R_0 R_\epsilon v\|_{L^2} \leq c\epsilon^{1/2k+2} \|v\|_{H^1}$. So the result follows. $\square$

Note that for R_0 given by (1.16), we have

$$R_0 B_\epsilon = R_0 B - \epsilon R_0 R_\epsilon T = I - T_\epsilon.$$

By Lemma 1.5, for ϵ sufficiently small $I - T_\epsilon$ is invertible in $L^2(0, 1)$, and

$$(I - T_\epsilon)^{-1} R_0 B_\epsilon = I.$$

Assumption 1.1 implies that B_ϵ has a one dimensional cokernel, call it $v_\epsilon(x)$, i.e.

$$(B_\epsilon u, v_\epsilon) = 0 \text{ for all } u \in L^2(0, 1).$$

Now we determine $v_\epsilon(x)$ in terms of $v_0(x)$. Let B_ϵ^* denote the adjoint of B_ϵ , given by (1.11). We have

$$\begin{aligned} B_\epsilon^* &: H^{-1}(0, 1) \rightarrow L^2(0, 1) \\ B_\epsilon^* &= B^* - \epsilon(R_\epsilon T)^*, \end{aligned}$$

and by the definition of $v_\epsilon(x)$,

$$B_\epsilon^* v_\epsilon(x) = 0.$$

Let R_0^* denote the adjoint of R_0 , defined by (1.9), which has the following properties:

$$\begin{aligned} R_0^* &: L^2(0, 1) \rightarrow H^1(0, 1), \\ R_0 B &= I_{L^2(0, 1)}, \\ (R_0 B)^* &= I_{L^2(0, 1)}. \end{aligned}$$

We will find an $h_\epsilon \in L^2(0, 1)$ such that

$$v_\epsilon(x) = v_0(x) + R_0^* h_\epsilon.$$

Let T_ϵ^* be the adjoint of T_ϵ , i.e.

$$T_\epsilon^* = \epsilon(R_0 R_\epsilon T)^*.$$

By Lemma 1.5,

$$\|T_\epsilon^* g\| \leq c\epsilon^{\frac{1}{2k+2}} \|g\|,$$

so for ϵ sufficiently small $(I - T_\epsilon^*)^{-1}$ exists. Now we compute h_ϵ :

$$\begin{aligned} 0 &= B_\epsilon^* v_\epsilon(x) = (B^* - \epsilon(R_\epsilon T)^*)(v_0(x) + R_0^* h_\epsilon) \\ &= (I - T_\epsilon^*) h_\epsilon - \epsilon(R_\epsilon T)^* v_0(x). \end{aligned}$$

Thus, we finally obtain

$$h_\epsilon = (I - T_\epsilon^*)^{-1} \epsilon(R_\epsilon T)^* v_0(x) \in L^2(0, 1).$$

Note that $v_\epsilon(x)$ is a small perturbation of $v_0(x)$.

Now we come to the crucial definition. Set

$$(1.24) \quad c_\epsilon = -\frac{(R_\epsilon f, v_\epsilon)}{(E_\epsilon(x, 0), v_\epsilon)}$$

provided that the denominator does not vanish. Note that since $R_\epsilon f = \frac{f}{a} - \frac{f}{a}(0)E_\epsilon(x, 0) - \epsilon R_\epsilon \frac{d}{dx} \frac{f}{a}$ then $c_\epsilon = \frac{f}{a}(0) - (\frac{f}{a}, v_\epsilon(x)) / (E_\epsilon(x, 0), v_\epsilon(x)) + (\epsilon R_\epsilon \frac{d}{dx} \frac{f}{a}, v_\epsilon(x)) / (E_\epsilon(x, 0), v_\epsilon(x))$.

Lemma 1.6. *When Assumption 1.1 holds, c_ϵ given by (1.24), is well defined and $c_\epsilon = -\frac{(\frac{f}{a}, v_0)}{c_1} + \frac{f}{a}(0) + O(\epsilon^{1/k+1})$.*

Proof. First we examine the denominator of c_ϵ (1.24) :

$$\begin{aligned} (E_\epsilon(x, 0), v_\epsilon) &= (E_\epsilon(x, 0), v_0 + R_0^* h_\epsilon) \\ &= c_1 + c_2 E_\epsilon(1, 0) + (E_\epsilon(x, 0), w) + (E_\epsilon(x, 0), R_0^* h_\epsilon). \end{aligned}$$

The first term is nonzero by Assumption 1.1. The second term is exponentially small. The third term is $O(\epsilon^{1/k+1})$ by Cauchy-Schwarz and Lemma 1.2. Now we show that $|(E_\epsilon(x, 0), R_0^* h_\epsilon)| \leq c\epsilon^{3/2k+2}$.

$$\begin{aligned} |(E_\epsilon(x, 0), R_0^* h_\epsilon)| &\leq \|R_0 E_\epsilon(x, 0)\|_{L^2} \|h_\epsilon\|_{L^2} \\ &\leq c\epsilon^{-\frac{1}{2k+2}} \|(1 - T_\epsilon^*)^{-1} \epsilon(R_\epsilon T)^* v_0\|_{L^2} \\ &\leq c\epsilon^{-\frac{1}{2k+2}} \|(1 - T_\epsilon^*)^{-1}\|_{op} \|\epsilon(R_\epsilon T)^* v_0\|_{L^2} \\ &\leq c\epsilon^{-\frac{1}{2k+2}} (1 - \|T_\epsilon\|_{op})^{-1} \|\epsilon(R_\epsilon T)^* v_0\|_{L^2} \\ &\leq c\epsilon^{-\frac{1}{2k+2}} \|\epsilon R_\epsilon^* v_0\|_{H^{-1}} \end{aligned}$$

Since $v_0(x) = c_1 \delta(x) + c_2 \delta(x-1) + w(x)$, with $c_1 \neq 0$ and $w(x) \in C^\infty[0, 1]$,

$$\begin{aligned} \|\epsilon R_\epsilon^* v_0(x)\|_{H^{-1}} &= \sup_{\|g\|_{H^1}=1} |(\epsilon R_\epsilon^* v_0(x), g)| \\ &\leq \sup_{\|g\|_{H^1}=1} |c_2 \epsilon R_\epsilon g(1)| + |(w, \epsilon R_\epsilon g)|. \end{aligned}$$

Now we will show that $|\epsilon R_\epsilon g(1)| \leq c\epsilon$.

$$\begin{aligned} |\epsilon R_\epsilon g(1)| &= \left| \int_0^1 \exp\left(-\int_y^1 \frac{a(s)}{\epsilon} ds\right) g(y) dy \right| \\ &\leq \|g\|_{L^\infty} \int_0^1 \exp\left(-c \frac{(1-y^{k+1})}{\epsilon}\right) dy. \end{aligned}$$

Let

$$\begin{aligned} I_1 &= \int_{1-\epsilon}^1 \exp\left(-c \frac{1-y^{k+1}}{\epsilon}\right) dy \\ I_2 &= \int_0^{1-\epsilon} \exp\left(-c \frac{1-y^{k+1}}{\epsilon}\right) dy \end{aligned}$$

Then it is easy to see that $I_1 \approx c\epsilon$ and $I_2 \approx c\epsilon$.

We also have the estimate

$$\begin{aligned} |(w, \epsilon R_\epsilon g)| &< \|w\|_{L^\infty} \|g\|_{L^\infty} \int_0^1 \int_0^x \exp\left(-\int_y^x \frac{a(s)}{\epsilon} ds\right) dy dx \\ &< \|w\|_{L^\infty} \|g\|_{L^\infty} \int_0^1 \int_0^x \exp\left(-c \frac{x^{k+1} - y^{k+1}}{\epsilon}\right) dy dx. \end{aligned}$$

Integrating over a small region where the integrand is near 1 and a larger region where the integrand is small, we get $|(w, \epsilon R_\epsilon g)| \leq c\epsilon^{3/2k+2}$.

Therefore, $|(E_\epsilon(x, 0), v_\epsilon)| = c_1 + O(\epsilon^{1/k+1})$, so $|(E_\epsilon(x, 0), v_\epsilon)|^{-1} = 1/c_1 + O(\epsilon^{1/k+1})$.

Now we consider the numerator, $(R_\epsilon f, v_\epsilon(x))$. By (1.7) $R_\epsilon f = \frac{f}{a} - \frac{f}{a}(0)E_\epsilon(x, 0) - \epsilon R_\epsilon \frac{d}{dx} \frac{f}{a}$. So

$$\begin{aligned} \left(\frac{f}{a}, v_\epsilon\right) &= \left(\frac{f}{a}, v_0\right) + \left(\frac{f}{a}, R_0^* h_\epsilon\right) \\ &= \left(\frac{f}{a}, v_0\right) + \left(R_0 \frac{f}{a}, h_\epsilon\right) \end{aligned}$$

Now

$$|(R_0 \frac{f}{a}, h_\epsilon)| \leq \|R_0 \frac{f}{a}\|_{L^2} \|h_\epsilon\|_{L^2} \leq c \|R_0 \frac{f}{a}\|_{L^2} \epsilon^{1/k+1}.$$

Also

$$\begin{aligned} |(\epsilon R_\epsilon \frac{d}{dx} \frac{f}{a}, v_\epsilon(x))| &\leq |(\epsilon R_\epsilon \frac{d}{dx} \frac{f}{a}, v_0)| + |(\epsilon R_\epsilon \frac{d}{dx} \frac{f}{a}, R_0^* h_\epsilon)| \\ &\leq \|\frac{d}{dx} \frac{f}{a}\|_{L^2} \|\epsilon R_\epsilon^* v_0\|_{L^2} + \|\epsilon R_0 R_\epsilon \frac{d}{dx} \frac{f}{a}\|_{L^2} \|h_\epsilon\|_{L^2} \\ &\leq c \|\frac{d}{dx} \frac{f}{a}\|_{L^2} \epsilon^{1/k+1} + c\epsilon^{1/2k+2} \epsilon^{1/k+1} \\ &\leq c\epsilon^{1/k+1}. \end{aligned}$$

Hence, $c_\epsilon = \frac{f}{a}(0) - (\frac{f}{a}, v_0(x))/c_1 + O(\epsilon^{1/k+1})$. $\square$

Theorem 1.1. Suppose that Assumption 1.1 holds, then there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, there exists a unique solution $u_\epsilon(x) \in L^2(0, 1)$, which is given by

$$(1.25) \quad u_\epsilon(x) = (I - T_\epsilon)^{-1} R_0(R_\epsilon f + c_\epsilon E_\epsilon(x, 0))$$

for each $f \in L^2(0, 1)$.

Proof. Let $f \in L^2(0, 1)$. Define R_0 by (1.16), R_ϵ by (1.4), $E_\epsilon(x, 0)$ by (1.5), and c_ϵ by (1.24). By Lemmas 1.1 - 1.6 for $\epsilon > 0$ and sufficiently small, $u_\epsilon(x)$ is in $L^2(0, 1)$. Now we will show that $u_\epsilon(x)$ is the solution of (1.2).

Apply $B_\epsilon = B - \epsilon R_\epsilon T$, defined by (1.11), to (1.25).

$$B_\epsilon u_\epsilon(x) = B_\epsilon(I - T_\epsilon)^{-1} R_0(R_\epsilon f + c_\epsilon E_\epsilon(x, 0)).$$

By the choice (1.24) of c_ϵ , we find

$$(R_\epsilon f + c_\epsilon E_\epsilon(x, 0), v_\epsilon) = 0.$$

Then

$$B_\epsilon u_\epsilon(x) = R_\epsilon f + c_\epsilon E_\epsilon(x, 0).$$

Evaluating this expression at $x = 0$, we get (1.13):

$$B u_\epsilon(0) = \int_0^1 Q(0, y) u_\epsilon(y) dy = c_\epsilon.$$

Finally, apply L_ϵ to recover (1.2). $\square$

Theorem 1.2. Suppose Assumption 1.1 holds. If $\frac{f}{a} \in H^1[0, 1]$ and orthogonal to the cokernel of B , then $u_\epsilon(x) \rightarrow u_0(x)$ in $L^2[0, 1]$ as $\epsilon \rightarrow 0$.

Proof. By (1.25) and (1.7),

$$\begin{aligned} u_\epsilon(x) &= (1 - T_\epsilon)^{-1} R_0(R_\epsilon f + c_\epsilon E_\epsilon(x, 0)) \\ &= (1 - T_\epsilon)^{-1} R_0\left(\frac{f}{a} - \frac{f}{a}(0) E_\epsilon(x, 0) - \epsilon R_\epsilon \frac{d}{dx} \frac{f}{a} + c_\epsilon E_\epsilon(x, 0)\right) \end{aligned}$$

By Lemma 1.6,

(1.26)

$$u_\epsilon(x) = (1 - T_\epsilon)^{-1} R_0\left(\frac{f}{a} - \frac{f}{a}(0) E_\epsilon(x, 0) - \epsilon R_\epsilon \frac{d}{dx} \frac{f}{a} + \left(\frac{f}{a}(0) - \left(\frac{f}{a}, v_0\right)/c_1 + O(\epsilon^{1/k+1})\right) E_\epsilon(x, 0)\right).$$

By the orthogonality condition $(\frac{f}{a}, v_0) = 0$. By Lemma 1.2, $\|R_0 E_\epsilon(x, 0)\| \leq c\epsilon^{-1/2k+2}$ and Lemma 1.5 implies that for ϵ sufficiently small $(1 - T_\epsilon)^{-1} = \sum_{j=0}^{\infty} T_\epsilon^j$ converges in operator norm and $\|T_\epsilon^l w\|_{L^2} \rightarrow 0$ in L^2 for $l = 1, 2, 3..$ So $\|u_\epsilon(x)\|_{L^2} \leq \|u_0(x)\|_{L^2} + c\epsilon^{1/2k+2}$ and $u_\epsilon(x) \rightarrow u_0(x) = R_0 \frac{f}{a}$ in L^2 as $\epsilon \rightarrow 0$. $\square$

Corollary 1.2. Suppose Assumption 1.1 holds. If $\frac{f}{a} \in H^1[0, 1]$ but not orthogonal to the cokernel of B , then for ϵ sufficiently small, $\|u_\epsilon(x)\|_{L^2} \geq c\epsilon^{-\frac{1}{2k+2}}$.

Proof. Since $\frac{f}{a}$ is not orthogonal to the cokernel of B , $(\frac{f}{a}, v_0) \neq 0$. From (1.26), we see $\|u_\epsilon(x)\|_{L^2} \geq -C + \|(\frac{f}{a}, v_0)/c_1 R_0 E_\epsilon(x, 0)\|_{L^2} + c\epsilon^{1/2k+2}$. Lemma 1.2 is sharp in the sense that $\|\frac{d}{dx} E_\epsilon(x, 0)\|_{L^2} \approx c\epsilon^{-1/2k+2}$. So for ϵ sufficiently small, we have $\|u_\epsilon(x)\|_{L^2}^2 \geq c\epsilon^{-1/2k+2}$. $\square$

There is a boundary layer at $x = 0$. It can be shown that if $\frac{f}{a} \in H^1[0, 1]$ and orthogonal to the cokernel of B , then $u_\epsilon(x) \rightarrow u_0(x)$ uniformly as $\epsilon \rightarrow 0$ on compact subsets of $[0, 1]$ not containing the origin. Also since it can be shown that $R_0 E_\epsilon(x, 0) \rightarrow 0$ uniformly as $\epsilon \rightarrow 0$ on closed subsets of $[0, 1]$ not containing the origin, it can be shown the if $\frac{f}{a} \in H^2$ but is not orthogonal to the cokernel of B then $u_\epsilon(x) \rightarrow u_0(x)$ as $\epsilon \rightarrow 0$ uniformly on closed subsets of $[0, 1]$ not containing the origin.

Remark:

The solution, $u_\epsilon(x)$, has complicated behavior as $\epsilon \rightarrow 0$ if f/a is not in H^1 . The reason is that R_0 contains terms which are integral operators while f/a is singular at the origin, which means that $R_0 R_\epsilon f$ may be undefined as $\epsilon \rightarrow 0$. However, Theorem 1.1 is true for any $f \in L^2$ for $\epsilon > 0$. These results are in marked contrast with those of the non-degenerate case discussed in [S] (c.f. also [E3]).

2. Degeneracy in an interior point with k odd: reduction to a regular perturbation problem.

We sketch the case of

$$(2.1) \quad \epsilon u_\epsilon(x) + a(x) \left(\int_{-1}^x u_\epsilon(y) dy + \int_{-1}^1 Q(x, y) u_\epsilon(y) dy \right) = f(x)$$

where $a(x) = x^k b(x)$, k is an odd positive integer, $b(x) \in C^\infty[-1, 1]$ and $b(x) \geq c > 0$. The kernel $Q(x, y)$ has the same form as (1.1) but is defined on $[-1, 1] \times [-1, 1]$.

The associated differential operator L_ϵ , (1.3), is now defined on $(-1, 1)$ and R_ϵ is given by

$$R_\epsilon f = \int_0^x \frac{E_\epsilon(x, y)}{\epsilon} f(y) dy,$$

$$E_\epsilon(x, y) = \exp\left(-\int_y^x \frac{a(s)}{\epsilon} ds\right).$$

The general relations (1.6a, 6b) hold for $w \in H^1[-1, 1]$ and $g \in L^2(-1, 1)$.

We apply R_ϵ to (2.1) to obtain

$$(2.2) \quad R_\epsilon f = \int_{-1}^x u_\epsilon(y) dy + \int_{-1}^1 Q(x, y) u_\epsilon(y) dy$$

$$- \epsilon R_\epsilon \int_{-1}^1 \frac{\partial Q}{\partial x}(x, y) u_\epsilon(y) dy - E_\epsilon(x, 0) \left(\int_{-1}^0 u_\epsilon(y) dy + \int_{-1}^1 Q(0, y) u_\epsilon(y) dy \right)$$

$$:= B_\epsilon u_\epsilon(x) - c_\epsilon E_\epsilon(x, 0).$$

Set

$$c_\epsilon = \int_{-1}^0 u_\epsilon(y) dy + \int_{-1}^1 Q(0, y) u_\epsilon(y) dy.$$

We will treat c_ϵ as an unspecified constant to be determined shortly.

The associated reduced equation to (2.1) is

$$\begin{aligned} Bu_0 &= \int_{-1}^x u_0(x)(y) dy + \int_{-1}^1 Q(x, y) u_0(x)(y) dy \\ &= g \end{aligned}$$

where $g \in H^1[-1, 1]$.

Let $(,)$ denote the natural pairing between $H^1(-1, 1)$ and its dual $H^{-1}(-1, 1)$.

Assumption 2.1: B has a one dimensional cokernel, spanned by $v_0(x)$, which means $(Bu, v_0(x)) = 0$, and $v_0(x)$ has the form

$$v_0(x) = c_1 \delta(x+1) + c_2 \delta(x-1) + w(x)$$

where we further assume that $w(0) \neq 0$.

The function w will be in $C^\infty[-1, 1]$, by the argument of Lemma 1.1.

The assumption implies that there exists R_0 such that $R_0 B = I$. We will call the reduced solution $u_0(x) = R_0 g$ if $g \in H^1$ and is orthogonal to the cokernel of B .

Let

$$T_\epsilon w = \epsilon R_0 R_\epsilon \int_{-1}^1 \frac{\partial Q}{\partial x}(x, y) w(y) dy \quad w \in L^2(-1, 1).$$

Apply R_0 to (2.2) to obtain

$$(I - T_\epsilon) u_\epsilon(x) = R_0(R_\epsilon f + c_\epsilon E_\epsilon(x, 0)).$$

Lemma 2.1. T_ϵ is a bounded operator from $L^2[-1, 1]$ to $H^1[-1, 1]$ which satisfies the estimate $\|T_\epsilon w\|_{L^2} \leq c\epsilon^{1/2k+1} \|w\|_{H^1}$.

Proof. The proof follows from Lemma 1.5 and the observation that when $c_1 x^k \leq a(x) \leq c_2 x^k$, so $d_1(x^{k+1} - y^{k+1}) \leq E_\epsilon(x, y) \leq d_2(x^{k+1} - y^{k+1})$ and a similar estimate holds for $E_\epsilon(-x, -y)$ since k is an odd integer. $\square$

Therefore for ϵ sufficiently small

$$(2.3) \quad u_\epsilon(x) = (I - T_\epsilon)^{-1} R_0(R_\epsilon f + c_\epsilon E_\epsilon(x, 0)).$$

For the same reasons as in §1, we pick

$$c_\epsilon = -\frac{(R_\epsilon f, v_\epsilon)}{(E_\epsilon(x, 0), v_\epsilon)}$$

where $(B_\epsilon u, v_\epsilon) = 0$ for all $u \in L^2(-1, 1)$ and

$$v_\epsilon = v_0(x) + R_0^*(I - T_\epsilon^*)^{-1} \epsilon (R_\epsilon T)^* v_0(x).$$

We claim that when Assumption 2.1 holds, $c_\epsilon = -\frac{(\frac{f}{a}, v_0)}{d_1} \epsilon^{-1/k+1} + \frac{f}{a}(0) + O(\epsilon^{1/k+1})$ by an argument similar to the one given in Lemma 1.6. However, the denominator $(E_\epsilon(x, 0), v_\epsilon)$ is composed of two exponentially small terms $c_1 E_\epsilon(1, 0)$ and $c_2 E_\epsilon(-1, 0)$ and by Assumption 2.1 there exist d_1 and d_2 such that

$$d_1 \epsilon^{1/k+1} \leq |(E_\epsilon(x, 0), w)| \leq d_2 \epsilon^{1/k+1}.$$

So for ϵ sufficiently small, $(E_\epsilon(x, 0), w)$ is the dominant term.

By arguments analogous to those given in Theorems 1.1 and 1.2, we can state the following theorems:

Theorem 2.1. *If Assumption 2.1 holds, then there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, there exists a unique solution in $L^2(-1, 1)$ to (2.1) given by (2.3) for each $f \in L^2(-1, 1)$.*

Theorem 2.2. *If Assumption 2.1 holds and if $\frac{f}{a} \in H^1[-1, 1]$ and orthogonal to the cokernel of B , then $u_\epsilon \rightarrow u_0 = R_0 \frac{f}{a}$ in L^2 as $\epsilon \rightarrow 0$.*

Corollary 2.2. *If Assumption 2.1 holds and if $\frac{f}{a} \in H^1[-1, 1]$ but is not orthogonal to the cokernel of B , then $\|u_\epsilon(x)\|_{L^2} \geq -C + c\epsilon^{-3/2k+2}$.*

3. Degeneracy at an interior point with k even: reduction to a regular perturbation problem.

We sketch the case of the singularly perturbed integral equation

$$(3.1) \quad \epsilon u_\epsilon(x) + a(x) \left(\int_{-1}^x u_\epsilon(y) dy + \int_{-1}^1 Q(x, y) u_\epsilon(y) dy \right) = f(x)$$

where $a(x) = x^k b(x)$, where k is an even positive integer and $b(x) \geq c > 0, b(x) \in C^\infty[-1, 1]$. The kernel $Q(x, y)$ is the same as in §2. We define the operator L_ϵ and R_ϵ :

$$\begin{aligned} L_\epsilon w(x) &= \epsilon \frac{d}{dx} w(x) + a(x) w(x) & w \in H^1[-1, 1], \\ R_\epsilon f(x) &= \int_{-1}^x \frac{E_\epsilon(x, y)}{\epsilon} f(y) dy & f \in L^2(-1, 1) \end{aligned}$$

which satisfy the general relations:

$$\begin{aligned} R_\epsilon L_\epsilon w(x) &= w(x) - w(-1) E_\epsilon(x, -1), \\ L_\epsilon R_\epsilon f(x) &= f(x). \end{aligned}$$

The reduced equation is given by

$$(3.2) \quad Bu_0 = \int_{-1}^x u_0(y) dy + \int_{-1}^1 Q(x, y) u_0(y) dy = g$$

where $g \in H^1[-1, 1]$.

By an argument similar to Lemma 1.1, the elements of the cokernel of B , ie the $v_0 \in H^{-1}$ with $(Bu, v_0(x)) = 0$ for all u , have the form

$$v_0(x) = c_1 \delta(x+1) + c_2 \delta(x-1) + w(x)$$

with $w \in C^\infty[0, 1]$.

Assumption 3.1: B has a one dimensional cokernel, and the constant c_1 is nonzero.

Let R_0 denote an operator such that $R_0 B = I$ and B_ϵ be of the form (2.2). As before we will call the reduced solution $u_0(x) = R_0 g$ when $g \in H^1$ and orthogonal to the cokernel of B .

Applying $R_0 R_\epsilon$ to (3.1) we obtain

$$u_\epsilon(x) + \epsilon R_0 R_\epsilon \int_{-1}^1 \frac{\partial Q}{\partial x}(x, y) u_\epsilon(y) dy = R_0(R_\epsilon f + c_\epsilon E_\epsilon(x, -1))$$

where

$$c_\epsilon = \int_{-1}^1 Q(-1, y) u_\epsilon(y) dy$$

is an unspecified constant to be determined later on. Set

$$T_\epsilon w = \epsilon R_0 R_\epsilon \int_{-1}^1 \frac{\partial Q}{\partial x}(x, y) w(y) dy \quad w \in L^2(-1, 1).$$

It can be shown that T_ϵ is a bounded operator from $L^2[-1, 1]$ into $H^1[-1, 1]$ which satisfies the estimate

$$\|T_\epsilon w\|_{L^2} \leq c\epsilon^{\frac{1}{2k+2}} \|w\|_{H^1}$$

by a similar argument to the one in the proof of Lemma 1.5.

Then for ϵ sufficiently small

$$(3.3) \quad u_\epsilon(x) = (I - T_\epsilon)^{-1} R_0(R_\epsilon f + c_\epsilon E_\epsilon(x, -1)).$$

The crucial step is to define

$$c_\epsilon = -\frac{(R_\epsilon f, v_\epsilon)}{(E_\epsilon(x, -1), v_\epsilon)}$$

where $(B_\epsilon u, v_\epsilon) = 0$ for all $u \in L^2(-1, 1)$ and

$$\begin{aligned} v_\epsilon(x) &= v_0(x) + R_0^* h_\epsilon, \\ h_\epsilon &= (I - T_\epsilon^*)^{-1} \epsilon (R_0^* T)^* v_0(x). \end{aligned}$$

As in Lemma 1.6, we have $c_\epsilon = -\frac{(\frac{1}{a} v_0(x))}{c_2} + \frac{f}{a}(-1) + O(\epsilon^{1/k+1})$ provided that Assumption 3.1 holds.

By arguments analogous to those given in Theorem 1.1 and Theorem 1.2, we can state the following theorems:

Theorem 3.1. If Assumption 3.1 holds, then there exists an $\epsilon_0 > 0$, such that for $0 < \epsilon < \epsilon_0$, there exists a unique $L^2(-1, 1)$ solution to (3.1) given by (3.3) for each $f \in L^2(-1, 1)$.

Theorem 3.2. If Assumption 3.1 holds and if $\frac{f}{a} \in H^1[-1, 1]$ and orthogonal to the cokernel of B , then $u_\epsilon \rightarrow u_0 = R_0 \frac{f}{a}$ in L^2 as $\epsilon \rightarrow 0$.

Corollary 3.2. If Assumption 3.1 holds and if $\frac{f}{a} \in H^1[-1, 1]$ but is not orthogonal to the cokernel of B , then $\|u_\epsilon(x)\|_{L^2} \geq -C + c\epsilon^{-1/2k+2}$.

4. A sign change in the degeneracy at an interior point with k odd.

In this section we examine the dual situation of the problem discussed in §2. Suppose we are given $f \in L^2(-1, 1)$. We consider the singularly perturbed integral equation

$$(4.1) \quad \epsilon u_\epsilon(x) - \int_{-1}^x a(y)u_\epsilon(y) dy - \int_{-1}^1 a(y)Q(x, y)u_\epsilon(y) dy = f(x)$$

where $a(x) = x^k b(x)$, $b(x) \in C^\infty[-1, 1]$, $b(x) > 0$ and k is an odd positive integer. The kernel $Q(x, y)$ is the same as in §2.

We find sufficient conditions for the existence and uniqueness of an $L^2(-1, 1)$ solution to (4.1). First we extend (4.1) to $\mathbf{R}$ in the following manner. Denote still by $u_\epsilon(x)$, $u_\epsilon(x) \in L^2(\mathbf{R})$, the zero extension of $u_\epsilon(x) \in L^2(-1, 1)$, i.e. the new u_ϵ is the same as the original u_ϵ for $x \in (-1, 1)$ and zero otherwise. Let $f_+(x)$ be the zero extension of f as an element in $L^2(\mathbf{R})$. Let p denote the restriction operator to $(-1, 1)$, then $pf_+(x) = f$. Take smooth bounded extensions of $Q(x, y)$ and $a(x)$.

Then (4.1) becomes

$$(4.2) \quad \epsilon u_\epsilon(x) - \int_{-\infty}^x a(y)u_\epsilon(y) dy - \int_{-\infty}^{\infty} a(y)Q(x, y)u_\epsilon(y) dy = f_+(x).$$

Let $\mathring{H}_s(-1, 1)$ denote the space of distributions which are in $H_s(R)$ with support contained in $[-1, 1]$. We define the operators L_ϵ , R_ϵ by

$$(4.3) \quad \begin{aligned} L_\epsilon w &= \epsilon \frac{d}{dx} w - a(x)w & w &\in \mathring{H}_0(-1, 1) \\ R_\epsilon g &= -\Theta(x) \int_x^1 \frac{E_\epsilon(x, y)}{\epsilon} g(y) dy + \Theta(-x) \int_{-1}^x \frac{E_\epsilon(x, y)}{\epsilon} g(y) dy, \\ &g \in \mathring{H}_{-1}(-1, 1) \end{aligned}$$

where

$$E_\epsilon(x, y) = \exp\left(\int_y^x \frac{a(s)}{\epsilon} ds\right).$$

It is easy to verify the general relations:

$$(4.4) \quad \begin{aligned} R_\epsilon L_\epsilon w(x) &= w(x), \\ L_\epsilon R_\epsilon g(x) &= g(x) & \text{if } (g, E_\epsilon(0, x)) &= 0 \end{aligned}$$

Set

$$(4.5) \quad Tu_\epsilon(x) = p \int_{-\infty}^{\infty} Q(x, y) \frac{d}{dy} u_\epsilon(y) dy.$$

Now we can better see the structure of (4.1) by writing it as

$$(4.6) \quad pBL_\epsilon u_\epsilon + Tu_\epsilon = f$$

The associated reduced equation of (4.1) is

$$(4.7) \quad \begin{aligned} pBv &= \int_{-\infty}^x v(y) dy + \int_{-\infty}^{\infty} Q(x, y)v(y) dy \\ &= f \end{aligned}$$

where $v = au_0$.

Since pB has index 1 as an operator from $\mathring{H}_{-1}(-1, 1)$ to $L^2(-1, 1)$, the following assumption on pB is natural as it is nothing more than the dual situation compared with §2.

Assumption 4.1: The operator $pB : \mathring{H}_{-1}(-1, 1) \rightarrow L^2(-1, 1)$ in (4.7) has a one dimensional kernel. $\square$

Let $k(x)$ span the kernel of pB , i.e. $pBk(x) = 0$. $k(x)$ has the form

$$(4.8) \quad k(x) = c_1\delta(x+1) + c_2\delta(x-1) + w(x)$$

where $w(x) \in C^\infty[-1, 1]$, by an argument similar to Lemma 1.1.

Assumption 4.2: We assume that the smooth term in the kernel of pB (4.8) does not vanish at the origin, i.e. $w(0) \neq 0$. $\square$

Let R_0 be the operator with the following properties:

$$(4.9a) \quad R_0 : L^2(-1, 1) \rightarrow \mathring{H}_{-1}(-1, 1) : \quad pBR_0g = g$$

provided that R_0g is not in the kernel of pB and

$$(4.9b) \quad R_0pBv_0(x) = v_0(x) + ck(x)$$

where c is an arbitrary constant.

Then if R_0f is not in the kernel of pB , the solution to the reduced equation (4.7) is $u_0 = \frac{R_0f}{\alpha}$.

After applying R_0 to (4.6), we have

$$(4.10) \quad L_\epsilon u_\epsilon(x) - \epsilon R_0 T u_\epsilon(x) = R_0 f + c_\epsilon k(x).$$

We will determine c_ϵ by requiring that the compatibility condition (4.4) should be satisfied:

$$(R_0 f + c_\epsilon k(x) + \epsilon R_0 T u_\epsilon(x), E_\epsilon(0, x)) = 0.$$

Thus, we take

$$c_\epsilon = -(k(x), E_\epsilon(0, x))^{-1}(\epsilon R_0 T u_\epsilon(x) + R_0 f, E_\epsilon(0, x))$$

and decompose it as a sum of two terms,

$$c_\epsilon = c_{1\epsilon} + c_{2\epsilon}$$

where

$$(4.11) \quad c_{1\epsilon} = -(k(x), E_\epsilon(0, x))^{-1}(\epsilon R_0 T u_\epsilon(x), E_\epsilon(0, x)),$$

$$(4.12) \quad c_{2\epsilon} = -(k(x), E_\epsilon(0, x))^{-1}(R_0 f, E_\epsilon(0, x)).$$

We see that $c_{1\epsilon}$ depends on the unknown function $u_\epsilon(x)$. It will be determined in terms of f shortly.

Apply R_ϵ to (4.10) to obtain

$$(4.13) \quad u_\epsilon(x) - \epsilon R_\epsilon R_0 T u_\epsilon(x) = R_\epsilon(R_0 f + c_\epsilon k(x)).$$

In order to prove that (4.13) can be viewed as a regular perturbation problem, we will show that

$$(4.14) \quad T_\epsilon u_\epsilon(x) := R_\epsilon(\epsilon R_0 T u_\epsilon(x) + c_{1\epsilon} k(x))$$

has small norm in $\mathring{H}_0(-1, 1)$.

Lemma 4.1. T_ϵ defined by (4.14) is a bounded operator from $\mathring{H}_0(-1, 1)$ into $\mathring{H}_0(-1, 1)$ which satisfies the estimate

$$(4.15) \quad \|T_\epsilon w\|_{\mathring{H}_0(-1, 1)} \leq c\epsilon^{\frac{1}{2k+2}} \|w\|_{\mathring{H}_0(-1, 1)}.$$

Proof. The first term in (4.14) satisfies the estimate

$$(4.16) \quad \|\epsilon R_\epsilon R_0 T w\|_{\mathring{H}_0(-1, 1)} \leq c\epsilon^{1/k+1} \|w\|_{\mathring{H}_0(-1, 1)}$$

which may be established by the similar argument as in the proof of Lemma 1.4. Let us have a look of the second term. Since

$$k(x) = c_1 \delta(x+1) + c_2 \delta(x-1) + w(x),$$

by Assumption 4.2, there exist β_1, β_2 such that

$$(4.17) \quad \beta_1 \epsilon^{1/k+1} \leq |(k(x), E_\epsilon(0, x))| \leq \beta_2 \epsilon^{1/k+1}.$$

It is easy to see that

$$(4.18) \quad \begin{aligned} |(R_0 T w, E_\epsilon(0, x))| &= |(T w, R_0^* E_\epsilon(0, x))| \\ &\leq c \epsilon^{\frac{1}{2k+2}} \|w\|_{\dot{H}_0(-1,1)}. \end{aligned}$$

Thus, by estimates similar to those given in the proof of Lemma 1.3, we have

$$(4.19) \quad \|R_\epsilon k(x)\|_{\dot{H}_0(-1,1)} \leq c \epsilon^{-\frac{k}{k+1}} \|w\|_{\dot{H}_0(-1,1)}.$$

Combining (4.16), (4.17), (4.18), and (4.19), we get (4.15). $\square$

Thus for ϵ sufficiently small, $(I - T_\epsilon)^{-1}$ exists and

$$(4.20) \quad u_\epsilon(x) = (I - T_\epsilon)^{-1} R_\epsilon(R_0 f - c_{2\epsilon} k(x)).$$

Now we substitute (4.21) into (4.12) to obtain

$$(4.21) \quad c_{1\epsilon} = -(k(x), E_\epsilon(0, x))^{-1} (\epsilon R_0 T (I - T_\epsilon)^{-1} R_\epsilon(R_0 f - c_{2\epsilon} k(x)), E_\epsilon(0, x)).$$

Lemma 4.2. *When Assumption 4.2 holds, c_ϵ is well defined and $O(\epsilon^{-\frac{k}{k+1}})$.*

Proof. We write $c_\epsilon = c_{1\epsilon} + c_{2\epsilon}$, where $c_{1\epsilon}, c_{2\epsilon}$ are given by (4.21) and (4.12), respectively. Let us examine $c_{2\epsilon}$ first. By Assumption 4.2, the numerator satisfies the following inequality:

$$\begin{aligned} |(R_0 f, E_\epsilon(0, x))| &\leq c \|R_0 f\|_{\dot{H}_{-1}(-1,1)} \|E_\epsilon(0, x)\|_{H_1(-1,1)} \\ &\leq c \epsilon^{-\frac{1}{2k+2}}. \end{aligned}$$

Therefore, $c_{2\epsilon} = O(\epsilon^{-\frac{1}{k+1}})$.

Now we consider $c_{1\epsilon}$. The denominator of $c_{1\epsilon}$ also satisfies (4.17). The numerator is composed of two terms:

$$\begin{aligned} A_1 &= \epsilon c_{2\epsilon} (R_0 T (I - T_\epsilon)^{-1} R_\epsilon k, E_\epsilon(0, x)), \\ A_2 &= \epsilon (R_0 T (I - T_\epsilon)^{-1} R_\epsilon R_0 f, E_\epsilon(0, x)). \end{aligned}$$

Using our assumption that $k(x)$ has the form (4.8), we have

$$\begin{aligned} |A_1| &\leq c \epsilon c_{2\epsilon} |(R_\epsilon k(x), E_\epsilon(0, x))| \\ &\leq c \epsilon c_{2\epsilon} |(R_\epsilon w(x), E_\epsilon(0, x))| \end{aligned}$$

because $R_\epsilon \delta(x \pm 1) = 0$. Thus,

$$\begin{aligned} |A_1| &\leq c\epsilon c_{2\epsilon} \|R_\epsilon w\|_{\dot{H}_0(-1,1)} \|E_\epsilon(0, x)\|_{\dot{H}_0(-1,1)} \\ &\leq c\epsilon^{\frac{1}{2k+2}} \end{aligned}$$

since

$$\begin{aligned} \|R_\epsilon w\|_{\dot{H}_0(-1,1)} &\leq c\epsilon^{-\frac{k}{k+1}}, \\ \|E_\epsilon(0, x)\| &\leq c\epsilon^{\frac{1}{2k+2}}. \end{aligned}$$

Finally, we also make an estimate on A_2 .

$$\begin{aligned} |A_2| &\leq c\epsilon |(R_\epsilon R_0 f, E_\epsilon(0, x))| \\ &\leq c\epsilon \|R_\epsilon R_0 f\|_{\dot{H}_0(-1,1)} \|E_\epsilon(0, x)\|_{L^2(-1,1)} \\ &\leq c\epsilon^{-\frac{k}{k+1}} \epsilon^{\frac{1}{2k+2}}. \end{aligned}$$

Therefore, $c_{1\epsilon} = O(\epsilon^{-\frac{k}{k+1}})$ which implies that $c_\epsilon = O(\epsilon^{-\frac{k}{k+1}})$. Thus, the lemma is proved. $\square$

Remark: Assumption 4.2 was necessary to ensure that the main term in the denominator of c_ϵ satisfies (4.17). If $w(x)$ vanishes to higher order, say $w^j(0) = 0$, $w^{j+1}(0) \neq 0$, then the denominator of c_ϵ is $O(\epsilon^{\frac{2j+1}{2k+2}})$. In the case when $Q(x, y) = 0$, then $w(x) = 0$, and the denominator is exponentially small which means that c_ϵ grows exponentially as $\epsilon \rightarrow 0$. In fact, it is not well defined if $c_1 E_\epsilon(0, 1) = c_2 E_\epsilon(0, -1)$.

Theorem 4.1. Suppose Assumptions 4.1 and 4.2 hold, then there exists an ϵ_0 such that for $0 < \epsilon < \epsilon_0$, (4.1) has a unique solution $u_\epsilon(x) \in \dot{H}_0(-1, 1)$, which is given by (4.20) for each $f \in L^2(-1, 1)$.

Proof. Let $f \in L^2(-1, 1)$. Suppose Assumption 4.1 and 4.2 hold and $c_\epsilon = c_{1\epsilon} + c_{2\epsilon}$ where $c_{1\epsilon}$ and $c_{2\epsilon}$ are given by (4.11) and (4.21), respectively. Lemma 4.1 shows that for ϵ sufficiently small, $(I - T_\epsilon)^{-1}$ exists. Let $u_\epsilon(x)$

$$u_\epsilon(x) = (I - T_\epsilon)^{-1} R_\epsilon(R_0 f + c_{2\epsilon} k(x))$$

where T_ϵ , R_ϵ , $k(x)$, and R_0 are given by (4.19), (4.3), (4.8), and (4.9), respectively. Then

$$(I - T_\epsilon)u_\epsilon(x) = R_\epsilon(R_0 f + c_{2\epsilon} k(x))$$

and

$$\begin{aligned} u_\epsilon(x) &= T_\epsilon u_\epsilon(x) + R_\epsilon(c_{2\epsilon} k(x) + R_0 f) \\ &= \epsilon R_\epsilon R_0 T u_\epsilon(x) - (k(x), E_\epsilon(0, x))^{-1} (\epsilon R_0 T u_\epsilon(x), E_\epsilon(0, x)) R_\epsilon k(x) \\ &\quad + R_\epsilon(c_{2\epsilon} k(x) + R_0 f). \end{aligned}$$

Apply L_ϵ , given by (4.3), to the above expression, to obtain

$$(4.22) \quad \begin{aligned} L_\epsilon u_\epsilon(x) &= \epsilon R_0 T u_\epsilon(x) + c_{2\epsilon} k(x) + R_0 f \\ &\quad - (k(x), E_\epsilon(0, x))^{-1} (\epsilon R_0 T u_\epsilon(x), E_\epsilon(0, x)) k(x). \end{aligned}$$

The compatibility condition (4.4) is satisfied by our choice of c_ϵ , i.e.

$$\begin{aligned} &(\epsilon R_0 T u_\epsilon(x) - (k(x), E_\epsilon(0, x))^{-1} (\epsilon R_0 T u_\epsilon(x), E_\epsilon(0, x)) k(x) \\ &\quad + c_{2\epsilon} k(x) + R_0 f, E_\epsilon(0, x)) = 0. \end{aligned}$$

Composing pB , (4.7) with L_ϵ (4.22) to obtain

$$pBL_\epsilon u_\epsilon(x) = \epsilon T u_\epsilon(x) - f,$$

since by Assumption 4.1, $pBk(x) = 0$. Thus, we have recovered (4.1),

$$pBL_\epsilon u_\epsilon(x) - \epsilon T u_\epsilon(x) = f.$$

Hence, the theorem is proved. $\square$

REFERENCES

- [D] A. Demidov, *Elliptic pseudodifferential boundary problems with a small parameter in the coefficient of the leading operator*, Math. USSR Sb. **20** (1973), 439–463.
- [E1] G. Eskin, *Asymptotics of solutions of elliptic pseudodifferential equations with a small parameter*, Dokl. Akad. Nauk USSR **211** (1973), 547–550, English translation Soviet Mathematics Doklady **14** (1973), 1080–1083.
- [E2] ———, *Boundary Value Problems for Elliptic Pseudodifferential Equations*, American Mathematical Society, Providence, 1981.
- [E3] ———, *Degenerate elliptic pseudodifferential equations of principal type*, Math. USSR Sbornik **11** (1970), 539–581.
- [EV] G. Eskin and M. Višik, *Convolution equations of variable order*, Trudy Moskov, Math. Obšč **16** (1967), 25–50, English translation Trans. Moscow Math. Soc. (1967), 27–52.
- [F] L. S. Frank, *Coercive singular perturbations I: a priori estimates*, Ann. Mat. Pura Appl. **119** (1979), 41–113.
- [FW1] L. S. Frank and W. D. Wendt, *Coercive singular perturbations II: reduction to regular perturbations and applications*, Comm. Part. Diff. Eqn **7** (1982), 469–535.
- [FW2] ———, *Coercive singular perturbations II: reduction and convergence*, J. Math Anal. Appl. **88** (1982), 463–535.
- [FW3] ———, *Coercive singular perturbations III: Wiener-Hopf operators*, J. d'Analyse Math. **43** (1984), 88–135.
- [G] G. Grubb, *Functional Calculus of Pseudo-differential Boundary Problems*, Birkhauser, Boston, 1986.

- [LS] C. Lange and D. Smith, *Singular Perturbation analysis of integral equations*, Stud. Appl. Math. **79** no. 1 (1988), 1-63.
- [MP] V. Mazja and B. Panejah, *Degenerate elliptic pseudodifferential operators and the oblique derivative problem*, Transactions of Moscow Math Society (1974).
- [S] C. A. Shubin, *Singularly perturbed integral equations*, preprint.
- [Sj] J. Sjöstrand, *Operators of principal type with interior boundary conditions*, Acta Math **130** (1973), 1-51.
- [T] F. Treves, *A new method of proof of the subelliptic estimates*, Comm. Pure Appl. Math **24** (1971), 71-115.
- [VG] M. Višik and V. Grusin, *On a class of higher order degenerate elliptic equations*, Math. USSR Sbornik **8** (1969), 1 - 32.

DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY, NORTHRIDGE
NORTHRIDGE, CA. 91330
AMS CLASSIFICATION: 45B11, 45E99, 34E15

Submitted: December 22, 1994
Revised: August 24, 1995