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Dyons in affine field theories

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By postulating that the gauge vector field and the Higgs field both lie in an affine algebra rather than a Lie algebra, an affine field theory with an expanded soliton sector is obtained. An infinite family of exact, special solutions of the new equations corresponding to the original 't Hooft–Polyakov solitonic solutions, is now found. A perturbation method for an arbitrary solution of the new equation is also proposed.

I. INTRODUCTION

Since the work of 't Hooft and Polyakov, it has been known that non-Abelian field theories may have a soliton sector. In particular, magnetic monopoles^{1,2} as well as dyons³ may arise in a spontaneously broken SU(2) gauge theory. Although there is no present evidence for these particles one can speculate that they are simply too heavy to have been observed or that they are bound preons.⁴ If one is interested in gravitationally coupled theories, however, and therefore in extremely high energies, perhaps one should think more about these objects.

The theoretically known Yang–Mills solitons are nodeless, but in principle there could be nodding solutions like the excited states of an atom or nucleus. In fact, one should in general expect higher modes of excitation in any spatially extended structure.⁵ It may be, however, that the ensemble of higher modes is realized physically only if one passes from the finite Lie algebra to an infinite algebra. Moreover, such a formal extension is natural since there exists an infinite-dimensional Lie algebra corresponding to every semisimple Lie algebra. This formal extension is also suggested by the idea of regarding the infinite algebra and the corresponding internal space as a surrogate for particle extension in space-time.

If one does pass from Yang–Mills theories to their affine extensions,⁶ one finds that the soliton sector is indeed correspondingly expanded. In this paper these new solitons will be studied. In taking this step our basic assumption is that the Higgs field, as well as the vector field, lies in a loop algebra rather than in a Lie or in a Kac–Moody algebra. In an earlier note⁷ we discussed the Higgs splitting of a Kac–Moody field and found that the mass of the vector particle was linear in the loop index n . That result provided a simple interpretation of n as a label of the heavy vectors, but our assumptions here will be slightly different: for simplicity we limit the discussion to the loop rather than the full Kac–Moody algebra.

II. FORMULATION

We assume that the vector and Higgs field lie in the loop algebra

$$W_\mu = \sum (W_\mu)_n^a T_n^a, \quad (2.1)$$

$$\Phi = \sum (\Phi)_n^a T_n^a, \quad (2.2)$$

where

$$(T_n^a, T_m^b) = f^{abc} T_{n+m}^c \quad (2.3)$$

and the structure constants are real. Then

$$(T_n^a)^+ = -T_{-n}^a. \quad (2.4)$$

We also take both fields to be anti-Hermitian:

$$W_\mu^+ = -W_\mu, \quad (2.5)$$

$$\Phi^+ = -\Phi. \quad (2.6)$$

Then

$$(W_\mu)_n^a = (W_\mu)_{-n}^a, \quad (2.7)$$

$$\Phi_n^a = \Phi_{-n}^a, \quad (2.8)$$

if we take the component fields to be real. We consider the action

$$S = \int d^4x L, \quad (2.9)$$

$$L = \langle -\frac{1}{4} G_{\mu\lambda} (G^{\mu\lambda})^+ + \frac{1}{2} (D_\mu \Phi) (D^\mu \Phi)^+ + \lambda V(\langle \Phi^+ \Phi \rangle) \rangle, \quad (2.10)$$

where $\langle \rangle$ is the invariant scalar product for the algebra, $G_{\mu\lambda}$ is the usual field strength, and $D_\mu \Phi$ is a covariant derivative:

$$G_{\mu\lambda} = (\nabla_\mu, \nabla_\lambda), \quad (2.11)$$

$$D_\mu \Phi = (\nabla_\mu, \Phi), \quad (2.12)$$

$$\nabla_\mu = \partial_\mu + W_\mu. \quad (2.13)$$

Here W_μ and Φ are the loop vector and scalar fields.

The equations of motion are formally the same as for a Lie algebra since $\langle \rangle$ shares with the trace the properties

$$\langle AB \rangle = \langle BA \rangle, \quad (2.14)$$

$$\langle (A, B) C \rangle = \langle (B, C) A \rangle. \quad (2.15)$$

Also

$$\langle T_n^a T_m^b \rangle = -\delta^{ab} \delta(n+m). \quad (2.16)$$

The equations of motion are

$$(\nabla_\lambda, G^{\mu\lambda}) = (\Phi, (\nabla^\mu, \Phi)), \quad (2.17)$$

$$(\nabla^\mu, (\nabla_\mu, \Phi)) = \lambda \frac{\partial V}{\partial \Phi}. \quad (2.18)$$

In addition there is the Bianchi identity

$$(\nabla_\mu, \tilde{G}^{\mu\lambda}) = 0, \quad (2.19)$$

where $\tilde{G}^{\mu\lambda}$ is the dual field. The symmetric energy momentum tensor is

$$T^{\mu\lambda} = \langle G^{\mu\sigma}(G_\sigma^\lambda)^+ + (D^\mu\Phi)(D^\lambda\Phi)^+ + g^{\mu\lambda}L \rangle \quad (2.20)$$

and the energy density is

$$T^{00} = \frac{1}{2} \langle \mathcal{E}^i(\mathcal{E}^i)^+ + \mathcal{B}^i(\mathcal{B}^i)^+ + (D^0\Phi)(D^0\Phi)^+ + (D^i\Phi)(D^i\Phi)^+ + \lambda V(\langle\Phi\Phi^+\rangle) \rangle, \quad (2.21)$$

where

$$G^{0i} = \mathcal{E}^i, \quad (2.22)$$

$$G^{ij} = \epsilon^{ijk} \mathcal{B}^k, \quad (2.23)$$

$$\lambda V > 0. \quad (2.24)$$

A field configuration for which $T^{00} = 0$ everywhere will be called a vacuum configuration. Then by (2.21) a vacuum configuration implies

$$G^{\mu\lambda} = 0, \quad (2.25)$$

$$D^\mu\Phi = 0, \quad (2.26)$$

$$V = 0. \quad (2.27)$$

A region of space-time will be described as a Higgs vacuum if (2.26) and (2.27) but not necessarily (2.25) are satisfied.⁸ The condition of finite energy for the entire soliton field will enforce the Higgs vacuum asymptotically at large distances.

One may ask for classical solitons that solve Eqs. (2.17)–(2.19) subject to the boundary conditions (2.26) and (2.27). Solutions are known where $G^{\mu\lambda}$ and Φ are both isotriplets lying in $SO(3)$. For definiteness we limit our work to the loop generalization of $SO(3)$. Therefore we shall investigate the generalization of the $SO(3)$ solitons to the corresponding loop algebra.

III. THE HIGGS VACUUM AND THE ELECTROMAGNETIC FIELD⁸

In the Higgs vacuum the $SO(3)$ symmetry is broken down to $SO(2)$ or equivalently to $U(1)$. Then the photon, remaining massless, is separated from the massive vectors which acquire mass $ae\hbar$, where $a = |\Phi|$ in the Higgs vacuum and e is the gauge coupling constant. The Higgs particle acquires the mass $a(2\lambda)^{1/2}\hbar$. In the same region of space-time where the Higgs field points in a fixed direction Φ , the electromagnetic potential may be identified with $\Phi W^\mu/a$ and the electric charge operator with $e\Phi T^a/a$, where T^a ($a = 1, 2, 3$) are the $SO(3)$ generators and $\Phi T/a$ is the equivalent $U(1)$ generator. In this region the electromagnetic field is well defined. However, the expression for it is not unique since the electromagnetic field is not well defined in the interior of the soliton, although all such expressions must agree in the Higgs vacuum.

The electromagnetic field tensor proposed by 't Hooft is¹

$$F_{\mu\lambda} = \hat{\phi}^a (G_{\mu\lambda}^a - e^{-1} \epsilon^{abc} D_\mu \hat{\phi}_b D_\lambda \hat{\phi}_c), \quad (3.1)$$

where

$$\hat{\phi}^a = \phi^a / |\Phi|. \quad (3.2)$$

For $SO(3)$ this tensor may be rewritten as

$$F_{\mu\lambda} = M_{\mu\lambda} + H_{\mu\lambda}, \quad (3.3)$$

where

$$M_{\mu\lambda} = \partial_\mu A_\lambda - \partial_\lambda A_\mu, \quad (3.4)$$

$$A_\lambda = \hat{\phi}^a W_\lambda^a, \quad (3.5)$$

and

$$H_{\mu\lambda} = e^{-1} \epsilon^{abc} \hat{\phi}^a \partial_\mu \hat{\phi}^b \partial_\lambda \hat{\phi}^c. \quad (3.6)$$

The magnetic current is

$$k_\mu = \partial^\lambda \tilde{F}_{\mu\lambda}, \quad (3.7)$$

where $\tilde{F}_{\mu\lambda}$ is the dual field. Then

$$k_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial^\nu M^{\rho\sigma} - (1/2e) \epsilon_{\mu\nu\rho\sigma} \epsilon^{abc} \partial_\nu \hat{\phi}^a \partial_\rho \hat{\phi}^b \partial_\sigma \hat{\phi}^c. \quad (3.8)$$

The magnetic current may arise from the first term if A_λ is singular on Dirac strings or entirely from the second term if A_λ has no string singularities. These equivalent descriptions of magnetic charge are connected by a singular gauge transformation. If there are no string singularities, the magnetic charge is

$$g = \frac{1}{4\pi} \int k_0 d^3x \quad (3.9)$$

$$= \frac{1}{8\pi e} \int \epsilon_{ijk} \epsilon_{abc} \partial_i \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c d^3x \quad (3.10)$$

or

$$eg = \lim_{R \rightarrow \infty} \frac{1}{8\pi} \int_{S_R^{(2)}} \epsilon_{ijk} \epsilon_{abc} \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c (d^2\sigma)_i. \quad (3.11)$$

The integral in the preceding section is a topological invariant that can assume only integral values. The total magnetic charge is therefore

$$g = n/e, \quad (3.12)$$

where n is the number of nodes in the Higgs function.⁹

Equation (3.1) for the electromagnetic field is gauge invariant and reduces by (2.26) to the required form in the Higgs vacuum. As already remarked, however, there is no unique definition of $F_{\mu\lambda}$ outside the Higgs vacuum. Another proposal that agrees with (3.1) in the Higgs vacuum is¹⁰

$$F_{\mu\lambda} = \Phi \cdot G_{\mu\lambda} / a \quad (3.13a)$$

or

$$F_{\mu\lambda} = \text{Tr}(\Phi G_{\mu\lambda} / a). \quad (3.13b)$$

We shall generalize (3.13b) for the loop algebra to

$$F_{\mu\lambda} = \langle \Phi^+ G_{\mu\lambda} \rangle / a, \quad (3.14)$$

where

$$a^2 = \lim_{r \rightarrow \infty} \langle \Phi^+ \Phi \rangle \quad (3.14a)$$

and $\langle \rangle$ is the invariant scalar product for the loop algebra. Here $F_{\mu\lambda}$ is the generalized electromagnetic field, but only $n = 0$ is the Maxwell field. The corresponding generalized magnetic current is

$$k^\mu = \partial_\lambda \langle \Phi^+ \tilde{G}^{\mu\lambda} \rangle / a \quad (3.15a)$$

or

$$= \langle D_\lambda \Phi^+ \tilde{G}^{\mu\lambda} \rangle / a, \quad (3.15b)$$

by the Bianchi identities (2.19).

The conservation of k^μ follows from its definition (3.15a) as the divergence of an antisymmetric tensor.

The generalized electric and magnetic charges following from (3.14) are

$$Q = \lim_{r \rightarrow \infty} \frac{r^2}{a} \int d\Omega \langle \Phi^+ \mathcal{E}_r \rangle \quad (3.16a)$$

$$= \frac{1}{a} \int d^3x \langle D_i \Phi^+ \mathcal{E}_i \rangle, \quad (3.16b)$$

$$M = \lim_{r \rightarrow \infty} \frac{r^2}{a} \int d\Omega \langle \Phi^+ \mathcal{B}_r \rangle \quad (3.17a)$$

$$= \frac{1}{a} \int d^3x \langle D_i \Phi^+ \mathcal{B}_i \rangle, \quad (3.17b)$$

where $\mathcal{E}$ and $\mathcal{B}$ are given by (2.22) and (2.23), respectively.

The actual electric and magnetic charges are, of course, given by the $n = 0$ components.

IV. DYONS^{3,11}

We consider those special solutions obeying the conditions

$$(W^0, \Phi) = 0, \quad (4.1)$$

$$\partial_0 W^\mu = \partial_0 \Phi = 0. \quad (4.2)$$

Then

$$D_0 W^0 = D_0 \Phi = 0. \quad (4.3)$$

With these assumptions it is shown in the Appendix that Eqs. (2.17)–(2.19) are equivalent to the set

$$\mathcal{E}_i = \sin \alpha D_i \Phi, \quad (4.4)$$

$$\mathcal{B}_i = \cos \alpha D_i \Phi, \quad (4.5)$$

in the limit $\lambda \rightarrow 0^+$. This reduction holds for any algebra including loop algebras.

The mass of the dyon is

$$m = \int T_{00} d^3x. \quad (4.6)$$

By (2.21) and (4.3),

$$m = \frac{1}{2} \int \langle \mathcal{E}_i \mathcal{E}_i^+ + \mathcal{B}_i \mathcal{B}_i^+ + D_i \Phi D_i \Phi^+ \rangle d^3x. \quad (4.7)$$

By (4.4) and (4.5),

$$m = \int \langle D_i \Phi D_i \Phi^+ \rangle d^3x \quad (4.8a)$$

$$= \int \langle \mathcal{E}_i \mathcal{E}_i^+ + \mathcal{B}_i \mathcal{B}_i^+ \rangle d^3x. \quad (4.8b)$$

From (3.16), (3.17) and (4.4), (4.5), one finds

$$Q \sin \alpha = \frac{1}{a} \int \langle \mathcal{E}_i \mathcal{E}_i^+ \rangle d^3x, \quad (4.9)$$

$$M \cos \alpha = \frac{1}{a} \int \langle \mathcal{B}_i \mathcal{B}_i^+ \rangle d^3x, \quad (4.10)$$

and

$$m = a(Q \sin \alpha + M \cos \alpha). \quad (4.11)$$

Also

$$aQ = \sin \alpha \int (D_i \Phi)(D_i \Phi^+) d^3x, \quad (4.12)$$

$$aM = \cos \alpha \int (D_i \Phi)(D_i \Phi^+) d^3x, \quad (4.13)$$

$$Q/M = \tan \alpha. \quad (4.14)$$

By (4.11) and (4.14),

$$m = a(Q^2 + M^2)^{1/2}. \quad (4.15)$$

This relation between mass and generalized electric and magnetic charge holds for an exact solution of the affine field equations, and in relation to the finite algebra it is known as the Bogomolny bound.^{8,12}

The solution found by Prasad and Sommerfield, a special solution of the general loop equations, is the following^{8,11}:

$$eW_a^i = [K(r) - 1] \epsilon_{aij} \hat{r}^j / r, \quad (4.16)$$

$$e\phi_a = (H(r)/r) \hat{r}^a, \quad (4.17)$$

$$W_a^0 = (J(r)/r) \hat{r}^a, \quad (4.18)$$

where

$$K(r) = \lambda r / \sinh(\lambda r), \quad (4.19)$$

$$H(r) \cos \alpha = \lambda r \coth(\lambda r) - 1, \quad (4.20)$$

$$J(r) = H(r) \sin \alpha, \quad (4.21)$$

and

$$\lambda = ae \cos \alpha. \quad (4.22)$$

Then

$$\lim_{r \rightarrow \infty} \phi(r) = a\hat{r}. \quad (4.23)$$

The differential equations for the functions J , H , and K are scale invariant. The scale of the soliton is fixed by the value of a , the vacuum expectation value of the Higgs field.

V. GENERALIZATION TO LOOP ALGEBRA

As already noted, Eqs. (4.4) and (4.5) hold for the loop algebra. We again assume (4.1) and (4.2) as well. Then

$$\mathcal{E}_k = G_{0k} = -D_k W_0. \quad (5.1)$$

By (4.4)

$$D_k W_0 = -\sin \alpha D_k \Phi, \quad (5.2)$$

so that we may take W_0 proportional to the Higgs potential as shown in (4.21). Then it is only necessary to solve (4.5).

We make the simplest generalization of the Yang–Wu ansatz¹³:

$$eW_k^{an} = g_n(r) \epsilon_{kl}^a X^l, \quad (5.3)$$

$$eW_0^{an} = J_n(r) X^a, \quad (5.4)$$

$$e\Phi^{an} = h_n(r) X^a. \quad (5.5)$$

Then

$$\mathcal{B}_k^{an} = \delta_k^a \mathcal{B}_1^n + X^a X_k \mathcal{B}_2^n, \quad (5.6)$$

where

$$e\mathcal{B}_1^n = -\left(r \frac{d}{dr} + 2\right) g_n(r), \quad (5.6a)$$

$$e\mathcal{B}_2^n = \frac{1}{r} \frac{dg_n}{dr} + \sum_p g_{n-p} g_p, \quad (5.6b)$$

while

$$(\nabla_k, \Phi^{an}) = \delta_k^a \Pi^n + X^a X_k \Pi^n, \quad (5.7)$$

$$e\Pi^n = h_n - r^2 \sum_p g_{n-p} h_p, \quad (5.7a)$$

$$e\Pi^n = \frac{1}{r} \frac{dh_n}{dr} + \sum_p g_{n-p} h_p. \quad (5.7b)$$

Then (4.5) becomes

$$rg_n' + 2g_n = \cos \alpha \left[r^2 \sum_p g_{n-p} h_p - h_n \right], \quad (5.8)$$

$$\frac{1}{r} g_n' + \sum_p g_{n-p} g_p = \cos \alpha \left[\frac{1}{r} h_n' + \sum_p g_{n-p} h_p \right]. \quad (5.9)$$

In order to describe a soliton solution, these equations must be satisfied subject to the boundary conditions that all the components of Φ and W_μ be regular at $r=0$ and that they approach the Higgs vacuum at $r=\infty$. We shall require finiteness and a flat tangent at the origin as well as

$$\lim_{r \rightarrow \infty} D_\mu \Phi = 0, \quad (5.10)$$

$$\lim_{r \rightarrow \infty} \Phi_m = a_m \hat{r}. \quad (5.11)$$

Then

$$\lim_{r \rightarrow \infty} \langle \Phi^\dagger \Phi \rangle = \sum a_m^2 = a^2. \quad (5.12)$$

VI. SPECIAL SOLUTIONS

Let us denote the n th soliton by the set $(\mathbf{g}, \mathbf{h})$, where $g_m = h_m = 0$ if $|m| > n$ and where

$$\mathbf{g} = (g_{-n} \cdots g_0 \cdots g_n), \quad (6.1)$$

$$\mathbf{h} = (h_{-n} \cdots h_0 \cdots h_n). \quad (6.2)$$

By (2.7) and (2.8), $g_m = g_{-m}$ and $h_m = h_{-m}$. According to (5.8) and (5.9) the components (g_0, h_0) satisfy

$$rg_0' = -2g_0 - \bar{h}_0 + r^2 \left(g_0 \bar{h}_0 + 2 \sum_1^n g_m \bar{h}_m \right), \quad (6.3)$$

$$r\bar{h}_0' = -2g_0 - \bar{h}_0 + r^2 \left(g_0^2 + 2 \sum_1^n g_m^2 \right). \quad (6.4)$$

The remaining components satisfy

$$rg_m' = -2g_m - \bar{h}_m + r^2 \sum_0^m g_{m-p} \bar{h}_p, \quad (6.5)$$

$$r\bar{h}_m' = -2g_m - \bar{h}_m + r^2 \sum_0^m g_{m-p} g_p. \quad (6.6)$$

Here

$$\bar{h}_m = \cos \alpha h_m. \quad (6.7)$$

The complete set of components $(g_0 \cdots g_n; h_0 \cdots h_n)$ is code-termined by the nonlinear equations (6.3)–(6.6). Notice that (h_m, g_m) cannot be renormalized independently. They can be rescaled together, however, if lengths are also rescaled.

The Prasad–Sommerfield solution is recognized in this class of solutions as

$$\bar{h}_m = g_m = 0, \quad m \neq 0, \quad (6.8)$$

$$\bar{h}_0(r) = (1/r^2) [1 - \lambda_0 \coth(\lambda_0 r)], \quad (6.9)$$

$$g_0(r) = (1/r^2) [1 - \lambda_0 r / \sinh(\lambda_0 r)], \quad (6.10)$$

where

$$\lambda_0 = ae \cos \alpha. \quad (6.11)$$

This solution will be described as a “singlet.”

Since all components are coupled nonlinearly, there is no obvious method to investigate the general solution of these equations. One may, however, obtain special solutions in the following way. With r corresponding to the time, the general solution will trace out an orbit in $(\mathbf{h}, \mathbf{g})$ space. We may find special solutions in this space by confining the motion to a subspace of the full space. In particular, we may ask for solutions lying in the following subspace:

$$\sum_1^n g_m h_m = Kg_0 h_0, \quad (6.12)$$

$$\sum_1^n g_m^2 = Kg_0^2. \quad (6.13)$$

Then (6.3) and (6.4) become

$$rg_0' = -2g_0 - \bar{h}_0 + (1 + 2K)r^2 g_0 \bar{h}_0, \quad (6.14)$$

$$r\bar{h}_0' = -2g_0 - \bar{h}_0 + (1 + 2K)r^2 g_0^2. \quad (6.15)$$

Now rescale the length by

$$x^2 = (1 + 2K)r^2. \quad (6.16)$$

Then

$$x \frac{dg_0}{dx} = -2g_0 - \bar{h}_0 + x^2 g_0 \bar{h}_0, \quad (6.17)$$

$$x \frac{d\bar{h}_0}{dx} = -2g_0 - \bar{h}_0 + x^2 g_0^2. \quad (6.18)$$

These rescaled equations have the explicit solutions for (g_0, h_0) given by (6.9) and (6.10) in which r is replaced by $(1 + 2K)^{1/2} r$. The rescaling, of course, depends on the other components $(g_1 \cdots g_n)$.

Since they are not integrals of the complete set of equations, (6.12) and (6.13) are nonlinear constraints on the set (6.5) and (6.6). These constraints may be satisfied for the special solutions

$$\begin{aligned} \mathbf{g}_n &= (g_{-n} \circ \cdots \circ g_0 \circ \cdots \circ g_n), \\ g_m &= 0, \quad |m| = 1, \dots, n-1, \end{aligned} \quad (6.19)$$

$$\begin{aligned} \mathbf{h}_n &= (h_{-n} \circ \cdots \circ g_0 \circ \cdots \circ h_n), \\ h_m &= 0, \quad |m| = 1, \dots, n-1. \end{aligned}$$

These special solutions will be termed “triplets.” Then (6.12) and (6.13) imply

$$g_n = kg_0, \quad (6.20)$$

$$\bar{h}_n = k\bar{h}_0,$$

where

$$k = \pm \sqrt{K} \quad (6.21)$$

and

$$\sum_0^n g_{n-p} \bar{h}_p = 2kg_0 \bar{h}_0, \quad (6.22)$$

$$\sum_0^n g_{n-p} g_p = 2kg_0^2. \quad (6.23)$$

Equations (6.5) and (6.6) for the components (g_n, h_n) become

$$rg_0' = -2g_0 - \bar{h}_0 + 2r^2 g_0 \bar{h}_0, \quad (6.24)$$

$$r\bar{h}_0' = -2g_0 - \bar{h}_0 + 2r^2 g_0^2. \quad (6.25)$$

These are identical with (6.14) and (6.15) for $(g_0, \bar{h}_0)$ with $1 + 2K = 2$,

$$k = \pm \frac{1}{2}\sqrt{2}. \quad (6.26)$$

Therefore $(g_0, \bar{h}_0)$ and $(g_n, \bar{h}_n)$ are given by the explicit solutions (6.9) and (6.10) in which r is replaced by $\sqrt{2}r$.

By (6.20) and (6.26), however, g_n and h_n are normalized differently from g_0 and h_0 :

$$g_n(x) = \frac{1}{2}\sqrt{2}g_0(x), \quad (6.27)$$

$$h_n(x) = \frac{1}{2}\sqrt{2}h_0(x). \quad (6.28)$$

We have now found an infinite set of exact solutions. Let us denote their h components by $h_m^{(n)}$. The simplest of these is the singlet for which

$$h_m^{(0)} = (r^2 \cos \alpha)^{-1} (1 - \lambda_0 r \coth(\lambda_0 r)) \delta_{m0}. \quad (6.29)$$

This is the previously obtained solution given by (6.8)–(6.11).

The new set contains the triplets for which

$$h_m^{(n)} = \delta_{m0} h_0^{(0)}(x) + \frac{1}{2}\sqrt{2} \delta_{n|m|} h_0^{(0)}(x), \quad n > 0, \quad (6.30)$$

where

$$x = \sqrt{2}r. \quad (6.31)$$

Let us label the boundary values by $a_m^{(n)}$. Then, by (5.11),

$$\lim_{r \rightarrow \infty} r |h_m^{(n)}(r)| = e a_m^{(n)}, \quad (6.32)$$

and, by (6.30), one finds

$$a_0^{(n)} = \frac{1}{2}\sqrt{2} a_0^{(0)}, \quad (6.33)$$

$$a_n^{(n)} = a_{-n}^{(n)} = \frac{1}{2} a_0^{(0)}. \quad (6.34)$$

We are also interested in the sums

$$(a^{(n)})^2 \equiv \lim_{r \rightarrow \infty} \langle \Phi^{(n)} (\Phi^{(n)})^+ \rangle \quad (6.35a)$$

$$= \sum_{-n}^n (a_m^{(n)})^2. \quad (6.35b)$$

For the triplets

$$(a^{(n)})^2 = (a_0^{(n)})^2 + 2(a_n^{(n)})^2. \quad (6.36)$$

Then by (6.33) and (6.34),

$$a^{(n)} = \pm a^{(0)}. \quad (6.37)$$

VII. ELECTRIC AND MAGNETIC CHARGES AND MASS

The formally generalized electric charge may be calculated by (3.17):

$$a^{(n)} Q^{(n)} = \lim_{r \rightarrow \infty} r^2 \int d\Omega \langle \Phi^+ \mathcal{E}_r \rangle \quad (7.1)$$

$$= 4\pi \lim_{r \rightarrow \infty} r^2 \langle \Phi^+ \mathcal{E}_r \rangle. \quad (7.2)$$

The scalar product is

$$\begin{aligned} \langle \Phi^+ \mathcal{E}_r \rangle &= \sin \alpha \langle \Phi^+ D_r \Phi \rangle \\ &= \sin \alpha \langle \Phi^+ \partial_r \Phi + \Phi^+ (W_r, \Phi) \rangle. \end{aligned}$$

The commutator term depends on

$$\langle \Phi^+ (W_r, \Phi) \rangle = - \langle W_r (\Phi, \Phi) \rangle = 0.$$

Then

$$\begin{aligned} \langle \Phi^+ \mathcal{E}_r \rangle &= \sin \alpha \langle \Phi^+ \partial_r \Phi \rangle \\ &= \frac{1}{2} \sin \alpha \sum_{an} \frac{d}{dr} (\Phi_n^a)^2. \end{aligned} \quad (7.3)$$

The total charge of the n th solution is

$$Q^{(n)} = \sum_{-n}^n Q_m^{(n)}, \quad (7.4)$$

where

$$a^{(n)} Q_m^{(n)} = \frac{2\pi}{e^2} \sin \alpha \lim_{r \rightarrow \infty} r^2 \frac{d}{dr} r^2 [h_m^{(n)}(r)]^2. \quad (7.5)$$

Then for the singlet

$$a^{(0)} Q^{(0)} = \frac{2\pi}{e^2} \sin \alpha \lim_{r \rightarrow \infty} r^2 \frac{d}{dr} r^2 [h_0^{(0)}(r)]^2, \quad (7.6)$$

where $h_0^{(0)}(r)$ is given by (6.29) and

$$\lambda_0 = a^{(0)} e \cos \alpha. \quad (7.7)$$

Taking the limit one finds the known result

$$Q^{(0)} = (4\pi/e) \tan \alpha. \quad (7.8)$$

For the triplets, one has

$$Q^{(n)} = Q_0^{(n)} + 2Q_n^{(n)}, \quad (7.9)$$

where

$$a^{(n)} Q_0^{(n)} = \frac{2\pi}{e^2} \sin \alpha \lim_{r \rightarrow \infty} r^2 \frac{d}{dr} r^2 (h_0^{(n)}(r))^2. \quad (7.10)$$

By (6.30),

$$\begin{aligned} a^n Q_0^{(n)} &= \frac{2\pi}{e^2} \sin \alpha \left(\frac{1}{\sqrt{2}} \right)^3 \lim_{x \rightarrow \infty} x^2 \frac{d}{dx} x^2 (h_0^{(0)}(x))^2 \\ &= (1/\sqrt{2})^3 a^{(0)} Q^{(0)} \end{aligned} \quad (7.11)$$

or

$$Q_0^{(n)} = \frac{1}{4} \sqrt{2} Q^{(0)}, \quad (7.12)$$

by (6.30) and (6.37). The additional contributions to the total triplet charge are $Q_{+}^{(n)}$ and $Q_{-}^{(n)}$. For these we have

$$\begin{aligned} a^{(n)} Q_{\pm}^{(n)} &= \frac{2\pi}{e^2} \sin \alpha \lim_{r \rightarrow \infty} r^2 \frac{d}{dr} r^2 (h_n^{(n)}(r))^2 \\ &= \left(\frac{1}{\sqrt{2}} \right)^3 \frac{1}{2} \frac{2\pi}{e^2} \sin \alpha \\ &\quad \times \lim_{x \rightarrow \infty} x^2 \frac{d}{dx} x^2 (h_0^{(0)}(x))^2 \\ &= (\sqrt{2}/8) a^{(0)} Q^{(0)}, \end{aligned} \quad (7.13)$$

$$\therefore Q_{\pm}^{(n)} = (\sqrt{2}/8) Q^{(0)}. \quad (7.14)$$

Then the total triplet charge is

$$Q^{(n)} = (\sqrt{2}/4 + 2\sqrt{2}/8) Q^{(0)} = \frac{1}{2} \sqrt{2} Q^{(0)} \quad (7.15a)$$

or

$$Q^{(n)} = (\sqrt{2}/2) ((4\pi/e) \tan \alpha). \quad (7.15b)$$

The corresponding magnetic charge is

$$M^{(n)} = (\sqrt{2}/2) (4\pi/e), \quad (7.16)$$

by (4.14).

The true electric and magnetic charges come from the Maxwell field only and for the singlet have the usual values:

$$q^{(0)} = Q_0^{(0)} = (4\pi/e) \tan \alpha, \quad (7.17)$$

$$g^{(0)} = 4\pi/e. \quad (7.18)$$

For the triplets these charges are reduced:

$$q^{(n)} = Q_0^{(n)} = (\sqrt{2}/4) q^{(0)}, \quad (7.19)$$

$$g^{(n)} = (\sqrt{2}/4) g^{(0)}. \quad (7.20)$$

The mass is determined by the total charge according to (4.14) and (4.15):

$$m = aQ/\sin \alpha. \quad (7.21)$$

Then

$$m^{(0)} = a^{(0)} Q^{(0)}/\sin \alpha = a^{(0)} q/\sin \alpha, \quad (7.22)$$

$$m^{(n)} = a^{(n)} Q^{(n)}/\sin \alpha.$$

By (7.15a),

$$m^{(n)} = (\sqrt{2}/2) a^{(0)} Q^{(0)}/\sin \alpha, \quad (7.23)$$

$$\therefore m^{(n)} = (\sqrt{2}/2) m^{(0)}.$$

The triplet dyons have less charge and also less mass than the singlet. All triplets have the same charges and mass.

VIII. GENERAL SOLUTIONS

The special solutions just discussed are not only exact but are also the simplest. They also span an infinite-dimensional function space, but a general solution of the soliton equations (5.8) and (5.9) does not lie in this space since the soliton equations are nonlinear.

We have also studied a slightly different set of equations obtained by dropping the terms in (6.3) and (6.4) coupling (h_0, g_0) to the higher modes (h_m, g_m) , namely, $\Sigma_1^n g_m^2$ and $\Sigma_1^n g_m h_m$. In this truncation the remaining equations are to

be left unchanged. We propose the so-modified set of equations as the basis of a general perturbation method for investigating arbitrary solutions of the exact equations, provided that the higher amplitudes (h_m, g_m) are small compared to the zero set (h_0, g_0) . The modified equations themselves may be solved exactly by the following procedure.

Let the general solution be

$$\begin{aligned} \vec{g} &= (g_{-n} g_{-n+1} \cdots g_0 \cdots g_{n-1} g_n), \\ \vec{h} &= (h_{-n} h_{-n+1} \cdots h_0 \cdots h_{n-1} h_n). \end{aligned} \quad (8.1)$$

where (g_0, h_0) is the exact Prasad-Sommerfield solution since (g_0, h_0) is now decoupled from the other modes. The remaining components may be shown to satisfy

$$g_m'' + P g_m' + Q_g g_m = S_m^{(g)}, \quad (8.2g)$$

$$h_m'' + P h_m' + Q_h h_m = S_m^{(h)}, \quad (8.2h)$$

where

$$P = (2/r)F, \quad (8.3)$$

$$Q_g = [F^2 - 3G^2 - 1]/r^2, \quad (8.4)$$

$$S_m^{(g)} = G \sum_{p=1}^{m-1} g_{m-p} g_p, \quad (8.5)$$

and

$$Q_h = 2[F - G^2 - 1]/r^2, \quad (8.6)$$

$$\begin{aligned} S_m^{(h)} \cos \alpha &= \left[\left(r \frac{d}{dr} + 2F \right) \sum_{p=1}^{m-1} g_{m-p} g_p + 2G \sum_{p=1}^{m-1} g_{m-p} h_p \right]. \end{aligned} \quad (8.7)$$

These expressions all depend on only the two functions F and H , which are linear functionals of h_0 and g_0 :

$$F = 2 - r^2 \cos \alpha h_0, \quad (8.8)$$

$$G = r^2 g_0 - 1. \quad (8.9)$$

From (8.2h) and (8.2g) one sees that (h_m, g_m) satisfy linear equations where the coefficients depend on the pair (h_0, g_0) , which in turn satisfy the nonlinear equations (6.14) and (6.15) with $K = 0$. Since these coefficients depend only on (h_0, g_0) , they are the same for all m . The different components are distinguished by only the source terms $(S_m^{(h)}, S_m^{(g)})$, which do not depend on (h_m, g_m) but do depend on all the lower components (h_p, g_p) , where $1 \leq p \leq m-1$. As a consequence of this structure these coupled equations may be solved by first solving the nonlinear equations for (h_0, g_0) to obtain (6.9) and (6.10), and then proceeding along the sequence $(h_1, g_1) \cdots (h_m, g_m) \cdots (h_n, g_n)$ by solving at each step equations that are only linear in the new variables (h_m, g_m) .

Since there is no source for the (h_1, g_1) equation, the required (h_1, g_1) functions satisfy homogeneous equations subject to the boundary conditions at the origin and infinity. To proceed to higher states we shall introduce the Green's function, which also satisfies the proper boundary conditions. The general solution of these equations is the sum of the solutions of the homogeneous equation and a special solution determined by the source:

$$g_m(r) = \hat{g}_m(r) + \int_0^\infty W(r) G_g(r, r') S_m^{(g)}(r') dr', \quad (8.10g)$$

$$h_m(r) = \hat{h}_m(r) + \int_0^\infty W(r) G_h(r, r') S_m^{(h)}(r') dr'. \quad (8.10h)$$

Since the differential equations are not Sturm–Liouville, it is necessary to introduce a weight function $W(r)$ independent of m :

$$W(r) = (r \sinh r)^2. \quad (8.11)$$

The Green's functions are also independent of m since they depend on only (h_1, g_1) as follows:

$$G_g(r, r') = \begin{cases} -g_{1<}(r') g_{1>}(r), & r > r', \\ -g_{1>}(r') g_{1<}(r), & r < r'; \end{cases} \quad (8.12g)$$

$$G_h(r, r') = \begin{cases} \frac{1}{2} h_{1<}(r') h_{1>}(r), & r > r', \\ \frac{1}{2} h_{1>}(r') h_{1<}(r), & r < r'; \end{cases} \quad (8.12h)$$

where a subscript $<$ refers to solutions of homogeneous equations satisfying boundary conditions at the origin and a subscript $>$ refers to an independent solution satisfying boundary conditions at infinity.

IX. THE LOWEST STATES OF THE MODIFIED EQUATIONS

The first excited state is characterized by (h, g) , where $h = (h_0, h_1)$ and $g = (g_0, g_1)$. The components (h_0, g_0) have already been given in (6.9) and (6.10). The new components satisfy the homogeneous linear equations

$$g_1'' + P g_1' + Q_g g_1 = 0, \quad (9.1g)$$

$$h_1'' + P h_1' + Q_h h_1 = 0, \quad (9.1h)$$

with solutions of the form

$$g_1(r) = \exp\left[-\frac{1}{2} \int^r P(s) ds\right] U_g(r), \quad (9.2g)$$

$$h_1(r) = \exp\left[-\frac{1}{2} \int^r P(s) ds\right] U_h(r), \quad (9.2h)$$

where

$$U_g'' + (Q_g - \frac{1}{2} P' - \frac{1}{4} P^2) U_g = 0, \quad (9.3g)$$

$$U_h'' + (Q_h - \frac{1}{2} P' - \frac{1}{4} P^2) U_h = 0. \quad (9.3h)$$

By (8.3), (8.4), and (8.6), one finds

$$P = (2/r)(1 + \lambda_0 r \coth \lambda_0 r), \quad (9.4)$$

$$Q_g - \frac{1}{2} P' - \frac{1}{4} P^2 = -2(\lambda_0 / \sinh \lambda_0 r)^2, \quad (9.5)$$

$$Q_h - \frac{1}{2} P' - \frac{1}{4} P^2 = -\lambda_0^2 - 2(\lambda_0 / \sinh \lambda_0 r)^2. \quad (9.6)$$

There are two independent solutions of both the h and g equations. These may be chosen to be

$$g_{1<} = \frac{1}{\sinh \lambda_0 r} \left(\coth \lambda_0 r - \frac{1}{\lambda_0 r} \right), \quad (9.7)$$

$$g_{1>} = \frac{1}{\lambda_0 r} \frac{\coth \lambda_0 r}{\sinh \lambda_0 r} \quad (9.8)$$

and

$$h_{1<} = \frac{1}{\lambda_0 r} \left[-\coth \lambda_0 r + \frac{\lambda_0 r}{\sinh^2 \lambda_0 r} \right], \quad (9.9)$$

$$h_{1>} = \frac{1}{\lambda_0 r} \frac{1}{\sinh^2 \lambda_0 r}. \quad (9.10)$$

Here $g_{1<}$ and $h_{1<}$ satisfy the boundary conditions of regularity at the origin; these functions also vanish properly at infinity. On the other hand, $g_{1>}$ and $h_{1>}$ vanish at infinity but blow up at the origin. The solutions $g_{1<}$ and $h_{1<}$ satisfy not only the second-order equations (9.1) but also the first-order equations (6.5) and (6.6), as required. The functions $g_{1<}$ and $h_{1<}$ therefore describe the approximate solitonic solution we seek.

Since both the first- and second-order equations are linear and homogeneous in g_1 and h_1 , these equations do not fix the normalization of these functions. However, since the first-order equations (6.5) and (6.6) relate the normalization of g_1 and h_1 , only one normalization condition needs to be supplied. We take this to be the boundary condition at infinity for $h_{1<}$. At the two limits we have

$$\lim_{r \rightarrow 0} g_{1<}(\lambda_0 r) = \frac{1}{3} A_1, \quad (9.11)$$

$$\lim_{r \rightarrow 0} h_{1<}(\lambda_0 r) = \frac{2}{3} A_1,$$

where A_1 normalizes $g_{1<}$ and $h_{1<}$ and

$$\lim_{r \rightarrow \infty} g_{1<}(\lambda_0 r) = A_1 e^{-\lambda_0 r}, \quad (9.12)$$

$$\lim_{r \rightarrow \infty} |h_{1<}(\lambda_0 r)| = A_1 / \lambda_0 r.$$

But we require, by (5.11),

$$\lim_{r \rightarrow \infty} r |h_n(r)| = e a_n \quad (9.13)$$

or

$$A_1 = \lambda_0 \lambda_1, \quad (9.14)$$

$$\lambda_1 = a_1 e \cos \alpha.$$

The approximate soliton $(h_0, g_0; h_1, g_1)$ is then entirely determined by the parameters appearing in the action and the constants $\{a_m\}$, the vacuum expectation values of the extended Higgs field.

The (h_2, g_2) solutions of the modified equations are

$$h_2 = \frac{1}{\cos \alpha} \frac{1}{y} \left[(A_2 + \lambda_1^2) \frac{y}{\sinh^2 y} - \lambda_1^2 \frac{y^2 \coth y}{\sinh^2 y} - A_2 \coth y \right], \quad (9.15h)$$

$$g_2 = (A_2 + \lambda_1^2) \frac{\coth y}{\sinh y} - A_2 \frac{1}{y \sinh y} - \frac{\lambda_1^2}{2} \left[\frac{y}{\sinh^3 y} + \frac{y \coth^2 y}{\sinh y} \right], \quad (9.15g)$$

where $y = \lambda_0 r$ and A_2 is the constant that multiplies the solution of the homogeneous equation. The value of A_2 is fixed by the limit of h_2 at large distances. By (9.13), $A_2 = \lambda_0 \lambda_2$, with $\lambda_2 = a_2 e \cos \alpha$.

There is no reason for this series of solutions to termi-

nate. The recursive procedure for generating new solutions is described in Sec. VIII. The explicit form of the Green's function appearing there is obtained by substituting the appropriate h and g solutions from (9.7)–(9.10) into (8.12). The general solution is the sum of solutions of the homogeneous equation and the special solution depending on the source and the Green's function.

At each stage of this recursive procedure one must determine the constant associated with the solution to the homogeneous equation. In the next step we encounter the constant $A_3 = \lambda_0 \lambda_3$ with $\lambda_3 = a_3 e \cos \alpha$.

Although we do not have a general formula for the solutions at any level n , only the contribution from the homogeneous solution is important in the calculation of (9.13).

For $n = 3$, we have

$$h_3 = \frac{1}{\cos \alpha} \left[\frac{1}{y} \left[(A_3 + 2\lambda_2 \lambda_1) \frac{y}{\sinh^2 y} - A_3 \coth y \right] - \left(\frac{\lambda_1^3}{\lambda_0} + 2\lambda_1 \lambda_2 \right) \frac{y \coth y}{\sinh^2 y} + \frac{\lambda_1^3}{\lambda_0} \left[\frac{y^2 \coth^2 y}{\sinh^2 y} - \frac{1}{3} \frac{y^2}{\sinh^2 y} \right] \right], \quad (9.16h)$$

$$g_3 = (A_3 + 2\lambda_2 \lambda_1) \frac{\coth y}{\sinh y} - A_3 \frac{1}{y \sinh y} - \left(\lambda_2 \lambda_1 + \frac{1}{2} \frac{\lambda_1^3}{\lambda_0} \right) \left[\frac{2y}{\sinh^3 y} + \frac{y}{\sinh y} \right] + \frac{\lambda_1^3}{\lambda_0} \left(\frac{y^2 \coth y}{\sinh^3 y} + \frac{1}{6} \frac{y^2 \coth y}{\sinh y} \right). \quad (9.16g)$$

For $n = 1$ there is a test of this perturbative procedure since we have both the exact solution and the approximate solution as determined from the modified equations. For small r the approximate solution (9.9) becomes

$$h_1^{(1)} \sim (1/(\lambda_0 r)^2) [(\lambda_0 r) \coth(\lambda_0 r) - 1]. \quad (9.17)$$

This expression is of the same form as the exact solution (6.30). At large distances, on the other hand, the exact and approximate solutions do not have the same form; but at large distances one has a linear approximation to the unmodified equations.

In general, if the higher n components are small compared to (h_0, g_0) , one would expect the preceding procedure based on the modified equations to be useful.

X. REMARKS

In order to investigate the dependence of the soliton sector on the affine index, we have studied a very special model that can be generalized in different ways. In particular, if the total field contained an additional component transforming according to some irreducible and higher-dimensional representation of $SU(2)$, this additional component carrying electric charge would interact with the magnetic charge so as to convert internal degrees of freedom into spin degrees of freedom with the result that the total angular momentum would no longer vanish; the total angular momentum would then be either integral or half integral, depending on the transformation properties of the additional field.^{14,15} One

would also expect that this additional structure might lift the infinite mass degeneracy of the triplet solitons. Then the members of the infinite family of dyons would differ not only in mass but also in angular momentum and statistics,¹⁶ and in a way that can be calculated once the model is made definite.

The refinements just described have not yet been investigated. We have shown elsewhere, however, how the theory may be made globally¹⁷ and also locally¹⁸ supersymmetric.

The preceding comments refer to the soliton sector of the theory, which one may expect to be excited only at very high energies. At currently accessible energies, on the other hand, the vacuum or "meson" sector provides the appropriate formalism. Here the particle states have spin 1 and 0. In this sector, where one has the standard procedures, we have previously studied the Higgs splitting and found a linear mass spectrum for the vectors if the Higgs has a component lying in the L_0 direction of the associated Virasoro algebra. In the present paper our model is different and our object has been to concentrate on the affine solitons rather than to provide a unified consistent model for both sectors. One may, however, plausibly assume that the soliton and vacuum sectors exhibit different representations of the Kac-Moody algebra and, in particular, that the central operator k and the Virasoro operator L_0 vanish for the soliton sector. A more realistic model would also address this question.

APPENDIX: DERIVATION OF EQS. (4.4) AND (4.5)

Equations (4.4) and (4.5) may be obtained from the field equations in the following way. If we distinguish space from time coordinates, the set (2.17)–(2.19) becomes

$$(\nabla_k, G^{0k}) = 0, \quad (A1)$$

$$(\nabla_k, \tilde{G}^{0k}) = 0, \quad (A2)$$

$$(\nabla_k, (\nabla^k, \Phi)) = 0, \quad (A3)$$

$$(\nabla_\lambda, \tilde{G}^{k\lambda}) = 0, \quad (A4)$$

$$(\nabla_\lambda, G^{k\lambda}) = (\Phi^+, (\nabla^k, \Phi)). \quad (A5)$$

Let us try

$$G_{k0} = a(\nabla_k, \Phi), \quad (A6)$$

$$\tilde{G}_{k0} = b(\nabla_k, \Phi). \quad (A7)$$

By (A6) and (A7), Eqs. (A1)–(A3) become equivalent. Then it is enough to check (A2). But (A2) and (A4) together comprise the Bianchi identity. Since both may be checked in the same way, we write out (A4) only.

One equation of the set (A4) is

$$(\nabla_0, \tilde{G}^{10}) + (\nabla_2, \tilde{G}^{12}) + (\nabla_3, \tilde{G}^{13}) = 0$$

or

$$(\nabla_0, \tilde{G}_{01}) + (\nabla_2, G_{03}) + (\nabla_3, G_{20}) = 0.$$

By (A6) and (A7),

$$-b(\nabla_0, (\nabla_1, \Phi)) - a(\nabla_2, (\nabla_3, \Phi)) + a(\nabla_3, (\nabla_2, \Phi)) = 0.$$

By the Jacobi identity,

$$-b(\nabla_0, (\nabla_1, \Phi)) + a(\Phi, (\nabla_2, \nabla_3)) = 0.$$

By the Jacobi identity and (4.3),

$$b(\Phi, (\nabla_0, \nabla_1)) + a(\Phi, (\nabla_2, \nabla_3)) = 0.$$

By (2.11),

$$b(\Phi, G_{01}) + a(\Phi, G_{23}) = 0. \quad (\text{A8})$$

Finally by (A6) and (A7) one sees that (A8) is satisfied identically, as it should.

By a similar reduction of (A5) one finds

$$a^2 + b^2 = 1. \quad (\text{A9})$$

The preceding argument holds any for any algebra including the loop algebra.

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