SPACELIKE FOLIATIONS OF ROBERTSON-WALKER
SPACETIME BY FERMI SPACE SLICES

A thesis submitted in partial fulfillment of the requirements
For the degree of Master of Science
in Mathematics

By

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Dedication

I dedicate this to my father, Guy Randles and my grandmother, Faye Randles.
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Explicit Fermi coordinates are given for geodesic observers comoving with the Hubble flow in expanding Robertson-Walker spacetimes, along with exact expressions for the metric tensors in these coordinates. For the case of non inflationary cosmologies, we show that Fermi coordinate charts are global, and spacetime is foliated by space slices of constant Fermi (proper) time that have finite extent. A universal upper bound for the proper radius of any leaf of the foliation is given. A general expression is derived for the geometrically defined Fermi relative velocity of a test particle (e.g. a galaxy cluster) comoving with the Hubble flow away from the observer. Least upper bounds of superluminal recessional Fermi velocities are given for spacetimes whose scale factors follow power laws, including matter-dominated and radiation-dominated cosmologies. Exact expressions for the proper radius of any leaf of the foliation for this same class of spacetimes are given. It is shown that the radii increase linearly with proper time of the observer moving with the Hubble flow. These results are applied to particular cosmological models.
Conventions

Throughout this document we will use Einstein’s summation notation: If an index appears twice in an equation, once as a superscript and once as a subscript, we will assume that the equation is summed over this index. For example,

\[ x^i y_i \equiv \sum_{i=1}^{n} x^i y_i . \]
Chapter 1

Introduction

The aim of this thesis is to explore the structure of Robertson-Walker spacetime by considering a particular foliation of the manifold. The leaves of this foliation are spacelike submanifolds called Fermi space slices. For a timelike observer in the spacetime, we determine the Fermi space slices by computing spacelike geodesics whose tangent vectors are orthogonal to the observer’s timelike path. From this construction we are then able to compute the Fermi coordinates for the observer, as well as determine a new notion of simultaneity in the spacetime. Most importantly, this foliation of Robertson-Walker spacetime will provide us with a way to study the expansion of space. Much of the work is based on a joint publication with Professor David Klein, Fermi Coordinates, Simultaneity, and Expanding Space in Robertson-Walker Cosmologies [21].
Chapter 2
Mathematical Preliminaries

The mathematical description of the gravity and universe, general relativity, is formulated in the language of differential geometry. This chapter is reserved for the mathematics needed for our work in relativity and Robertson-Walker cosmology. The reader should already be familiar with multivariable calculus and point-set topology.

2.1 Smooth Manifolds

Smooth manifolds are essentially topological spaces that locally “look like” $\mathbb{R}^n$. Let us make this notion precise.

**Definition 1.** Let $\mathcal{M}$ be a Hausdorff space. A coordinate system, (or chart) $\varphi$ is a homeomorphism from an open set $U \subseteq \mathcal{M}$ to an open set $\varphi(U) \subseteq \mathbb{R}^n$ for some $n$. For $p \in U$, $\varphi(p) = (x^1(p), x^2(p), \ldots, x^n(p))$. Each $x^i$ is called a coordinate function of $\varphi$. In the case that $U = \mathcal{M}$ we say that the coordinate system is global.

**Definition 2.** Let $\varphi : U \to \mathbb{R}^n$ and $\phi : V \to \mathbb{R}^n$ be coordinate systems. We say that $\varphi$ and $\psi$ overlap smoothly provided,

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

and

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \subseteq \mathbb{R}^n \to \mathbb{R}^n$$

are smooth functions, i.e. functions with continuous partial derivatives of all orders. We take this condition to hold vacuously if $U \cap V = \emptyset$.

Eqs. (2.1) and (2.2) will often be referred to as transition functions. If two coordinate systems overlap smoothly, it follows that the transition functions must be diffeomorphisms from open subsets of $\mathbb{R}^n$ into $\mathbb{R}^n$.

In order to study the entire space $\mathcal{M}$ we want every point $p$ on $\mathcal{M}$ to be in the domain of some coordinate system. This necessitates the following definition.

**Definition 3.** Let $\mathcal{M}$ be a Hausdorff space. An atlas $A$ on the space $\mathcal{M}$ is a collection of coordinate systems such that

1. every point of $\mathcal{M}$ is contained in the domain of a coordinate system in $A$ and,

2. any two coordinate systems in $A$ overlap smoothly.

We say the atlas $A$ is complete if it is maximal in the sense that any given coordinate system that overlaps smoothly with the members of $A$ must be itself a member of $A$. 

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We may now define the smooth manifold. We will assume that the topological spaces in question are connected. From this it follows that each coordinate system on $\mathcal{M}$ maps into $\mathbb{R}^n$ for some unique $n$; thus allowing us to assign a dimension to $\mathcal{M}$ (see pg. 5 of Ref. [13]).

**Definition 4.** A smooth manifold $\mathcal{M}$ is a Hausdorff space together with a complete atlas. We define the dimension of the manifold $\mathcal{M}$ to be the dimension of its coordinate systems.

Given a smooth manifold, we will find it useful to study certain subsets of this space in that they acquire their smooth structure from the manifold. This is the notion of a submanifold. Although we will be studying a certain type of submanifold in this thesis, a general definition will not be needed. The interested reader can look at Refs. [29],[13] and [23]. For our work in Robertson-Walker spacetime we will look at submanifolds called coordinate slices. It follows from the theory that each coordinate slice of a smooth manifold is a smooth manifold in its own right.

2.2 Tangent Vectors, Tangent Spaces and Vector Fields

**Definition 5.** Let $\mathcal{M}$ be an $n$ dimensional smooth manifold and $f : \mathcal{M} \to \mathbb{R}$ be a function. We say that $f$ is a smooth function on $\mathcal{M}$ if $f \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth for any coordinate
We will denote the collection of smooth functions on $\mathcal{M}$ by $C^\infty(\mathcal{M})$. A noteworthy example of a smooth function can be seen as follows. Let $\varphi = (x^1, x^2, ..., x^n)$ be a coordinate system on $\mathcal{M}$. Notice that the projection map, $x^k = x^k \circ \varphi^{-1}(x^1, x^2, ..., x^n)$ is a smooth function. We will now define tangent vectors on the manifold $\mathcal{M}$.

**Definition 6.** Let $\mathcal{M}$ be a smooth manifold and let $p \in \mathcal{M}$. A tangent vector to $\mathcal{M}$ at $p$ is a map $v : C^\infty(\mathcal{M}) \to \mathbb{R}$ that satisfies,

1. $v(\alpha f + \beta g) = \alpha v(f) + \beta v(g)$ and,
2. $v(fg) = v(f)g(p) + f(p)v(g)$

for $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(\mathcal{M})$.

Let $u$ and $v$ be tangent vectors to $\mathcal{M}$ at $p$ and let $\alpha$ be a real number. Under the usual definition of addition and scalar multiplication,

$$(v + u)(f) \equiv v(f) + u(f) \quad (2.3)$$

and

$$(\alpha v)(f) \equiv \alpha v(f), \quad (2.4)$$

it is clear that the set of tangent vectors to $\mathcal{M}$ at $p$ is a vector space. We call this vector space the tangent space to $\mathcal{M}$ at $p$ and denote it by $T_p \mathcal{M}$. In $T_p \mathcal{M}$, the zero vector is the unique vector that takes all smooth functions to 0. Notice that addition between tangent vectors at different points on $\mathcal{M}$ is not defined. Such an operation does not make sense in that the vectors “live” in different spaces. This fact will later generate discussion as to why relative velocity between distant observers in the universe has no a priori definition.

At this point it is natural to ask, “Can we find a basis for $T_p \mathcal{M}$?” This question is answered after defining some notation. Given a coordinate system on $\mathcal{M}$, $\psi = (x^1, x^2, ..., x^n)$ and a smooth function $f$ we will write

$$\frac{\partial f}{\partial x^i} \bigg|_p \equiv \frac{\partial (f \circ \psi^{-1})}{\partial x^i}(\psi(p)). \quad (2.5)$$

Under this convention we notice that the map $\frac{\partial}{\partial x^i} \bigg|_p : C^\infty(\mathcal{M}) \to \mathbb{R}$ satisfies (1) and (2) of Definition 6 and is therefore a tangent vector to $\mathcal{M}$ at $p$ for every $i = 1, ..., n$. We call this vector a *coordinate vector*. The following theorem whose proof can be found in ref. [29] pg. 8 shows that the collection of coordinate vectors forms a basis for $T_p \mathcal{M}$.
Proposition 1. Let $\psi = (x^1, x^2, \ldots, x^n)$ be a coordinate system on $M$ at $p$. The collection of its coordinate vectors,
\[
\left\{ \frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \ldots, \frac{\partial}{\partial x^n}|_p \right\}
\]
is a basis for $T_p M$ and for any $v \in T_p M$
\[
v = v(x^\alpha) \frac{\partial}{\partial x^\alpha}|_p.
\]

The preceding proposition easily yields a change-of-basis formula for coordinate vectors. To see this, take a coordinate vector $v = \frac{\partial}{\partial y^i}|_p$ in one coordinate system and apply it to the coordinate function $x^k$ of another. The resultant expression will be found useful when changing coordinate systems. The reader should note that the formula is simply a restatement of the chain rule in $\mathbb{R}^n$. We state this result in the following corollary:

Corollary 1. Let $\psi = (x^1, x^2, \ldots, x^n)$ and $\varphi = (y^1, y^2, \ldots, y^n)$ be coordinate systems on $M$ with domains $U$ and $V$ respectively. Let $p \in U \cap V$, the coordinate vector $\frac{\partial}{\partial y^i}|_p$ of the coordinate system $\varphi$ expressed as a linear combination of the coordinate basis
\[
\left\{ \frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \ldots, \frac{\partial}{\partial x^n}|_p \right\}
\]
of the coordinate system $\psi$ is,
\[
\frac{\partial}{\partial y^i} = \frac{\partial x^1}{\partial y^i}|_p \frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial y^i}|_p \frac{\partial}{\partial x^2} + \cdots + \frac{\partial x^n}{\partial y^i}|_p \frac{\partial}{\partial x^n}.
\]

To extend the notion of tangent vectors and tangent spaces to the entire manifold we define the tangent bundle to $M$ as $T \mathcal{M} = \{T_p \mathcal{M}|p \in \mathcal{M}\}$. We now introduce the notion of a vector field. Vector fields on manifolds are generalizations of those seen in the theory of electricity and magnetism.

Definition 7. Let $\mathcal{M}$ be a smooth manifold. A vector field $X$ is a function that assigns to every point $p$ of $\mathcal{M}$, a tangent vector $X_p \in T_p \mathcal{M}$. We say that the vector field $X$ is smooth if $(X f)(p) \equiv X_p(f)$ is a smooth function on $\mathcal{M}$. We will denote the set of smooth vector fields by $\mathfrak{X}(\mathcal{M})$.

In light of this definition, we see that the coordinate vectors $\frac{\partial}{\partial x^i}|_p$ become smooth vector fields if we do not fix the point $p$. We appropriately call these coordinate vector fields and denote them by $\frac{\partial}{\partial x^i}$. Under this observation, Proposition 1 guarantees that the coordinate fields generate a basis for each tangent space $T_p \mathcal{M}$ where $p$ is in the domain of the coordinate system. In the case that we are working in a particular coordinate system and can avoid ambiguity, we will often write $\partial_i$ in place of $\frac{\partial}{\partial x^i}$.

The final topic that needs to be addressed in this section is that of the Lie bracket. This operation on two smooth vector fields will be used in the construction of the Levi-Civita connection. It is also a crucial tool when studying curvature.

Definition 8. Let $X, Y$ be two smooth vector fields on a manifold $\mathcal{M}$. The Lie Bracket of
the vector fields $X$ and $Y$ is the map $[X, Y] : C^\infty(M) \to C^\infty(M)$ defined by,

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

(2.9)

for any smooth function $f$.

The reader should note that $[X, X] = 0$, that is the map that takes all functions to zero. Also for any coordinate vector fields, $\partial_i$ and $\partial_j$, we have that

$$\partial_i(\partial_j(f)) - \partial_j(\partial_i(f)) = \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} \equiv 0$$

(2.10)

by the requirement that $f$ is smooth. Thus $[\partial_i, \partial_j](f) = 0$ for all smooth functions $f$.

### 2.3 The Dual Space, $T_pM^*$

For any vector space $W$, the dual space $W^*$ is the collection of all linear transformations $T : W \to \mathbb{R}$. It follows from linearity that $W^*$ is a vector space. We now consider the dual space of $T_pM$.

**Definition 9.** Let $M$ be a smooth manifold. The dual space to $T_pM$ is the set of all functions $T : T_pM \to \mathbb{R}$ such that

$$T(\alpha v + \beta v) = \alpha T(v) + \beta T(u)$$

(2.11)

for $u, v \in T_pM$ and $\alpha, \beta \in \mathbb{R}$. We denote the dual space by $T_pM^*$ and call the functions $T \in T_pM^*$, cotangent vectors.

It turns out that finding a basis for this dual space is easy once we extend the notion of the dual space to the entire manifold as we did by extending tangent vectors to vector fields.

**Definition 10.** A one-form $\omega$ on a smooth manifold $M$ is a function that takes a point $p$ in the manifold and assigns a cotangent vector in $T_pM^*$.

Take a smooth function $f$ and define the map $df : \mathcal{X}(M) \to \mathbb{R}$ by

$$df(v) \equiv v(f)$$

(2.12)

for any $v \in T\mathcal{M}$. This map is called the differential of the function $f$ and it follows from the definition of its action on $v$ that it is a one-form on $M$. Take a coordinate system $\psi = (x^1, x^2, \ldots x^n)$ on $M$. As we previously noted, the coordinate function $x^i$ is smooth and therefore its differential, $dx^i$ is a one-form. We call $dx^i$ a coordinate one-form. Notice that

$$dx^i(\partial_j) = \frac{\partial x^i}{\partial x^j} = \delta^i_j$$

(2.13)

where $\delta^i_j$ is the Kronecker-delta function.
The following proposition shows the collection of coordinate one-forms is a basis for the dual space. Its proof can be found in Ref. [29].

**Proposition 2.** Let \( \psi = (x^1, x^2, ..., x^n) \) be a coordinate system on \( \mathcal{M} \). The collection of its coordinate one-forms

\[
\{dx^1, dx^2, ..., dx^n\}
\]

is a basis for \( T_p\mathcal{M}^* \) for every \( p \) in the domain of \( \psi \). Also, for every smooth function \( f \),

\[
\frac{df}{dx^j} = \frac{\partial f}{\partial x^j}.
\]

2.4 Smooth Paths

Smooth paths on a manifold are generalizations of smooth paths in \( \mathbb{R}^n \). As smooth paths in euclidean space can describe the trajectory of an object governed by the laws of Newtonian mechanics, a smooth path on a spacetime will be used to describe the trajectory of an observer governed by the laws of general relativity. A major section of this thesis will concern a special type of smooth path called a geodesic.

**Definition 11.** Let \( \mathcal{M} \) be a smooth manifold. A smooth path (or path) \( \gamma \) is a function \( \gamma : I \subseteq \mathbb{R} \rightarrow \mathcal{M} \) such that for any coordinate system \( \psi = (x^1, x^2, ..., x^n) \), \( \psi^{-1} \circ \gamma : I \rightarrow \mathbb{R}^n \) is a smooth path.

Let us alert the reader of some rather common and abusive notation for paths. When a coordinate system \( \psi \) on \( \mathcal{M} \) is specified it is rather common to write \( \psi(\gamma(\rho)) = (x^1(\rho), x^2(\rho), ..., x^n(\rho)) \) when one really means \( \psi^{-1} \circ \gamma(\rho) \).

**Definition 12.** Let \( \gamma \) be a path on a smooth manifold \( \mathcal{M} \) and suppose that \( \psi \) is a coordinate system whose domain contains the range of \( \gamma \), the velocity vector to \( \gamma \) is the vector field,

\[
\gamma' \equiv \frac{d(\psi^{-1} \circ \gamma)^j}{d\rho} \frac{\partial}{\partial x^j}
\]

where \( (\psi^{-1} \circ \gamma)^j \) is the \( j \)th component of the path \( (\psi^{-1} \circ \gamma) \) in \( \mathbb{R}^n \). The velocity vector is sometimes also referred to as the tangent vector to the path \( \gamma \). It is also sometimes written as \( \frac{d}{d\rho} \).

Using the notation discussed prior to this definition, we will write Eq. (2.16) as

\[
\gamma' = \frac{dx^j(\rho)}{d\rho} \frac{\partial}{\partial x^j}.
\]

2.5 The Metric Tensor

The dot product in \( \mathbb{R}^n \) is a very useful operation in that it allows us to measure angles and compute distances. In a sense, it determines the geometric structure of euclidean space. We want to generalize this notion to a smooth manifold; this is the metric.
Definition 13. Let $V$ be a vector space and let $V^*$ be its dual. An $(l,k)$ tensor is a map
\[ T : V \times V \times \cdots \times V \times V^* \times V^* \times \cdots V^* \to \mathbb{R} \] (2.18)
that is linear in each slot. For example, let $\alpha, \beta \in \mathbb{R}$ and $v, u \in V$ (or $V^*$) then
\[ T(\cdot, \cdot, \cdots, \alpha v + \beta u, \cdots, \cdot) = \alpha T(\cdot, \cdot, \cdots, v, \cdots, \cdot) + \beta T(\cdot, \cdot, \cdots, u, \cdots, \cdot). \] (2.19)

Definition 14. A symmetric bilinear form $b$ on a vector space $V$ is a $(0,2)$ tensor that is symmetric, i.e.
\[ b(v, u) = b(u, v) \] (2.20)
for all $u, v \in V$. We say that the bilinear form is non-degenerate if $b(v, v) \neq 0$ for all non-zero vectors $v$. We say that $b$ is positive definite if $b(v, v) > 0$ or negative definite if $b(v, v) < 0$ for all non-zero vectors $v$.

We need one last definition before we may define the metric on a smooth manifold.

Definition 15. Let $V$ be a finite dimensional vector space and $b$ a symmetric bilinear form. The index of $b$ is the dimension of largest subspace $W \subseteq V$ such that $b$ is negative definite when restricted to vectors in $W$.

Definition 16. Let $\mathcal{M}$ be a smooth manifold with tangent bundle $T\mathcal{M}$. A metric $g$ of index $n$ on $\mathcal{M}$ is a non-degenerate symmetric bilinear form on $T_p\mathcal{M}$ for every $p$ such that

1. $g(X,Y)$ is a smooth function for any smooth vector fields $X$ and $Y$,
2. for every $p \in \mathcal{M}$, the index of $g$ restricted to $T_p\mathcal{M}$ is a constant $n$.

For the sake of clarity, let us rephrase the above definition. Let $\mathcal{M}$ be a smooth manifold. A metric $g$ is a map that assigns (smoothly) a non-degenerate symmetric bilinear form $g|_p : T_p\mathcal{M} \to \mathbb{R}$ of constant index $n$ to every point $p$ of $\mathcal{M}$. Let us discuss some properties of the metric $g$.

As linear transformations on a vector space are completely determined by their actions on a basis, it follows from linearity that the same notion holds true for metrics on manifolds. This is summarized in the proposition below.

Proposition 3. Let $\mathcal{M}$ be a smooth manifold with metric $g$. If $\psi = (x^1, x^2, \ldots x^n)$ is a coordinate system on $\mathcal{M}$, then on the domain of the coordinate system
\[ g(\cdot, \cdot) = g_{ij}dx^i(\cdot)dx^j(\cdot), \] (2.21)
where $g_{ij} \equiv g(\partial_i, \partial_j)$. Using the notation of the tensor product (which we have not defined) this is commonly written by the line element
\[ ds^2 \equiv g_{ij}dx^i \otimes dx^j \equiv g_{ij}dx^i dx^j. \] (2.22)
We see that this proposition gives us the following: for any two smooth vector fields $V = V^\alpha \partial_\alpha$ and $W = W^\beta \partial_\beta$,

$$\begin{align*}
g(V, W) &= g_{ij} dx^i (V^\alpha \partial_\alpha) dx^j (W^\beta \partial_\beta) \quad (2.23) \\
&= g_{ij} V^\alpha W^\beta dx^i (\partial_\alpha) dx^j (\partial_\beta) \quad (2.24) \\
&= g_{ij} V^\alpha W^\beta \delta^i_\alpha \delta^j_\beta \quad (2.25) \\
&= g_{\alpha \beta} V^\alpha W^\beta. \quad (2.26)
\end{align*}$$

Let us note one last thing about the metric tensor. At $p \in \mathcal{M}$ the collection $\{g_{ij}|i, j = 1, ..., n\} = \{g_{ij}\}$ forms an $n \times n$ matrix. It follows by symmetry and non-degeneracy of the tensor $g$ that the corresponding matrix $\{g_{ij}\}$ is symmetric and invertible. We denote its inverse by $\{g^{ij}\}$ and have

$$g^{ij} g_{jk} = \delta^i_k. \quad (2.27)$$

### 2.6 Lorentzian Manifolds and Spacetimes

We are finally at the point where we can provide a broad classification for manifolds. Riemannian manifolds are smooth manifolds with a metric of index 0, i.e. the metric is positive definite on every tangent space. We will not be primarily concerned with Riemannian manifolds. The manifolds of interest in relativity are called spacetimes, which are generalizations of Lorentzian manifolds.

**Definition 17.** A Lorentzian manifold $\mathcal{M}$ is a smooth manifold of dimension $n \geq 2$ equipped with a metric $g$ of index 1. We will denote a Lorentzian manifold by the pair $(\mathcal{M}, g)$.

The index of 1 in the previous definition can be loosely understood by requiring that the manifold have a dimension which is different in some sense from the rest. This dimension is associated with time. Lorentzian manifolds also provide us with a way to classify vector fields. This classification is the basis for the notion of causality.

**Definition 18.** Let $\mathcal{M}$ be a Lorentzian manifold with metric $g$. We say that the vector field $X$ is

1. timelike provided $g(X, X) < 0$
2. spacelike provided $g(X, X) > 0$
3. lightlike provided $g(X, X) = 0$.

With this classification we define spacetime.

**Definition 19.** A spacetime $(\mathcal{M}, g)$ is a 4-dimensional Lorentzian manifold that contains at least one timelike vector field.
2.7 Levi-Civita Connection

The way in which a given vector field changes in some direction is described by a connection. Loosely speaking, a connection is a directional derivative for vector fields. Let us define this.

**Definition 20.** Let $\mathcal{M}$ be a smooth manifold. A connection $\nabla$ on $\mathcal{M}$ is a map $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \rightarrow \mathfrak{X}(\mathcal{M})$ that satisfies:

\[
\nabla_{fX+gY}W = f\nabla_XW + g\nabla_YW, \tag{2.28}
\]

\[
\nabla_X\alpha Y + \beta W = \alpha \nabla_XY + \beta \nabla_XW \tag{2.29}
\]

and

\[
\nabla_XfY = X(f)Y + f\nabla_XY \tag{2.30}
\]

for real numbers $\alpha, \beta$, smooth functions $f, g$ and smooth vector fields $X, Y, W$. $\nabla_XY$ is called the covariant derivative of $Y$ in the direction $X$.

There are many different connections that one can define on a smooth manifold. On Lorentzian manifolds, we will only be concerned with one particular connection. This connection, the Levi-Civita connection, turns out to be unique on any Lorentzian manifold. That is, there is only one connection to each Lorentzian manifold that satisfies the properties of the definition below. This is proved in Theorem 11 of Ref. [29].

**Definition 21.** The Levi-Civita connection on the Lorentzian manifold $(\mathcal{M}, g)$ is the unique connection that satisfies,

\[
[V, W] = \nabla_VW - \nabla_WV \tag{2.31}
\]

and

\[
X(g(V, W)) = g(\nabla_XV, W) + g(V, \nabla_XW) \tag{2.32}
\]

for smooth vector fields $X, V, W$ on $\mathcal{M}$.

It is a consequence of the proof of uniqueness that these properties completely characterize the connection and thus determine its form in terms of a coordinate system $\psi$. See Koszul’s formula in Refs. [13, 29, 23]. This is summarized as follows: Let $\psi = (x^1, x^2, ..., x^n)$ be a coordinate system on a Lorentzian manifold $\mathcal{M}$, Given vector fields $V = v^\alpha \partial_\alpha$ and $W = w^\beta \partial_\beta$,

\[
\nabla_WV = \left[w^\beta \left(\frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta \gamma}v^\gamma\right)\right] \partial_\alpha \tag{2.33}
\]

where $\Gamma^k_{ij}$ are called the Christoffel symbols and are given by,

\[
\Gamma^k_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right). \tag{2.34}
\]
It is important to note that the Cristoffel symbols above, as derivatives of the metric tensor, measure the manifold’s deviation from flat. Appropriately, the Cristoffel symbols in euclidean space are all zero.

### 2.8 Parallel Translation

In this thesis we will construct a set of coordinates relative to an observer in Robertson-Walker spacetime called the Fermi coordinates. To do this we will need to define a way to determine whether two vectors are “parallel” at different points on a manifold. As discussed earlier, we cannot directly compare vectors at different points on the manifold because the vectors live in different tangent spaces. It is still however possible to compare these vectors. To do this, let’s discuss parallel translation.

Let \( p \) be a point on a Lorentzian manifold \( (\mathcal{M}, g) \) with connection \( \nabla \) and let \( \gamma \) be a smooth path whose image contains \( p \). Let \( v_p \in T_p\mathcal{M} \), parallel translation is the process by which \( v_p \) is moved along the path \( \gamma \) and does not change. This process creates a vector field \( v \) along the path. This situation is depicted in figure 2.2.

![Figure 2.2: Parallel translation of \( v_p \)](image)

**Definition 22.** Let \( (\mathcal{M}, g) \) be a Lorentzian manifold with connection \( \nabla \). Let \( \gamma : I \rightarrow \mathcal{M} \) be a smooth path with velocity vector \( \gamma' \). A vector field \( v \) is said to be parallel along \( \gamma \) if

\[
\nabla_\gamma v \equiv 0
\]

at all points on the path \( \gamma \).

It follows from the theory of ordinary differential equations that given a vector \( v_p \in T_p\mathcal{M} \) and a path \( \gamma \) containing \( p \) there exists a unique vector field \( v \) that is parallel along \( \gamma \) with \( v(p) = v_p \). We say \( v \) is the parallel translation of the vector \( v_p \) along \( \gamma \) [23].

**Proposition 4.** Let \( (\mathcal{M}, g) \) be a Lorentzian manifold with connection \( \nabla \). Let \( \gamma : I \rightarrow \mathcal{M} \) be a smooth path with velocity vector \( \gamma' \). If \( X \) and \( Y \) are two vector fields parallel along \( \gamma \) then \( g(X, Y) \) is constant along \( \gamma \).
Proof. Applying Eq. (2.32) to the vector fields $\gamma', X$ and $Y$ gives,

$$
\gamma'(g(X, Y)) = g(\nabla_{\gamma'}X, Y) + g(X, \nabla_{\gamma'}Y).
$$

By hypothesis, the previous equation yields,

$$
\gamma'(g(X, Y)) = g(0, Y) + g(X, 0) = 0 + 0.
$$

Therefore $g(X, Y)$ is constant along the path.

The previous proposition guarantees the following: if two vectors are orthogonal in a tangent space $T_p\mathcal{M}$ then the parallel translation of each vector will remain orthogonal to the other. This will be used in the construction of Fermi coordinates.

### 2.9 Geodesics

The generalization of straight lines on Lorentzian (more generally Semi-Riemannian) manifolds is of great importance. These lines are geodesics.

**Definition 23.** Let $\mathcal{M}$ be a Lorentzian manifold with metric $g$ and Levi-Civita connection $\nabla$. A geodesic on $\mathcal{M}$ is a path $Y : I \rightarrow \mathcal{M}$ with velocity vector $X$ that satisfies

$$
\nabla_X X \equiv 0.
$$

We will refer to Eq.(2.38) as the geodesic equation. The geodesic $Y$ is called spacelike provided its velocity vector is spacelike, i.e.,

$$
g(X, X) \geq 0
$$
on its domain. Analogously we will call the geodesic $Y$ timelike provided its velocity vector is timelike.

**Remark 1.** Eq.(2.38) is a second order ordinary differential equation. From the theory of ordinary differential equations, given initial conditions, a unique solution is guaranteed at least locally on the manifold. The topic of completeness in Riemannian and Semi-Riemannian geometry involves how far one can extend these geodesics on $\mathcal{M}$. This can be found in any standard text, i.e. [29, 23].

It can be seen as a consequence of the Levi-Civita connection, $\nabla$, that $g(X, X)$ is constant for any geodesic $Y$. It follows that one can parameterize any spacelike geodesic $Y$ by the affine parameter $\rho$ such that $g(X, X) = 1$ and any timelike geodesic $Y$ by the parameter $\tau$ such that $g(X, X) = -1$ [29, 23]. This necessitates the following definition.

**Definition 24.** Let $(\mathcal{M}, g)$ be a Lorentzian manifold with connection $\nabla$. Let $Y$ be a spacelike geodesic with velocity vector $X$. The affine parameter $\rho$ that gives $g(X(\rho), X(\rho)) = 1$ is called the proper distance along the path $Y$. Analogously let $\gamma$ be a spacelike geodesic with velocity vector $\gamma'$. The affine parameter $\tau$ that gives $g(\gamma'(\tau), \gamma'(\tau)) = -1$ is called the proper time of the path $\gamma$. 

13
In practice, geodesics are found by rewriting Eq.(2.38) in terms of a coordinate system. Given a coordinate system on \( U \subseteq \mathcal{M}, \psi = (x^1, x^2, ..., x^n) \), it follows from Eq. (2.33) that a path \( Y = (x^1(\rho), x^2(\rho), ..., x^n(\rho)) \) is a geodesic if and only if it satisfies

\[
\frac{d^2 x^k(\rho)}{d\rho^2} + \Gamma^k_{ij} \frac{dx^i(\rho)}{d\rho} \frac{dx^j(\rho)}{d\rho} = 0
\]

for \( k = 1, ..., n \).

### 2.10 The Exponential Map

The map defined in this section, the *exponential map* (see fig 2.3), is a crucial tool in the construction of coordinate systems in Riemannian and Semi-Riemannian geometry. On Riemannian manifolds these coordinate systems are referred to as *Riemannian Normal Coordinates*. In this thesis we will use the exponential map to construct Fermi coordinates in Robertson-Walker spacetime. The following fact follows from the theory of ordinary differential equations. Its proof may be found in any standard text [13, 29, 23].

**Proposition 5.** Let \((\mathcal{M}, g)\) be a Lorentzian manifold with Levi-Civita connection \( \nabla \). For every \( v \in T_p\mathcal{M} \) there is a unique geodesic \( \gamma_v : I \to \mathcal{M} \) such that \( \gamma(0) = p \) and the velocity vector \( \gamma'_v(0) = v \). This unique geodesic is guaranteed to exist locally around the point \( p \).

**Definition 25.** Let \((\mathcal{M}, g)\) be a Lorentzian manifold with Levi-Civita connection \( \nabla \). Let \( p \in \mathcal{M} \), the function \( \exp_p : T_p\mathcal{M} \to \mathcal{M} \) is called the exponential map at \( p \) and is given by,

\[
\exp_p(v) = \gamma_v(1)
\]

where \( \gamma_v \) is the unique geodesic guaranteed by proposition 5.

![Figure 2.3: The exponential map at \( p \)](image)

It can be shown that on sufficiently small neighborhoods of \( p \) that the above function is invertible (see Ref. [29] pg. 71). In the case that these neighborhoods are star-shaped they are called *normal neighborhoods* of \( p \).

The final proposition of this chapter will be used in the construction of Fermi coordinates.
Proposition 6. Let $v \in T_pM$. Then for all $t \in \mathbb{R}$ for which the exponential map is well-defined

$$exp_p(tv) = \gamma_v(t).$$ (2.42)

Proof. Fix $t \in \mathbb{R}$ and let $\gamma_v(\rho)$ be the path guaranteed by proposition 5. By the chain rule,

$$\frac{d}{d\rho} \gamma_v(t\rho) = t\gamma'_v(\rho) = tv.$$ (2.43)

Therefore by uniqueness $\gamma_{t\rho}(\rho) = \gamma_v(t\rho)$ for all $\rho$. In particular for $\rho = 1$ we have,

$$exp_p(tv) = \gamma_{t\rho}(1) = \gamma_v(t).$$ (2.44)
Chapter 3
General Relativity and Cosmology

The aim of the present chapter is to introduce the reader to the general theory of relativity, Robertson-Walker cosmology and some special topics therein. These special topics will include Fermi space slices, Fermi coordinates and the Fermi-relative velocity.

3.1 General Relativity

The general theory of relativity, famously developed by Albert Einstein in 1915, is the theory of gravitation; it is far-reaching and well-confirmed experimentally. It is beyond the scope of this thesis to formally discuss much of this theory. The reader is encouraged to read any of the treatises on the subject, for example Ref. [27]. Consequently, we will only informally discuss the rudiments of this theory as they pertain to the description of the universe.

We postulate that the universe in which we live is a 4-dimensional Lorentzian manifold, \((\mathcal{M}, g)\) with a causal structure, i.e. a spacetime. The points on each spacetime are called events. The metric on \(\mathcal{M}\) is determined by a collection of partial differential equations called the field equations. Given a distribution of matter in the universe, one can in principle solve the field equations and determine the metric on that spacetime.

Concerning the dynamical laws of general relativity, we postulate that all observers (massive bodies) in the absence of force travel along timelike geodesics. Here gravity is not regarded as a force. We will refer to these observers as geodesic observers, their timelike paths as worldlines and their velocity vectors as 4-velocities.

Together these postulates are probably best summarized in the words of John Archibald Wheeler: “Matter tells space how to curve, and curved space tells matter how to move” ([27], pg 5). Before discussing the principle of equivalence on which the entire theory rests, we must first discuss Einstein’s previous and earth-shattering theory, the special theory of relativity.

Appropriately so, the special theory of relativity is a special case of the general theory. The special theory, introduced by Einstein in 1906, does not take into account the presence of matter in spacetime. Its original postulates are the following:

1. Light travels at a constant speed \(c\) with respect to all inertial frames\(^1\)
2. The laws of physics are the same in all inertial frames [12]

As a consequence of these postulates, the relative speed between observers is limited by the speed of light. This subluminal bound will be discussed in more detail in Section 3.3.

\(^1\)Throughout this thesis we will use units such that \(c = 1\).
It follows from the field equations that the spacetime of special relativity is exactly that of $\mathbb{R}^4$ with spacetime metric,

$$\{\eta_{\mu\nu}\} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

expressed in the standard coordinate system $(t, x^1, x^2, x^3)$.

The principle of equivalence states: *in small enough regions of spacetime, the laws of physics reduce to those of special relativity* [6]. Here we will be concerned with the mathematics contained in this statement. Let us rephrase it in those terms: *for every geodesic observer there is a system of coordinates in which the metric is that of special relativity, $\{\eta_{\mu\nu}\}$ in a neighborhood of the observer’s worldline.* In this thesis we construct the system of coordinates consistent with this principle. These are the Fermi coordinates.

The theory of relativity introduces some concepts that are often found to be nonintuitive. One such concept is that of simultaneity. Given a spacetime $(M, g)$, suppose that $p \in M$ is in the domain of two coordinate systems, $\psi = (t, x^1, x^2, x^3)$ and $\varphi = (\tau, y^1, y^2, y^3)$. Our typical notion of simultaneity comes from fixing the time coordinates of the event $p$. In this spirit consider the coordinate slices,

$$M_{t(p)} = \{q \in M | t(q) = t(p)\}$$

and,

$$M_{\tau(p)} = \{q \in M | \tau(q) = \tau(p)\}.$$ 

These two coordinate slices are generally unequal as seen in Figure 3.1. We conclude that simultaneity is coordinate dependent and not absolute, perhaps against our intuition. The natural question to ask: is there a preferred system of coordinates from which we may define simultaneity? We wish to argue the affirmative.
3.2 Fermi Space Slices

To begin answering the question posed in the last section we define Fermi space slices. Let $\gamma$ be the worldline of a geodesic observer in spacetime $(M, g)$. We will consider all spacelike geodesics orthogonal to the observer’s path. To this end, consider the map $\varphi : M \to \mathbb{R}$ defined by,

$$\varphi(p) = g(exp^{-1}_\gamma \gamma(\tau), \gamma'(\tau)).$$

(3.4)

**Definition 26.** We call the set,

$$M_\tau \equiv \varphi^{-1}(0),$$

(3.5)

the Fermi space slice of $\tau$-simultaneous events relative to geodesic observer $\gamma$ at proper time $\tau$. Any $p \in M_\tau$ is said to be simultaneous to the the event $\gamma(\tau)$ on the geodesic observer’s worldline.

This notion of simultaneity is not new. It was previously discussed in Refs. [3] and [28]. It will soon be shown that $M_\tau$ is a coordinate slice of the Fermi coordinate system relative to the geodesic observer. This observation will, in some sense, justify our definition of simultaneity as the natural one.

The restriction $g_\tau$ of the metric $g$ to each space slice makes the pair $(M_\tau, g_\tau)$ a Riemannian manifold. The collection of Fermi space slices foliates the spacetime in a neighborhood of the observer’s path. This foliation will be denoted by $\{M_\tau\}$. The reader should think of this foliation as a division of spacetime into disjoint 3-dimensional Riemannian manifolds, also called leaves. For an extensive treatment on foliations of smooth manifolds, the reader is encouraged to consult Candel and Conlon’s text *Foliations I.* [4].

In the case of Robertson-Walker spacetimes, the Fermi space slices have often been referred to as the private space of the observer [33, 10]. Let us now define relative distance on this space slice as depicted in Figure 3.2. We note that the following definition is consistent with Ref. [3].

**Definition 27.** Consider a geodesic observer at event $\gamma(\tau)$ with worldline $\gamma$. Let $p \in M_\tau$ be an event simultaneous to $\gamma(\tau)$. The relative distance (or Fermi distance) $\rho_{Fermi}(\gamma(\tau), p)$ of event $p$ with respect to $\gamma(\tau)$ is

$$\rho_{Fermi}(\gamma(\tau), p) \equiv \sqrt{g(exp_{\gamma(\tau)}^{-1} p, exp_{\gamma(\tau)}^{-1} p)}.$$  

(3.6)

We will soon study Fermi space slices relative to specific geodesic observers in Robertson-Walker spacetime. It is important to note that our entire course of study will depend on this foliation of Robertson-Walker spacetime. Every topic herein: simultaneity, relative position, relativity velocity and the Fermi coordinate system are dependent on the definition of the Fermi space slice in one way or another. This notion is therefore a crucial one.
3.3 Relative Velocity

According to the principle of equivalence, the laws of general relativity must reduce to those of special relativity at any spacetime point. Consequently the relative speed between observers is required to be subluminal at that point. To restate this rigorously, suppose that two observers at \( p \in \mathcal{M} \) have 4-velocities, \( u \) and \( u' \in T_p \mathcal{M} \) respectively. The laws of special relativity require that the relative velocity, \( v \) of \( u' \) observed by \( u \) is determined by the equation

\[
v = \frac{-1}{g(u', u)} u' - u. \tag{3.7}
\]

As a consequence we have,

\[
||v|| = \sqrt{g(v, v)} = \sqrt{\eta_{\mu\nu} v^\mu v^\nu} < 1. \tag{3.8}
\]

As discussed in the previous chapter, if the two observers are located at different spacetime points, their 4-velocities live in different tangent spaces and cannot be subtracted as in Eq. (3.7). Accordingly, the theory of general relativity provides no \textit{a priori} definition of relative velocity, and hence no upper bounds on speeds. Distant particles may have superluminal or subluminal speeds, depending on the coordinate system used for the calculations, and on the particular definition of relative velocity.

Mitigating such ambiguities, four geometrically defined, non-equivalent notions of relative velocity were introduced and developed in a series of papers by V. Bolós (c.f. [2, 3] and the references therein). In this thesis we will concentrate on one particular definition, the Fermi relative velocity. It will not be necessary to look at the vector quantity, for this reason we state the definition of the Fermi speed between two observers. This is consistent with Bolós’ definition (See [3]).

**Definition 28.** Let \( \gamma \) and \( \beta \) be the worldlines of two geodesic observers in spacetime \( (\mathcal{M}, g) \) at proper time \( \tau \) of the observer \( \gamma \). Let \( p = \beta(t) \cap \mathcal{M}_\tau \) be the unique event on \( \beta \)'s worldline.
simultaneous to $\gamma(\tau)$. The Fermi speed $v_{\text{Fermi}}$ of the observer $\beta$ relative to $\gamma$ is

$$v_{\text{Fermi}} = \frac{d}{d\tau} [\rho_{\text{Fermi}}(\gamma(\tau), p)]$$

(3.9)

where $\rho_{\text{Fermi}}(\gamma(\tau), p)$ is given by Eq. (3.6).

### 3.4 Fermi Coordinates

In this section we define and discuss the Fermi coordinate system. As we previously remarked, the existence of this coordinate system is intricately related to the equivalence principle. For a geodesic observer this coordinate system provides the most natural frame in which to study physics because it is the frame that the observer experiences [27]. For this reason it is natural to define simultaneity in terms of coordinate slices of this coordinate system.

Fermi coordinates are associated with the foliation $\{\mathcal{M}_r\}$ in a natural way. Each spacetime point on $\mathcal{M}_r$ is assigned time coordinate $\tau$, and the spatial coordinates are defined relative to a parallel transported orthonormal reference frame (see section 2.8). Specifically, a Fermi coordinate system [34, 27, 25, 24] along $\gamma$ is determined by an orthonormal tetrad of vectors, $e_0(\tau), e_1(\tau), e_2(\tau), e_3(\tau)$ parallel along $\gamma$, where $e_0(\tau) = \partial/\partial \tau$ is the 4-velocity of the Fermi observer, i.e., the unit tangent vector of $\gamma(\tau)$. Fermi coordinates $x^0, x^1, x^2, x^3$ relative to this tetrad are defined by,

$$x^0 \left( \exp_{\gamma(\tau)}(\lambda^j e_j(\tau)) \right) = \tau$$

$$x^k \left( \exp_{\gamma(\tau)}(\lambda^j e_j(\tau)) \right) = \lambda^k,$$

(3.10)

where Latin indices run over 1, 2, 3 (and Greek indices run over 0, 1, 2, 3). Here it is assumed that the $\lambda^j$ are sufficiently small so that the exponential maps in Eq.(3.10) are defined. This situation is illustrated in Figure 3.4.

From the theory of differential equations, a solution to the geodesic equations depends smoothly on its initial data so it follows that Fermi coordinates are smooth, and it may be shown in general that there exists a neighborhood $U$ of $\gamma$ on which the map $(x^0, x^1, x^2, x^3) : U \rightarrow \mathbb{R}^4$ is well-defined and is a diffeomorphism onto its image, [29] (p. 200). We refer to $(x^\alpha, U)$ as a Fermi coordinate system for $\gamma$.

A particularly useful feature of Fermi coordinates is that the metric tensor expressed in these coordinates is Minkowskian to first order near the path of the Fermi observer, with second order corrections involving only the curvature tensor [25]. General formulas in the form of Taylor expansions for coordinate transformations to and from more general Fermi-Walker coordinates were given in [17] and exact transformation formulas for a class of spacetimes were given in [7, 19]. Applications of these coordinate systems are voluminous. They include the study of tidal dynamics, gravitational waves, statistical mechanics, and the influence of curved spacetime on quantum mechanical phenomena [7, 16, 26, 11, 18, 22, 1, 31, 32].
In this thesis we will construct the Fermi coordinates for a geodesic observer in a particular class of spacetimes. It follows from the definition of the Fermi coordinates by the exponential map that the proper distance from the geodesic observer to an event \( p \) on a coordinate slice \( \mathcal{M}_\tau \) is exactly that of \( \rho_{\text{Fermi}} \) given in Eq. (3.6). Consequently we will construct both the Fermi space slices and the Fermi coordinate system simultaneously in this thesis. From this construction the Fermi speed between geodesic observers will be found.

### 3.5 Robertson-Walker Cosmology

One of the triumphs of the general theory of relativity is its accurate description of the universe. As Einstein stated: “From my point of view one cannot arrive, by way of theory, at any at least somewhat reliable results in the field of cosmology if one makes no use of the principle of general relativity” [27]. This application of relativity was pioneered by Friedman, Lemaître, Robertson and Walker in the 1920’s. Their solutions to the field equations have laid the foundation for the study of modern cosmology. Particularly so, this study has led to the prediction of the expansion of space and the initial cosmic singularity also known as the \textit{big bang}. In this section we will discuss Robertson-Walker spacetime.

On the largest scales, the distribution of matter in the universe appears to be homogeneous and isotropic. One can construct a spacetime that is consistent with these two properties and satisfies the field equations for different configurations of matter and energy. We will not rigorously construct this spacetime but will only discuss it informally. A well-written and
extensive treatment of this particular solution is given in *Exact Space-Times in Einstein’s General Relativity* [15]. The construction of Robertson-Walker spacetime begins with a collection of 3-dimensional Riemannian manifolds, \{\Sigma_t\} that incorporate the assumptions of spatial homogeneity and isotropy. The mathematical manifestation of this requirement is maximal symmetry and therefore constant spatial curvature. This highly specific requirement only allows for three possibilities: 3-dimensional euclidean space \((k = 0)\), the 3-sphere \((k = 1)\) and 3-dimensional hyperbolic space \((k = -1)\). The metric in these cases can be written in spherical coordinates as,

\[
ds^2 = a^2 \left( d\chi^2 + S_k^2(\chi) \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right)
\]  

(3.11)

where

\[
S_k(\chi) = \begin{cases} 
\sin(\chi) & \text{if } k = 1 \\
\chi & \text{if } k = 0 \\
\sinh(\chi) & \text{if } k = -1 
\end{cases}
\]  

(3.12)

and \(a\) corresponds to the “size” of the manifold. In the case that \(k = 1\), it is the radius of the 3-sphere. We now introduce time.

Let us now allow these Riemannian manifolds to foliate spacetime by introducing the coordinate \(t\) which we will call cosmic time. We will also allow the parameter \(a\) to be a smooth function of time, \(a(t)\); we will call it the scale factor. In these coordinates, \(\{t, \chi, \theta, \varphi\}\), the Robertson-Walker metric is given by the line element,

\[
ds^2 = -dt^2 + a(t)^2 \left( d\chi^2 + S_k^2(\chi) \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right)
\]  

(3.13)

where \(S_k\) is defined above. This situation is depicted in Figure 3.4. Let us now discuss the scale factor \(a(t)\).

Inserting the Eq.(3.13) into the field equations while assuming different cases for the matter content, radiation content and existence of cosmological constant \(\Lambda\) yields different forms of \(a(t)\). We will discuss further aspects of certain scale factors throughout the thesis. We summarize the cases to be studied in Table 3.5.

<table>
<thead>
<tr>
<th>Name</th>
<th>Matter</th>
<th>Radiation</th>
<th>Cosmological Constant</th>
<th>(k)</th>
<th>(a(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milne</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>-1</td>
<td>(t)</td>
</tr>
<tr>
<td>de Sitter</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>0</td>
<td>(e^{H_0t})</td>
</tr>
<tr>
<td>Matter-Dominated</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>0</td>
<td>(t^{2/3})</td>
</tr>
<tr>
<td>Radiation-Dominated</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>0</td>
<td>(t^{1/2})</td>
</tr>
</tbody>
</table>

This cosmological model has provided a very useful description for the universe in which we live. It has also been shown to be consistent with the famous observations made by Edwin Hubble in 1929. These observations found that the universe was not only expanding, but this expansion was accelerating [27]. Let us explore this and introduce Hubble’s Law.

We will define the geodesic observer at the origin of the standard Robertson-Walker coor-
Figure 3.4: The standard foliation of Robertson-Walker spacetime [15]

dinate system as the Fermi Observer. All measurements involving Hubble’s law will be made with respect to this observer. We will say that observers or test particles (such as galaxy clusters) with fixed Robertson Walker coordinates are comoving or moving with the Hubble flow. Take any leaf of the standard foliation $\Sigma_t$ and define the distance $d$ between the Fermi observer and a comoving observer to be the proper distance between them along this Riemannian manifold. This allows us to define Hubble’s Law,

$$\dot{d}(t) \equiv v_H = H d.$$  \hspace{1cm} (3.14)

Here $H$ is the Hubble parameter defined by

$$H \equiv \frac{\dot{a}(t)}{a(t)}$$  \hspace{1cm} (3.15)

and the overdot on $d$ signifies differentiation with respect to $t$. If $H > 0$ and the distance $d$ is sufficiently large, the Hubble speed $v_H$ exceeds the speed of light. It is argued largely on this basis that in a physical sense, the universe is expanding, c.f., for example, [35, 5, 8, 9]. We notice that Hubble’s law lends itself for direct comparison with the Fermi speed defined in Eq. (3.9). We will pursue this similarity throughout our work.
Chapter 4

Spacelike Geodesics in Robertson-Walker Spacetime

4.1 Spacelike Geodesics in Robertson-Walker Spacetime

Given an observer in Robertson-Walker spacetime, one may consider spacelike geodesics orthogonal to the observer’s path as illustrated in Figure 4.1. These geodesics are the defining geodesics of the Fermi space slices. As we remarked earlier, we will later use these to construct the Fermi coordinates. In this section we compute these spacelike geodesics.

![Figure 4.1: Spacelike geodesic $Y(\rho)$ orthogonal to timelike path $\gamma$.](image)

The Robertson-Walker metric on spacetime $\mathcal{M} = \mathcal{M}_k$ is given by the line element,

$$ds^2 = -dt^2 + a(t)^2 \left( d\chi^2 + S_k^2(\chi)d\Omega^2 \right), \tag{4.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$, $a(t)$ is the scale factor, and

$$S_k(\chi) = \begin{cases} 
\sin(\chi) & \text{if } k = 1 \\
\chi & \text{if } k = 0 \\
\sinh(\chi) & \text{if } k = -1. 
\end{cases} \tag{4.2}$$

The coordinate $t > 0$ is synchronous proper time and $\chi, \theta, \varphi$ are dimensionless. The values $1, 0, -1$ of the parameter $k$ distinguish the three possible maximally symmetric space slices for constant values of $t$ with positive, zero, and negative curvatures respectively.

There is a coordinate singularity in Eq.(4.1) at $\chi = 0$, but this will not affect the calculations.
that follow. Consider the submanifold $\mathcal{M}_{\theta_0,\varphi_0} = \mathcal{M}_{\theta_0,\varphi_0,k}$ determined by $\theta = \theta_0$ and $\varphi = \varphi_0$. The restriction of the metric to $\mathcal{M}_{\theta_0,\varphi_0}$ is given by,

$$ds^2 = -dt^2 + a(t)^2 d\chi^2.$$  \hfill (4.3)

On $\mathcal{M}_{\theta_0,\varphi_0}$, the coordinate $\chi$ can be extended to take all real values if $k = 0$ or $-1$, and for the case that $k = 1$, the range of $\chi$ is an interval centered at zero, so there is no coordinate singularity at $\chi = 0$ on the submanifold (see e.g., [15]).

Using Eq.(2.34), the non-zero Cristoffel symbols are

$$\Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{a}(t)}{a(t)}, \quad \Gamma^0_{11} = \dot{a}(t)a(t),$$ \hfill (4.4)

where $\dot{a}(t)$ denotes $da/dt$.

Consider the observer with timelike geodesic path, $\gamma(t) = (t, 0)$ in $\mathcal{M}_{\theta_0,\varphi_0}$. Our immediate aim is to find expressions for all spacelike geodesics orthogonal to $\gamma(t)$. Inserting Eq.(4.4) into the geodesic equation, Eq.(2.40), we have,

$$\frac{d^2x^1}{d\rho^2} + 2\Gamma^1_{01} \frac{dx^1}{d\rho} \frac{dx^2}{d\rho} = \frac{d^2\chi}{d\rho^2} + 2 \frac{\dot{a}(t)}{a(t)} \frac{dt}{d\rho} \frac{d\chi}{d\rho} = 0.$$ \hfill (4.5)

Rewriting the above equation gives

$$a^2(t) \frac{d^2\chi}{d\rho^2} + 2\dot{a}(t)a(t) \frac{dt}{d\rho} \frac{d\chi}{d\rho} = \frac{d}{d\rho} \left( a^2(t) \frac{d\chi}{d\rho} \right) = 0.$$ \hfill (4.6)

It follows that $a(t)^2 d\chi/d\rho$ is a constant $C$ along geodesics parametrized by arc length $\rho$. We will henceforth use the notation $dt/d\rho \equiv \dot{t}$ and $d\chi/d\rho \equiv \dot{\chi}$. Since the tangent vector to the geodesic has unit length,

$$g(X, X) = -1(\dot{t})^2 + a^2(t)(\dot{\chi})^2 = 1,$$ \hfill (4.7)

and therefore

$$(\dot{t})^2 = \frac{C^2}{a(t)^2} - 1.$$ \hfill (4.8)

The requirement that $X = (\dot{t}) \partial/\partial t + (\dot{\chi}) \partial/\partial \chi$ is orthogonal to $\partial/\partial t$, the tangent vector to $\gamma(t)$ at $t = \tau$, forces $C = a(\tau) \equiv a_0$. This yields,

$$\dot{\chi} = \frac{a_0}{a(t)}.$$ \hfill (4.9)
We will assume throughout that $a(t)$ is an increasing function of $t$. To ensure $(\dot{t})^2 \geq 0$ in Eq.(4.8) we must have $a(t) \leq a_0$ and thus $\dot{t} \leq 0$. Together, Eqs.(4.8) and (4.9) give the tangent vector,

$$X = -\sqrt{\left(\frac{a_0}{a(t)}\right)^2 - 1} \frac{\partial}{\partial t} + \frac{a_0}{a^2(t)} \frac{\partial}{\partial \chi}. \quad \text{(4.10)}$$

**Remark 2.** The arc length parameter $\rho$ for the geodesic $Y(\rho) = (t(\rho), \chi(\rho))$ with tangent vector $X$ may be chosen so that $Y(0) = (t(0), \chi(0)) = \gamma(\tau)$. With this convention, which we assume throughout, it follows from symmetry that $\chi(\rho)$ is an odd function of $\rho$.

In light of Remark 2 there is no loss of generality in restricting our attention to those spacetime points with space coordinate $\chi \geq 0$ corresponding to $\rho \geq 0$ for the purpose of finding spacelike geodesics orthogonal to $\gamma(t)$ with initial point on $\gamma(t)$.

**Remark 3.** A general expression for $X$ in Cartesian coordinates was given in [3] and may be deduced from Eq.(4.10) using the transformation of space coordinates, $S_k(\chi) = x/(1 + \frac{1}{4}kx^2)$.

The vector field $X$ can be integrated to give explicit formulas for the geodesic, $Y(\rho)$, for the special cases that $a(t) = \exp(H_0 t)$, where $H_0$ is Hubble constant, i.e., for the de Sitter universe [7], and for the Milne universe (see, e.g., [8]), i.e., for $a(t) = t$. To obtain integral expressions for the general case, we introduce a change of parameter from $\rho$ to $\sigma$. For the sake of clarity of exposition, we assume henceforth that $k = 0$ or $-1$ so that the range of $\chi$ is unrestricted. The minor modifications needed to deal with the case $k = 1$ would include altering the range of $\sigma$ indicated in Eq.(4.11), but the methods are the same. Let,

$$\sigma = \left(\frac{a_0}{a(t)}\right)^2 = a_0 \dot{\chi}, \quad \sigma \in [1, \sigma_\infty(\tau)), \quad \sigma_\infty(\tau) \equiv \begin{cases} (a(t)/a_{\inf})^2 & \text{if } a_{\inf} \equiv \lim_{t \to 0^+} a(t) > 0 \\ \infty & \text{if } \lim_{t \to 0^+} a(t) = 0. \end{cases} \quad \text{(4.12)}$$

Using Eq.(4.10), it follows that,

$$\dot{t} = -\sqrt{\sigma - 1}, \quad \text{(4.13)}$$

and then differentiating Eq.(4.11) gives,

$$\frac{d\sigma}{d\rho} = a_0 \ddot{\chi} = 2 \frac{\dot{a}(t)}{a(t)} \frac{\sigma \sqrt{\sigma - 1}}, \quad \text{(4.14)}$$

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where again $\dot{a}(t)$ denotes $da/dt$. From the chain rule,

$$\frac{d\chi}{d\rho} = \frac{\sigma}{a_0} = \frac{d\chi}{d\sigma} \frac{d\sigma}{d\rho},$$  \hspace{1cm} (4.15)

and combining this with Eq.(4.14) gives,

$$\frac{d\chi}{d\sigma} = \frac{a(t)}{2a_0\dot{a}(t)\sqrt{\sigma - 1}}.$$  \hspace{1cm} (4.16)

We end this section with a theorem and corollary that give explicit integral formulas for spacelike geodesics orthogonal to the timelike path $\gamma(t)$.

**Theorem 1.** Let $a(t)$ be a smooth, increasing function of $t$ with inverse function $b(t)$. Then the spacelike geodesic orthogonal to $\gamma(t)$ at $t = \tau$ and parametrized by the (non-affine) parameter $\sigma$ is given by $Y_\tau(\sigma) = (t(\tau, \sigma), \chi(\tau, \sigma))$ where,

$$t(\tau, \sigma) = b \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)$$  \hspace{1cm} (4.17)

$$\chi(\tau, \sigma) = \frac{1}{2} \int_1^{\sigma} \dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \frac{1}{\sqrt{\sigma} \sqrt{\sigma - 1}} d\tilde{\sigma},$$  \hspace{1cm} (4.18)

and where the overdot on $b$ denotes differentiation. Moreover, for fixed $\tau$, the arc length $\rho$ along $Y_\tau(\sigma)$ is given by,

$$\rho = \rho_\tau(\sigma) = \frac{a(\tau)}{2} \int_1^{\sigma} \dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \frac{1}{\sigma^{3/2} \sqrt{\sigma - 1}} d\tilde{\sigma}.$$  \hspace{1cm} (4.19)

**Proof.** Eq.(4.17) follows immediately from Eq.(4.11). To prove the other two equations, observe that by the inverse function theorem,

$$\dot{b}(a(t)) = \frac{1}{\dot{a}(t)}.$$  \hspace{1cm} (4.20)

From Eq.(4.11), $a(t) = a_0/\sqrt{\sigma}$ and combining this with Eq.(4.20) gives,

$$\frac{a(t)}{\dot{a}(t)} = \frac{a(\tau)}{\sqrt{\sigma}} \dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right).$$  \hspace{1cm} (4.21)

Substituting Eq.(4.21) into Eqs.(4.16) and (4.14) and integrating yields Eqs.(4.18) and (4.19).
From Eq.(4.19) and the assumption that $a(t)$ is increasing it follows that for a fixed value of $\tau$, $\rho$ is a smooth, increasing function of $\sigma$ with a smooth inverse which we denote by,

$$\sigma_\tau(\rho) = \sigma(\rho).$$ \hspace{1cm} (4.22)

Combining Eq.(4.22) with Theorem 1 immediately gives the following corollary.

**Corollary 2.** Let $a(t)$ be a smooth, increasing function of $t$ with inverse function $b(t)$. Then the spacelike geodesic orthogonal to $\gamma(t)$ at $t = \tau$, and parametrized by arc length $\rho$, is given by

$$Y_\tau(\rho) = (t(\tau, \sigma(\rho)), \chi(\tau, \sigma(\rho))).$$ \hspace{1cm} (4.23)
Chapter 5
Fermi Coordinates

The goal of this chapter is to find explicit Fermi coordinates for a timelike geodesic comoving observer in Robertson-Walker cosmologies and to compute the metric in these coordinate systems. We will show that with suitable assumptions on the scale factor \( a(t) \), Fermi coordinates cover the entire spacetime.

We begin constructing this coordinate system in Robertson-Walker spacetime relative to a comoving geodesic observer. To this end, we first construct a coordinate transformation from the standard Robertson-Walker coordinates, \( \{ t, \chi, \theta, \varphi \} \), to a set of coordinates which we will call the Fermi polar coordinates. We will here address how the scale factor \( a(t) \) determines the extent to which this coordinate system is valid. The metric will also be computed in Fermi polar coordinates. Section 3 will conclude this chapter with the a simple transformation that gives us the general Fermi coordinates for a comoving observer and the metric in this form.

5.1 Fermi Polar Coordinates

In this section, we use the following notation,

\[
U = \{ (\tau, \sigma) : \tau > 0 \text{ and } \sigma \in (1, \sigma_\infty(\tau)) \} \tag{5.1}
\]

and

\[
U_1 = \{ (\tau, \sigma) : \tau > 0 \text{ and } \sigma \in [1, \sigma_\infty(\tau)) \}, \tag{5.2}
\]

where \( \sigma_\infty(\tau) \) is given by Eq.(4.12). Observe that \( U \) is an open subset of \( \mathbb{R}^2 \).

**Lemma 1.** In addition to the hypotheses of Theorem 1, assume that \( a(t) \) is unbounded and \( \dot{b}(t) \geq 0 \) for all \( t > 0 \). Then the map \( F : U_1 \to (0, \infty) \times [0, \infty) \) given by,

\[
F(\tau, \sigma) = (t(\tau, \sigma), \chi(\tau, \sigma)) = Y_\tau(\sigma), \tag{5.3}
\]

is a bijection, and \( F : U \to (0, \infty) \times (0, \infty) \) is a diffeomorphism. Here, the functions \( t \) and \( \chi \) are defined by Eqs.(4.17) and (4.18) respectively.

**Proof.** Let \( (t_1, \chi_1) \in (0, \infty) \times [0, \infty) \) be arbitrary but fixed. We show that \( F(\tau_1, \sigma_1) = (t_1, \chi_1) \) for a uniquely determined pair \( (\tau_1, \sigma_1) \in U_1 \). From Eq.(4.17) we must have

\[
\sigma_1 = \left( \frac{a(\tau_1)}{a(t_1)} \right)^2. \tag{5.4}
\]
It remains to find $\tau_1$. To that end, define the function $\sigma(\tau)$ by

$$\sigma(\tau) \equiv \left( \frac{a(\tau)}{a(t_1)} \right)^2. \quad (5.5)$$

From the hypotheses of the lemma, $\sigma(\tau)$ is an unbounded increasing function of $\tau$. Then by Eq.(4.18),

$$\chi(\tau) \equiv \chi(\tau, \sigma(\tau)) = \frac{1}{2} \int_{0}^{\sigma(\tau)} \hat{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \frac{1}{\sqrt{\sigma} \sqrt{\sigma - 1}} d\tilde{\sigma}. \quad (5.6)$$

We want to show that this expression is strictly increasing. Applying Liebnitz’ Rule we have

$$\frac{d\chi(\tau)}{d\tau} = \hat{a}(\tau) \left( \frac{\hat{b}(a(t_1)) a(\tau)}{\sqrt{\sigma(\tau)} \sqrt{\sigma(\tau) - 1}} \right) + \frac{1}{2} \int_{0}^{\sigma(\tau)} \hat{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \frac{1}{\sqrt{\sigma} \sqrt{\sigma - 1}} d\tilde{\sigma}. \quad (5.7)$$

It then follows from the hypotheses that the above expression is positive for all values of $\tau$ and thus $\chi(\tau)$ is strictly increasing. It is also not bounded above. To see this observe,

$$0 < \hat{b}(a(t_1)) = \hat{b} \left( \frac{a(\tau)}{\sqrt{\sigma(\tau)}} \right) \leq \hat{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right). \quad (5.8)$$

Thus by monotonicity of integration,

$$\frac{1}{2} \int_{0}^{\sigma(\tau)} \hat{b}(a(t_1)) \frac{1}{\sqrt{\sigma} \sqrt{\sigma - 1}} d\tilde{\sigma} \leq \chi(\tau) \quad (5.9)$$

which implies

$$\hat{b}(a(t_1)) \ln \left( \sqrt{\sigma(\tau)} + \sqrt{\sigma(\tau) - 1} \right) \leq \chi(\tau). \quad (5.10)$$

Since the left side of the above expression diverges as $\sigma(\tau) \to \infty$, $\chi(\tau)$ is not bounded above. Thus, by continuity and the intermediate value theorem there must exist a unique $\tau_1 \geq t_1$ such that $\chi(\tau_1) = \chi_1$. It now follows from Eq.(5.4) that $F(\tau_1, \sigma_1) = (t_1, \chi_1)$ and $F$ is a bijection. Now consider the restricted map,

$$F : U \to (0, \infty) \times (0, \infty). \quad (5.11)$$

By direct calculation, the Jacobian determinant $J(\tau, \sigma)$ is given by

$$J(\tau, \sigma) = \frac{\hat{a}(\tau)}{2\sigma} \hat{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \left( \frac{\hat{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sqrt{\sigma - 1}} + \frac{a(\tau)}{2\sqrt{\sigma}} \int_{\sigma}^{\infty} \hat{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \frac{1}{\sqrt{\sigma} \sqrt{\sigma - 1}} d\tilde{\sigma} \right). \quad (5.12)$$

Since $a, \hat{a}, \hat{b}$ are positive and $\hat{b} \geq 0$, it follows immediately that $J(\sigma, \tau) > 0$ on its domain, and by the inverse function theorem $F$ is a diffeomorphism. $\Box$
Let $G(\tau, \sigma) = (\tau, \rho(\sigma))$ where $\rho(\sigma)$ is given by (4.19). Then $G$ is a diffeomorphism with inverse, $G^{-1}(\tau, \rho) = (\tau, \sigma(\rho))$ and non vanishing Jacobian. Using the notation of Lemma 1 define,

$$H(t, \chi) = G \circ F^{-1}(t, \chi).$$

(5.13)

Then $H$ is a diffeomorphism from $(0, \infty) \times (0, \infty)$ onto an open subset of $(0, \infty) \times (0, \infty)$ and may be extended to a bijection with domain $(0, \infty) \times [0, \infty)$. We state this result as a corollary:

**Corollary 3.** Let $a(t)$ be a smooth, increasing, unbounded function on $(0, \infty)$ with inverse function $b(t)$ satisfying $\dot{b}(t) \geq 0$ for all $t > 0$. Then the function $(\tau, \rho) = H(t, \chi)$ given by Eq.(5.13) is a diffeomorphism from $(0, \infty) \times (0, \infty)$ onto an open subset of $(0, \infty) \times (0, \infty)$ and $H$ may be extended to a bijection with domain $(0, \infty) \times [0, \infty)$.

**Remark 4.** Using the notation of Corollary 2 we may write $Y_{\tau}(\rho) = H^{-1}(\tau, \rho)$.

**Remark 5.** The condition $\ddot{a}(t) \leq 0$ for all $t$ for a Robertson-Walker spacetime to be non inflationary is equivalent to $\dot{b}(t) \geq 0$ for all $t$.

The previous corollary guarantees that $\{\tau, \rho, \theta, \varphi\}$ are the coordinates of a smooth chart for the Robertson-Walker metric given by Eq.(4.1). We will now find the metric in this coordinate system.

**Theorem 2.** Let $a(t)$ be a smooth, increasing, unbounded function on $(0, \infty)$ with inverse function $b(t)$ satisfying $\dot{b}(t) \geq 0$ for all $t > 0$. In $\{\tau, \rho, \theta, \varphi\}$ coordinates the metric of Eq.(4.1) is given by,

$$ds^2 = g_{\tau\tau} d\tau^2 + d\rho^2 + \frac{a^2(\tau)}{\sigma(\rho)} S^2_k(\chi(\tau, \sigma(\rho))) d\Omega^2,$$

(5.14)

where,

$$g_{\tau\tau} = -\left(\dot{a}(\tau)\right)^2 \left(b \left(\frac{a(\tau)}{\sqrt{\sigma(\rho)}}\right) + a(\tau) \frac{\sqrt{\sigma(\rho)} - 1}{2\sqrt{\sigma(\rho)}} \int_{\sigma(\rho)}^{\sigma(\rho)} \frac{\dot{b}(\frac{a(\tau)}{\sqrt{\sigma}})}{\sqrt{\sigma - 1}} d\tilde{\sigma}\right)^2,$$

(5.15)

and where $\sigma(\rho)$ and $\chi(\tau, \sigma(\rho))$ are given by Eqs. (4.22) and (4.18).

**Proof.** By the chain rule, the derivative of $H^{-1}$ is given by $D_{H^{-1}} = D_{F}D_{G}^{-1}$, i.e.,

$$
\left(\begin{array}{c}
\frac{\partial t}{\partial \tau} \\
\frac{\partial t}{\partial \rho} \\
\frac{\partial \chi}{\partial \tau} \\
\frac{\partial \chi}{\partial \rho}
\end{array}\right) = 
\left(\begin{array}{cc}
\frac{\partial F_1}{\partial \tau} & \frac{\partial F_1}{\partial \sigma} \\
\frac{\partial F_2}{\partial \tau} & \frac{\partial F_2}{\partial \sigma}
\end{array}\right) 
\left(\begin{array}{cc}
\frac{\partial G_1}{\partial \tau} & \frac{\partial G_1}{\partial \sigma} \\
\frac{\partial G_2}{\partial \tau} & \frac{\partial G_2}{\partial \sigma}
\end{array}\right)^{-1}.
$$

(5.16)
The second column of $D_{H^{-1}}$ is given directly by Eqs.(4.13) and (4.15). The entries in the first column are,

$$ \frac{\partial t}{\partial \tau} = \frac{\partial F_1}{\partial \tau} + D_{G^{-1}}^{21} \frac{\partial F_1}{\partial \sigma} $$

(5.17)

and

$$ \frac{\partial \chi}{\partial \tau} = \frac{\partial F_2}{\partial \tau} + D_{G^{-1}}^{21} \frac{\partial F_2}{\partial \sigma}, $$

(5.18)

where $D_{G^{-1}}^{21}$ is the $(2, 1)$ entry of $D_{G^{-1}}$. Inverting the matrix $D_G$ yields,

$$ D_{G^{-1}}^{21} = -\frac{\dot{a}(\tau) \sigma^{3/2}(\rho) \sqrt{\sigma(\rho)}}{a(\tau) \dot{b}(\sigma(\rho))} \int_{\sigma(\rho)}^{1} \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \sqrt{\sigma} + \ddot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) a(\tau)}{\sigma^2 \sqrt{\sigma} - 1} d\sigma. $$

(5.19)

Using Eqs.(5.3), (4.17), and (4.18), the entries of $D_F$ may be calculated directly and are given by,

$$ \frac{\partial F_1}{\partial \tau} = \dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma(\rho)}} \right) \frac{\dot{a}(\tau)}{\sqrt{\sigma(\rho)}}, $$

(5.20)

$$ \frac{\partial F_1}{\partial \sigma} = -\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma(\rho)}} \right) \frac{a(\tau)}{2 \sigma^{3/2}(\rho)}, $$

(5.21)

$$ \frac{\partial F_2}{\partial \tau} = \frac{1}{2} \int_{\sigma(\rho)}^{1} \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \sqrt{\sigma} + \dot{a}(\tau)}{\sigma/\sqrt{\sigma} - 1} d\sigma, $$

(5.22)

and

$$ \frac{\partial F_2}{\partial \sigma} = \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma(\rho)}} \right)}{2 \sqrt{\sigma(\rho)} \sqrt{\sigma(\rho)} - 1}. $$

(5.23)

Using the definition of $\sigma$ given in Eq.(4.11), the first two metric components of Eq.(4.1) may be expressed as functions of $\tau$ and $\rho$ as follows,

$$ g_{tt} = -1 \quad \text{and} \quad g_{xx} = a(t(\tau, \rho)) = \frac{a^2(\tau)}{\sigma(\rho)}. $$

(5.24)

In what follows, let $\{x^0, x^1, x^2, x^3\}$ denote $\{\tau, \rho, \theta, \varphi\}$. Then using Eqs.(4.13), (4.15) and (5.24) the coefficient $g_{\rho\rho}$ of $d\rho^2$ in the metric tensor is given by,

$$ g_{\rho\rho} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \rho} \frac{\partial x^\beta}{\partial \rho} = 1, $$

(5.25)

which may also be deduced by noting that the tangent vector $\partial/\partial \rho$ of the geodesic $Y_\rho(\rho)$ has unit length (see Corollary 2). Similarly, a calculation using Eqs.(5.17),(5.19),(5.20),(5.21) and (5.24) gives,
\[ g_{\tau\tau} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tau} \frac{\partial x^\beta}{\partial \tau} = \dot{a}^2(\tau) \times \]
\[ \left[ a^2(\tau) \left( \int_1^\sigma \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sqrt{\sigma} - 1} d\bar{\sigma} - \sigma \int_1^\sigma \frac{\ddot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma^2 \sqrt{\sigma} - 1} d\bar{\sigma} - \sigma \int_1^\sigma \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma^{3/2} \sqrt{\sigma} - 1} d\bar{\sigma} \right)^2 \right. \]
\[ \left. - \left( 2\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) + \sqrt{\sigma} \sqrt{\sigma} - 1 \int_1^\sigma \frac{\ddot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma^2 \sqrt{\sigma} - 1} d\bar{\sigma} \right) \right] . \tag{5.26} \]

Applying integration by parts to the integral,
\[ \int_1^\sigma \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma^{3/2} \sqrt{\sigma} - 1} d\bar{\sigma} \tag{5.27} \]
results in simplification of Eq.(5.26) and yields Eq.(5.15). For the off-diagonal components, a calculation using Eqs.(5.18), (5.19), (5.22), (5.23) and (5.24) results in,
\[ g_{\tau\rho} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tau} \frac{\partial x^\beta}{\partial \rho} = \dot{a}(\tau) \left( \sqrt{\sigma - 1} \dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) + \frac{a(\tau)}{2} \int_1^\sigma \frac{\ddot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma \sqrt{\sigma} - 1} d\bar{\sigma} \right. \]
\[ \left. - \frac{1}{2} \int_1^\sigma \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma^2 \sqrt{\sigma} - 1} d\bar{\sigma} + \ddot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) a(\rho) \right) . \tag{5.28} \]

Analogous simplification using integration by parts shows that the right hand side of Eq.(5.28) is identically zero. \( \Box \)

The following corollary is used in the proof of Theorem 3.

**Corollary 4.** With the same assumptions as in Theorem 2 and using \( \{\tau, \rho, \theta, \varphi\} \) coordinates, for fixed \( \theta_0, \varphi_0, \) and \( \tau > 0, \) the path \( Y_\tau(\rho) = (\tau, \rho, \theta_0, \varphi_0) \) is a spacelike geodesic with parameter \( \rho > 0. \)

**Proof.** We first express the geodesic \( Y_\tau(\rho) \) on \( M_{\theta_0, \varphi_0} \) given in Corollary 2 in terms of \( \tau, \rho \) coordinates. Since
\[ \frac{\partial}{\partial \rho} = \frac{\partial t}{\partial \rho} \frac{\partial}{\partial t} + \frac{\partial \chi}{\partial \rho} \frac{\partial}{\partial \chi}, \tag{5.29} \]
the tangent vector field along \( Y_\tau(\rho) \) coincides with the tangent vector field along the path \( (\tau, \rho) \) with parameter \( \rho \geq 0 \) and \( \tau \) fixed. Since these paths have the same initial data at \( \gamma(\tau), \) they represent the same geodesic in the two respective coordinate systems.
It then follows that the function $Y_\tau(\rho) = (\tau, \rho, \theta_0, \varphi_0)$ on $\mathcal{M}$ is a spacelike geodesic. That the angular coordinates are constant follows from symmetry or by solving the geodesic equations directly.

5.2 The Fermi Coordinates

We refer to the coordinates, $\{\tau, \rho, \theta, \varphi\}$, of Theorem 2 as Fermi polar coordinates. The following theorem justifies this terminology and also gives us the metric in the rectangular system, the Fermi coordinates.

**Theorem 3.** Let $a(t)$ be a smooth, increasing, unbounded function on $(0, \infty)$ with inverse function $b(t)$ satisfying $\ddot{b}(t) \geq 0$ for all $t > 0$. Define $x^0 = \tau$, $x = \rho \sin \theta \cos \varphi$, $y = \rho \sin \theta \sin \varphi$, $z = \rho \cos \theta$. Then the coordinates $\{\tau, x, y, z\}$ may be extended to a chart that includes the path $\gamma(\tau) = (\tau, 0, 0, 0)$ and $\{\partial/\partial \tau, \partial/\partial x, \partial/\partial y, \partial/\partial z\}$ is a parallel tetrad along $\gamma(\tau)$. With respect to this tetrad, $\tau, x, y, z$ are global Fermi coordinates for the observer $\gamma(\tau)$. Expressed in these Fermi coordinates, the metric of Eq. (5.14) is given by,

\[
\begin{align*}
\text{ds}^2 &= g_{\tau\tau} d\tau^2 + dx^2 + dy^2 + dz^2 \\
&\quad + \lambda_k(\tau, \rho)[(y^2 + z^2)dx^2 + (x^2 + z^2)dy^2 + (x^2 + y^2)dz^2] \\
&\quad - xy(dx dy + dy dx) - xz(dx dz + dz dx) - yz(dy dz + dz dy),
\end{align*}
\]

where $g_{\tau\tau}$ is given by Eq. (5.15), $\rho = \sqrt{x^2 + y^2 + z^2}$, and,

\[
\rho^4 \lambda_k(\tau, \rho) = \frac{a^2(\tau)}{\sigma(\rho)} S_k(\chi(\tau, \sigma(\rho))) - \rho^2.
\]

The smooth function $\lambda_k(\tau, \rho)$ is a function of $\tau$ and $\rho^2$, and the notation in Eq. (5.31) is the same as in Theorem 1.

**Proof.** We begin by showing that the indicated transformation of coordinates applied to Eq. (5.14) results in Eq. (5.30). Using Eq. (4.15), Eq. (5.14) may be rewritten as,

\[
\begin{align*}
\text{ds}^2 &= g_{\tau\tau} d\tau^2 + (d\rho^2 + \rho^2 d\Omega^2) + \left[ a(\tau) \frac{S_k(\chi(\tau, \sigma(\rho)))}{\dot{\chi}} - \rho^2 \right] d\Omega^2, \\
&= g_{\tau\tau} d\tau^2 + dx^2 + dy^2 + dz^2 \\
&\quad + \frac{Q_k(\tau, \rho)}{\rho^4} [(y^2 + z^2)dx^2 + (x^2 + z^2)dy^2 + (x^2 + y^2)dz^2] \\
&\quad - xy(dx dy + dy dx) - xz(dx dz + dz dx) - yz(dy dz + dz dy),
\end{align*}
\]

where for convenience we take $\dot{\chi} \equiv \partial \chi/\partial \rho$. Applying the change of variables in the statement of the theorem results in,

\[
\begin{align*}
\text{ds}^2 &= g_{\tau\tau} d\tau^2 + dx^2 + dy^2 + dz^2 \\
&\quad + \frac{Q_k(\tau, \rho)}{\rho^4} [(y^2 + z^2)dx^2 + (x^2 + z^2)dy^2 + (x^2 + y^2)dz^2] \\
&\quad - xy(dx dy + dy dx) - xz(dx dz + dz dx) - yz(dy dz + dz dy),
\end{align*}
\]

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where,
\[ Q_k(\tau, \rho) = a(\tau) \frac{S_k^2(\chi(\tau, \sigma(\rho)))}{\chi} - \rho^2. \] (5.34)

From Remark 2, it follows that \( \dot{\chi} \) is an even function of \( \rho \), and thus \( Q_k(\tau, \rho) \) has a smooth extension to an even function of \( \rho \). Repeated use of Eqs. (4.13) and (4.14) yields, \( \chi(\tau, \sigma(0)) = 0, \dot{\chi}(\tau, \sigma(0)) = a(\tau)^{-1}, \ddot{\chi}(\tau, \sigma(0)) = 2 \ddot{a}(\tau)/a^3(\tau), \) and \( \dddot{\chi}(\tau, \sigma(0)) = 0, \) where as above the overdots signify differentiation with respect to \( \rho \). Using these results, it follows by direct calculation that \( Q_k(\tau, 0) = 0 \) and each of the first three derivatives of \( Q_k \) with respect to \( \rho \) vanish when evaluated at \( (\tau, 0) \), for \( k = 1, 0, -1 \). Thus, writing \( Q_k(\tau, \rho) \) as a Taylor polynomial in powers of \( \rho^2 \), we have,
\[ Q_k(\tau, \rho) = \rho^4 \lambda_k(\tau, \rho), \] (5.35)
where \( \lambda_k(\tau, \rho) \) is smooth and a function of \( \rho^2 \), establishing Eq. (5.30), which extends by continuity to the path \( \gamma(\tau) = (\tau, 0, 0, 0) \), where the metric is Minkowskian. It now follows by calculation that all first derivatives with respect to \( \tau, x, y, \) or \( z \) of the metric tensor components vanish on \( \gamma(\tau) \), forcing the connection coefficients also to vanish on \( \gamma(\tau) \). Thus, each of the vectors in the tetrad \( \{ \partial/\partial \tau, \partial/\partial x, \partial/\partial y, \partial/\partial z \} \) is parallel along \( \gamma(\tau) \).

Expressing the geodesic \( Y_\tau(\rho) \) in Corollary 4 in terms of \( \{ \tau, x, y, z \} \) gives,
\[ Y_\tau(\rho) = (\tau, a^1, a^2, a^3), \] (5.36)
where \( a^1 = \sin \theta_0 \cos \varphi_0, a^2 = \sin \theta_0 \sin \varphi_0, a^3 = \cos \theta_0 \). The geodesic of Eq. (5.36) may be extended to \( \gamma(t) \) and is orthogonal to \( \gamma(t) \). It now follows from Eq. (3.10) that \( \tau, x, y, z \) are global Fermi coordinates for the observer \( \gamma(\tau) \).

The following definition makes some of the notation in the introduction more precise and will be useful in what follows.

**Definition 29.** We refer to the observer following the path \( \gamma(\tau) = (\tau, 0, 0, 0) \) given in the statement of Theorem 3 as the Fermi observer. A test particle with fixed spatial Robertson-Walker coordinates \( \chi_0, \theta_0, \varphi_0 \) and with world line \( \gamma_0(\tau) = (\tau, \chi_0, \theta_0, \varphi_0) \) is said to be comoving. The Fermi observer is also defined to be comoving.

**Remark 6.** The unit tangent vector field, \( \partial/\partial t \), of the fundamental (comoving) observers of Robertson-Walker cosmologies — i.e., the direction of time in Robertson-Walker coordinates — may be expressed in Fermi coordinates via,
\[ \frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial \rho}{\partial t} \frac{\partial}{\partial \rho}. \] (5.37)

The partial derivatives in Eq. (5.37) are entries of the matrix \( D_H(t, \chi) = D_G \circ D_F^{-1}(t, \chi) \).
(see Eq.(5.13) and Corollary 3) and may be found explicitly as integral expressions from Eqs.(5.20) – (5.23), and Theorem 1.

**Remark 7.** The conclusions of Theorems 2 and 3 continue to hold even when the hypothesis that $\ddot{b} \geq 0$ is violated, i.e., for inflationary cosmologies, but for non global charts. The forms of the metric given by Eqs. (5.30) and (5.14) are valid in that case on some neighborhood of the Fermi observer’s path $\gamma(t)$. 
Chapter 6

Simultaneity, the proper radius of $\mathcal{M}_\tau$ and Fermi relative velocities

In this chapter we find a general bound for the finite proper radius of the Fermi space slice of $\tau$-simultaneous events, $\mathcal{M}_\tau$, and we obtain expressions for Fermi velocities of comoving test particles relative to the Fermi observer. Exact results are given for the case that the scale factor has the form $a(t) = t^\alpha$ for some $0 < \alpha \leq 1$.

6.1 Simultaneity and The Proper Radius of $\mathcal{M}_\tau$

Definition 30. Define the proper radius, $\rho_{\mathcal{M}_\tau}$, of the Fermi space slice of $\tau$-simultaneous events, $\mathcal{M}_\tau$, by,

$$\rho_{\mathcal{M}_\tau} = \frac{a(\tau)}{2} \int_1^{\sigma_\infty(\tau)} \dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \frac{1}{\sigma^{3/2}\sqrt{\sigma-1}} d\sigma,$$

where $\sigma_\infty(\tau)$ is given by Eq. (4.12).

The Hubble parameter, $H$ is defined by,

$$H = \frac{\dot{a}(\tau)}{a(\tau)}$$

and the Hubble radius is defined to be $1/H$, i.e., the speed of light divided by $H$.

Theorem 4. Let $a(t)$ be a smooth, increasing, unbounded function on $(0, \infty)$ with inverse function $b(t)$ satisfying $\dot{b}(t) \geq 0$ for all $t > 0$. Then,

(a) at proper time $\tau$ of the Fermi observer, the proper distance $\rho$ to any spacetime point along a geodesic on the space slice, $\mathcal{M}_\tau$, satisfies the inequality,

$$\rho < \rho_{\mathcal{M}_\tau} \leq \frac{1}{H},$$

and $\rho_{\mathcal{M}_\tau}$ is a monotone increasing function of time $\tau$.

(b) synchronous time $t$ decreases to zero along any spacelike geodesic, $Y_\tau(\rho)$, orthogonal to the path of the Fermi observer at fixed proper time $\tau$, as the proper distance $\rho \to \rho_{\mathcal{M}_\tau}$, and $t$ is strictly decreasing as a function of $\rho$.

Proof. By hypothesis, $\dot{b}$ is an increasing function, so from Eq.(4.19), the proper distance from the observer to a spacetime point on $\mathcal{M}_\tau$ corresponding to any parameter value $\sigma$ satisfies,
\[ \rho = \frac{a(\tau)}{2} \int_{1}^{a} b \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \frac{1}{\sqrt{\sigma}^{3/2} - 1} d\sigma \]

\[ < \frac{a(\tau)}{2} \int_{1}^{a} b \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \frac{1}{\sqrt{\sigma}^{3/2} - 1} d\sigma \leq \rho_{M}\tau \]

\[ \leq \frac{a(\tau)}{2} b(a(\tau)) \int_{1}^{a} \frac{1}{\sqrt{\sigma}^{3/2} - 1} d\sigma \]

\[ = \frac{a(\tau)}{2} \frac{1}{\sqrt{\sigma}^{3/2} - 1} = \frac{1}{H}, \quad (6.4) \]

It follows immediately from Eqs.(4.12) and (6.4) that \( \rho_{M}\tau \) increases with \( \tau \). This proves part (a). To prove (b), observe that since \( a : (0, \infty) \rightarrow (a_{\text{inf}}, \infty) \), then \( b : (a_{\text{inf}}, \infty) \rightarrow (0, \infty) \) and by hypothesis \( b \) is an increasing function. Thus, \( \lim_{a \rightarrow a_{\text{inf}}} b(a) \equiv b(a_{\text{inf}}^{+}) = 0 \).

Using the notation of Theorem 1, Eq.(4.22), and Corollary 2, we have,

\[ \lim_{\rho \rightarrow \rho_{M}\tau} t(\tau, \sigma(\rho)) = \lim_{\sigma \rightarrow a_{\text{inf}}(\tau)} t(\tau, \sigma) = \lim_{\sigma \rightarrow a_{\text{inf}}(\tau)} b \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) = b(a_{\text{inf}}^{+}) = 0. \quad (6.5) \]

It follows from Eq.(4.13) that \( dt/d\rho < 0 \) except at \( \rho = 0 \), so \( t \) is strictly decreasing as a function of \( \rho \).

\[ \square \]

The following corollary is an immediate consequence of Theorem 4b.

**Corollary 5.** Under the hypotheses of Theorem 4, no two distinct spacetime points are simultaneous with respect to both synchronous time \( t \) and Fermi time \( \tau \) when \( t = \tau \).

**Remark 8.** The Fermi hypersurfaces of \( \tau \)-simultaneous events, \( \{ \mathcal{M}_{\tau} \} \), are isotropic with respect to the Fermi observer, but they fail to be homogeneous (in contrast to the \( t \)-simultaneous hypersurfaces, \( \{ \Sigma_{t} \} \), for synchronous time \( t \). This may be explained by Corollary 5 and Theorem 4b which shows that \( t \) decreases with proper distance along each \( \mathcal{M}_{\tau} \). We elaborate on this point in the concluding chapter.

### 6.2 Fermi Relative Velocities of Comoving Observers

It is well-known that the motion of a comoving test particle follows Hubble’s Law,

\[ v_{H}(\chi_{0}) = a(\tau)\chi_{0} = Hd. \quad (6.6) \]

Here the Hubble speed \( v_{H}(\chi_{0}) \equiv \dot{d} \), the Hubble parameter \( H \) is given by Eq.(6.2), and \( d = a(\tau)\chi_{0} \) is the proper distance along the spacelike path \( Z_{\tau}(\chi) = (\tau, \chi, \theta_{0}, \varphi_{0}) \) as \( \chi \) varies from 0 to \( \chi_{0} \). Both \( Z_{\tau}(\chi) \) and the path \( Y_{\tau}(\rho) \) given by Eq.(5.36) — described in different coordinate systems — are orthogonal to the path of the Fermi observer at the
From Eqs. (4.18) and (4.19), it follows the the coordinate \( \chi \) is a smooth, increasing function of \( \rho \) along the geodesic \( Y_\tau(\rho) \). We denote the inverse of that function (with fixed \( \tau \)) by \( \rho(\tau, \chi) \). The Fermi speed, \( v_F(\chi_0) \), of the radially receding, comoving test particle with world line \( \gamma_0(\tau) = (\tau, \chi_0, \theta_0, \varphi_0) \), relative to the observer \( \gamma(\tau) \), is given by,

\[
v_F(\chi_0) = \frac{d}{d\tau} \rho(\tau, \chi_0) \equiv \dot{\rho}.
\]

Eq.(6.7) follows from Prop. 3 in [3] and is a special case of the more generally defined Fermi relative velocity for test particles and observers following arbitrary world lines.

**Remark 9.** In analogy to a well-known expression for the Hubble speed of a test particle with peculiar velocity, the following identity holds for the Fermi relative speed of a comoving particle,

\[
v_F(\chi_0) = H(\tau) \rho + a(\tau) \frac{d}{d\tau} \left( \frac{\rho}{a(\tau)} \right),
\]

as may be verified by direct calculation. The second term on the right side of Eq.(6.8) is roughly analogous to the peculiar velocity in Robertson-Walker coordinates.

The following Theorem provides a general expression for the Fermi speed \( v_F(\chi_0) \).

**Theorem 5.** Let \( a(t) \) be a smooth, increasing function of \( t \) with inverse function \( b(t) \). The Fermi speed, \( v_F(\chi_0) \), of the comoving test particle with world line \( \gamma_0(\tau) \), relative to the Fermi observer, is given by,

\[
v_F(\chi_0) = \frac{\dot{a}(\tau)}{2} \left( \int_1^{\sigma_0} \frac{b \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma^{3/2} \sqrt{\sigma - 1}} d\sigma \right. \\
\left. + a(\tau) \int_1^{\sigma_0} \frac{\ddot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma^2 \sqrt{\sigma - 1}} d\sigma - a(\tau) \int_1^{\sigma_0} \frac{b \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma \sqrt{\sigma - 1}} d\sigma \right),
\]

where \( \sigma_0 \) is the unique solution to \( \chi(\tau, \sigma_0) = \chi_0 \) in Eq.(4.18).
Proof. With \( \chi_0 \) fixed, differentiating Eq.(4.19) with respect to \( \tau \) yields,

\[
\frac{d}{d\tau} \rho(\tau, \chi_0) = \frac{\dot{a}(\tau)}{2} \left[ \int_1^{\sigma_0} \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma^{3/2} \sqrt{\sigma - 1}} d\sigma + a(\tau) \int_1^{\sigma_0} \frac{\ddot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma^2 \sqrt{\sigma - 1}} d\sigma \right] + \frac{a(\tau)}{2} \dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma_0}} \right) \frac{1}{\sigma_0^{3/2} \sqrt{\sigma_0 - 1}} \frac{d\sigma_0}{d\tau}. \tag{6.10}
\]

Now differentiating Eq.(4.18) with respect to \( \tau \) gives,

\[
\frac{d\chi_0}{d\tau} = 0 = \frac{\dot{a}(\tau)}{2} \int_1^{\sigma_0} \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma \sqrt{\sigma - 1}} d\sigma + \frac{1}{2} \frac{b(\tau)}{\sqrt{\sigma_0}} \frac{1}{\sqrt{\sigma_0 \sigma_0 - 1}} \frac{d\sigma_0}{d\tau}. \tag{6.11}
\]

Multiplying Eq.(6.11) by \( a(\tau)/\sigma_0 \) and solving for the last term on the right hand side of Eq.(6.10) gives the desired result.

Corollary 6. Let \( a(t) \) be a smooth, increasing function of \( t \) with inverse \( b(t) \) such that \( \dot{b} \geq 0 \), then \( v_F(\chi_0) \) is a monotone increasing function of \( \chi_0 \).

Proof. Differentiating Eq.(6.9) with respect to \( \sigma_0 \) gives,

\[
\frac{\dot{a}(\tau)}{2\sigma_0^2} \left( \frac{\sqrt{\sigma_0}}{\sigma_0 - 1} \dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma_0}} \right) + a(\tau) \int_1^{\sigma_0} \frac{\ddot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma \sqrt{\sigma - 1}} d\sigma \right). \tag{6.12}
\]

With the assumption that \( \dot{b} \geq 0 \), we see this expression is positive on its domain. Since \( \sigma_0 \) is an increasing function of \( \chi_0 \), the result follows from the chain rule.

In the next corollary we consider the class of Robertson-Walker spacetimes for which the scale factor has the form,

\[
a(t) = t^\alpha \quad 0 < \alpha \leq 1. \tag{6.13}
\]

It is easily checked that for these models, \( \dot{b}(t) \geq 0 \), and therefore by Theorem 3, Fermi coordinates are global. This class of spacetimes includes the Milne universe (\( \alpha = 1 \)), radiation-dominated universe (\( \alpha = 1/2 \)), and matter-dominated universe (\( \alpha = 2/3 \)) considered in the next chapter. The following corollary is a consequence of Theorem 5.

Corollary 7. In Robertson-Walker spacetimes with \( a(t) = t^\alpha \) for \( 0 < \alpha \leq 1 \), the Fermi speed of the comoving test particle with world line \( \gamma_0(\tau) \), relative to the Fermi observer is time independent and is given by,

\[
v_F(\chi_0) = \frac{1}{2\alpha} \left( \int_1^{\sigma_0} \frac{1}{\sigma^{1+1/\alpha} \sqrt{\sigma - 1}} d\sigma + \frac{\alpha - 1}{\sigma_0} \int_1^{\sigma_0} \frac{1}{\sigma^{1+1/\alpha} \sqrt{\sigma - 1}} d\sigma \right). \tag{6.14}
\]
The least upper bound for \( \{ v_F(\chi_0) \} \) is given by,
\[
\lim_{\chi_0 \to \infty} v_F(\chi_0) = \sqrt{\frac{\pi}{\alpha}} \frac{\Gamma\left(\frac{1}{2\alpha} + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2\alpha} + 1\right)}.
\] (6.15)

The right side of Eq. (6.15) is bounded above by \( 1/\alpha \) with equality only for \( \alpha = 1 \).

**Proof.** By Lemma 6, \( v_F(\chi_0) \) is a strictly increasing function. It follows from Lemma 1 that the limit on the left hand side of Eq.(6.15) may be found by taking the limit of Eq.(6.14) as \( \sigma_0 \to \sigma_\infty(\tau) \equiv \infty \). An application of L’Hôpital’s rule shows that the limit of the second term in Eq.(6.14) is zero. Consider the first integral in Eq. (6.14). For \( 0 < \alpha \leq 1 \),
\[
\frac{1}{\sigma^{\frac{1}{2\alpha} + 1} \sqrt{\sigma - 1}} \leq \frac{1}{\sigma^{\frac{3}{2} \sqrt{\sigma - 1}}},
\] (6.16)
with equality only for \( \alpha = 1 \). Thus,
\[
\int_1^{\sigma_0} \frac{1}{\sigma^{\frac{1}{2\alpha} + 1} \sqrt{\sigma - 1}} d\sigma \leq 2 \sqrt{\frac{\sigma_0 - 1}{\sigma_0}}.
\] (6.17)
Therefore,
\[
\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{\alpha} + \frac{1}{2}\right)} = \frac{1}{\sigma^{\frac{1}{2\alpha} + 1} \sqrt{\sigma - 1}} d\sigma \leq 2.
\] (6.18)
The limit and the upper bound follow by multiplying the above result by \( 1/2\alpha \).

The proof of the following corollary follows by direct calculation.

**Corollary 8.** In Robertson-Walker spacetimes with \( a(t) = t^\alpha \) for \( 0 < \alpha \leq 1 \), the Fermi speed of the comoving test particle with world line \( \gamma_0(\tau) \) satisfies the following relationship,
\[
v_F(\chi_0) = \frac{\rho}{\tau} + \frac{\alpha - 1}{2\alpha \sigma_0} \int_1^{\sigma_0} \frac{1}{\sigma^{\frac{1}{2\alpha} \sqrt{\sigma - 1}}} d\sigma,
\] (6.19)
where the proper distance \( \rho \) of the test particle from the Fermi observer is given by Eq.(4.19).

**Corollary 9.** In Robertson-Walker spacetimes with \( a(t) = t^\alpha \) for \( 0 < \alpha \leq 1 \), the proper radius \( \rho_{\mathcal{M}_\tau} \) of the Fermi space slice of \( \tau \)-simultaneous events, \( \mathcal{M}_\tau \), is a linear function of \( \tau \) and is given by,
\[
\rho_{\mathcal{M}_\tau} = \frac{\tau \sqrt{\pi}}{2\alpha \Gamma\left(\frac{1}{\alpha} + \frac{1}{2}\right)}.
\] (6.20)

**Proof.** The result follows by taking the limit as \( \sigma_0 \to \infty \) of of both sides of Eq.(6.19), applying Eq.(6.15), and observing as in the proof of Corollary 7, that the second term on the right side of Eq.(6.19) converges to zero.
Remark 10. Since the coefficient of $\tau$ in Eq. (6.20) is a decreasing function of $\alpha$, the proper radii, at fixed $\tau$, of the $\tau$-simultaneous events, $M_\tau$, decrease as functions of $\alpha$ in Robertson-Walker spacetimes with $a(t) = t^\alpha$ for $0 < \alpha \leq 1$. Conversely, $\rho_{M_\tau} \to \infty$ for any fixed $\tau > 0$ as $\alpha \to 0^+$. 
Chapter 7

Particular Cosmologies

In this chapter we apply results of the previous chapters to particular cosmologies: the Milne universe, de Sitter universe, radiation-dominated universe, and matter-dominated universe. We include the first of these for purposes of illustration only, as the results are well known (see e.g., [8]). In the inflationary de Sitter universe, the expressions for the metric components in Fermi coordinates for a timelike geodesic observer are also known [7] and [19], but we show here that co-moving particles necessarily recede from the observer only with Fermi velocities less than the speed of light, in contrast to their Hubble velocities (see Eq.(6.6)). To our knowledge, the results below for the radiation-dominated and matter-dominated universes are new.

7.1 The Milne Universe

The Milne Universe is a special case of a Robertson-Walker spacetime and a useful prototype cosmology. It is a solution to the field equations with no matter, radiation, or vacuum energy. For this spacetime, \( k = -1 \) and

\[
a(t) = t, \tag{7.1}
\]

and we have for the inverse function \( b \),

\[
b(t) = t \quad \dot{b} = 1 \quad \ddot{b} = 0 \geq 0. \tag{7.2}
\]

To find Fermi coordinates for a comoving observer, we first integrate Eq.(4.19) with the result,

\[
\rho = \tau \sqrt{\frac{\sigma - 1}{\sigma}}, \tag{7.3}
\]

and thus,

\[
\sigma = \frac{1}{1 - \left(\frac{\rho}{\tau}\right)^2}. \tag{7.4}
\]

It now follows from Eqs.(4.17) and (4.18) that,

\[
t = \frac{\tau}{\sqrt{\sigma}} = \sqrt{\tau^2 - \rho^2}, \tag{7.5}
\]

and

\[
\chi = \ln(\sqrt{\sigma} + \sqrt{\sigma - 1}) = \tanh^{-1} \sqrt{\frac{\sigma - 1}{\sigma}} = \tanh^{-1} \left(\frac{\rho}{\tau}\right). \tag{7.6}
\]
Eqs. (7.5) and (7.6) are easily inverted to give,

\[ \tau = t \cosh \chi \]
\[ \rho = t \sinh \chi. \]

(7.7)

The proper radius, \( \rho_{\mathcal{M}_\tau} \), of the Fermi space slice of \( \tau \)-simultaneous events, \( \mathcal{M}_\tau \) given by Eq.(6.20) for this example is,

\[ \rho_{\mathcal{M}_\tau} = \frac{\tau \sqrt{\pi} \Gamma(1)}{2 \Gamma(\frac{3}{2})} = \tau = \frac{1}{H}, \]

(7.8)

which shows that the upper bound given by Theorem 4 is sharp. Theorem 3 guarantees that the Fermi coordinates defined by the coordinate transformation determined by Eqs.(7.5) and (7.6) are global, and Eq.(5.14) for this case reduces to the polar form of the Minkowski line element,

\[ ds^2 = -d\tau^2 + d\rho^2 + \rho^2 d\Omega^2. \]

(7.9)

We can thus recover from Eqs.(7.8) and (7.9) the well known result that the Milne universe may be identified as the forward light cone in Minkowski spacetime, foliated by negatively curved hyperboloids orthogonal to the time axis. The orthogonal spacelike geodesic \( Y_\tau(\rho) \) expressed in Fermi (i.e., Minkowski) coordinates has the form given by Eq.(5.36):

\[ Y_\tau(\rho) = (\tau, a^1 \rho, a^2 \rho, a^3 \rho), \]

(7.10)

where \( (a^1, a^2, a^3) \) is any unit vector in \( \mathbb{R}^3 \), i.e., \( Y_\tau(\rho) \) is a horizontal line segment orthogonal to the vertical time axis in Minkowski space. Now from Eqs.(6.14) and (7.6), the Fermi speed of a comoving test particle, with fixed spatial coordinate \( \chi \) at proper time \( \tau \), corresponding to parameter \( \sigma \), is given by,

\[ v_F = \sqrt{\frac{\sigma - 1}{\sigma}} = \tanh \chi = \frac{\rho}{\tau}, \]

(7.11)

which by Eq.(7.8) cannot reach or exceed the speed of light. Comoving test particles have constant Fermi speeds proportional to their distances from the observer.

Although not new for this example, Fermi coordinates for a Fermi observer in the Milne universe lead to an interpretation of the Milne universe that is not immediately available via the original Robertson-Walker coordinates. The “big bang” may be identified as the origin of Minkowski coordinates, and spacetime itself may be defined as the set of all possible spacetime points in Minkowski space that can be occupied by a test particle whose world line includes the origin of coordinates, i.e., the big bang. Space does not expand, rather, idealized test particles from an initial “explosion” merely fly apart from the Fermi observer in all directions with sub light Fermi velocities. Similar interpretations were given
7.2 The de Sitter Universe

The line element for the de Sitter Universe with Hubble’s constant $H_0 > 0$ is given by Eq.(4.1) with $k = 0$ and,

$$ a(t) = e^{H_0 t}, \quad \dot{a}(t) = H_0 e^{H_0 t}, $$

and thus,

$$ b(t) = \frac{1}{H_0} \ln(t), \quad \dot{b}(t) = \frac{1}{H_0} \quad \ddot{b}(t) = -\frac{1}{H_0^2} < 0. $$

The de Sitter universe is a vacuum solution to the field equations with cosmological constant $\Lambda = 3H_0^2$. Eqs.(4.17),(4.18) and (4.19) yield,

$$ \rho = \frac{1}{H_0} \sec^{-1}(\sqrt{\sigma}), $$

and,

$$ t = \tau - \frac{1}{H_0} \ln \sqrt{\sigma} = \tau + \frac{\ln(\cos(H_0 \rho))}{H_0}, $$

$$ \chi = \frac{\sqrt{\sigma} - 1}{H_0 e^{H_0 \tau}} = \frac{e^{-H_0 \tau} \tan(H_0 \rho)}{H_0}. $$

It follows immediately from Eq.(7.14) that $H_0 \rho < \pi/2$ (note that the hypothesis to Theorem 4 is violated here). If synchronous time $t$ is required to be positive, then it follows from Eq.(7.14) together with Eqs.(4.11) and (4.12) that $\sqrt{\sigma} < \exp(H_0 \tau)$, and therefore,

$$ e^{-H_0 \tau} < \cos(H_0 \rho). $$

Thus, along the spacelike geodesics,

$$ \chi < \frac{\sin(H_0 \rho)}{H_0} < \frac{1}{H_0}. $$

Although Fermi coordinates are not global for this example, in light of Remark 7 we may calculate the metric coefficients in $\{\tau, \rho, \theta, \varphi\}$ coordinates by inserting Eqs.(7.13), (7.14), and (7.16) into Eqs.(5.14) and (5.15). The result is,

$$ ds^2 = -\cos^2(H_0 \rho) d\tau^2 + d\rho^2 + \frac{\sin^2(H_0 \rho)}{H_0^2} d\Omega^2, $$

which is the same expression obtained in [7]. A description of the way in which Fermi coordinates break down at the boundary of the Fermi chart in de Sitter space was included in [19]. It is intriguing to observe that maximal Fermi charts in both the Milne and de Sitter universes each occupy a single “quadrant” of larger embedding spacetimes, [14].
The Fermi relative speed of a comoving test particle at time $\tau$ with fixed space coordinate $\chi$, corresponding to the parameter $\sigma$, is given by Theorem 5 and reduces to,

$$v_F = \frac{\sqrt{\sigma - 1}}{\sigma}. \quad (7.20)$$

Combining this with Eq.(7.16) gives,

$$v_F(\chi) = \frac{H_0 e^{H_0 \tau} \chi}{1 + (H_0 e^{H_0 \tau} \chi)^2}. \quad (7.21)$$

Thus, the Fermi relative speed of a comoving test particle is bounded by one-half the speed of light for all values of $H_0$, $\chi$ and $\tau$. Although Hubble and Fermi speeds are not directly comparable, as we discuss in the concluding chapter, the above expression is strikingly different from the standard formula for the Hubble speed of a comoving test particle,

$$v_H(\chi) = H_0 e^{H_0 \tau} \chi, \quad (7.22)$$

which is unbounded.

### 7.3 Radiation-Dominated Universe

The radiation-dominated universe is characterized by $k = 0$ and the scale factor,

$$a(t) = \sqrt{t}, \quad (7.23)$$

and thus,

$$b(t) = t^2 \quad \dot{b} = 2t \quad \ddot{b} = 2 \geq 0. \quad (7.24)$$

Eqs.(4.17), (4.18) and (4.19) yield,

$$t = \frac{\tau}{\sigma}, \quad (7.25)$$

$$\chi = 2\sqrt{\tau} \sec^{-1} \sqrt{\sigma}, \quad (7.26)$$

and

$$\rho = \tau \left( \frac{\sqrt{\sigma - 1}}{\sigma} + \sec^{-1} \sqrt{\sigma} \right). \quad (7.27)$$

By Theorem 4 the proper radius of the space slice of $\tau$-simultaneous events, $\mathcal{M}_\tau$, is bounded by $2\tau$. The exact value, given by Eq.(6.20), is,

$$\rho_{\mathcal{M}_\tau} = \tau \frac{\sqrt{\pi} \, \Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} = \frac{\pi}{2} \tau. \quad (7.28)$$

By Corollary 3, $t$ and $\chi$ are smooth functions of $\tau$ and $\rho$. As in Eq.(4.22) we write $\sigma = \sigma_{\tau}(\rho)$. Then from Eq.(5.14), the line element for the radiation-dominated universe in polar
Fermi coordinates is,
\[ ds^2 = -\frac{1}{\sigma} \left( 1 + \sqrt{\sigma - 1} \sec^{-1} \sqrt{\sigma} \right)^2 d\tau^2 + d\rho^2 + \frac{1}{\sigma} \left( 2 \tau \sec^{-1} \sqrt{\sigma} \right)^2 d\Omega^2. \] (7.29)

The Fermi relative speed of a comoving test particle with fixed coordinate corresponding to parameter \( \sigma \) may be calculated from Eq. (6.14) as,
\[ v_F = \frac{\sqrt{\sigma - 1}}{\sigma} + \frac{\sigma - 1}{\sigma} \sec^{-1} \sqrt{\sigma}. \] (7.30)

Using (7.26), \( v_F \) can also be expressed in terms of \( \chi \) as,
\[ v_F = \frac{1}{2} \sin \left( \frac{\chi}{\sqrt{\tau}} \right) + \frac{\chi}{4\sqrt{\tau}} \left( 1 - \cos \left( \frac{\chi}{\sqrt{\tau}} \right) \right). \] (7.31)

Applying Corollary 7, we find that the asymptotic limit of the Fermi relative speed of a comoving test particle is \( \pi/2 \) times the speed of light.

It follows from Corollary 8 or by directly comparing Eqs. (7.27) and (7.30) that,
\[ v_F = \frac{\rho}{\tau} - \frac{\sec^{-1} \sqrt{\sigma}}{\sigma}, \] (7.32)
so that for large \( \sigma \), or equivalently for large proper distance \( \rho \), \( v_F \approx \rho/\tau \), in analogy to the Milne (or Minkowski) universe. However, in the Milne universe, the proper distance from the Fermi observer at time \( \tau \) is bounded by, and asymptotically equal to, \( \tau \). Thus, \( v_F = \rho/\tau < 1 \). By contrast, the corresponding bound in the radiation-dominated universe, i.e., the radius of \( \mathcal{M}_\tau \), is \( (\pi/2)\tau \) so that \( v_F = \rho/\tau < \pi/2 \), with asymptotic equality. From this point of view, the existence of superluminal Fermi velocities in the radiation-dominated universe may be attributed to the greater diameters of the Fermi space slices, \( \{ \mathcal{M}_\tau \} \), in comparison to the Milne spacetime.

### 7.4 Matter-Dominated Universe

The final case we consider is the matter-dominated universe. For this spacetime \( k = 0 \) and the scale factor is given by
\[ a(t) = t^\alpha, \] (7.33)
where \( \alpha = 2/3 \). The inverse of \( a \) and its derivatives are given by,
\[ b(t) = t^{3/2} \quad \dot{b}(t) = \frac{3}{2} t^{1/2} \quad \ddot{b}(t) = \frac{3}{4} t^{-1/2} \geq 0. \] (7.34)

By Theorem 1,
\[ t = \frac{\tau}{\sigma^{3/4}}. \] (7.35)
\[ \chi = 3^\tau^{1/3} \left[ \frac{\sqrt{\pi} \Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} - \frac{2F_1\left(\frac{1}{4}; \frac{1}{2}; \frac{5}{4}; \frac{1}{\sigma}\right)}{\sigma^{1/4}} \right], \quad (7.36) \]

and
\[ \rho = \tau \left[ \frac{\sqrt{\pi} \Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} - \left( \frac{\sigma - 1}{\sqrt{\sigma}} \right)^{3/2} + \frac{2F_1\left(-\frac{3}{4}; \frac{1}{2}; \frac{1}{4}; \frac{1}{\sigma}\right)}{\sigma^{3/4}} \right], \quad (7.37) \]

where \( 2F_1(\cdot; \cdot; \cdot) \) is the Gauss hypergeometric function. It follows from Theorem 3 that the metic in Fermi polar coordinates is given by,
\[ ds^2 = g_{rr} d\tau^2 + d\rho^2 + \frac{\tau^{4/3}}{\sigma^2} \chi(\tau, \sigma)^2 d\Omega^2, \quad (7.38) \]

where \( \sigma = \sigma(\rho) \), as in Eq.(4.22), is the inverse of the function given by Eq.(7.37), \( \chi(\tau, \sigma) \) is given by Eq.(7.36), and,

From Eq.(6.14), the Fermi speed of a comoving test paricle with worldline \( \gamma_0 \) is given by,
\[ g_{rr} = -\frac{\tau^{4/3}}{\sqrt{\sigma}} \left[ 1 + \frac{\sqrt{\sigma - 1}}{\sigma^{1/4}} \left( \frac{\sqrt{\pi} \Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} - \frac{2F_1\left(\frac{1}{4}; \frac{1}{2}; \frac{5}{4}; \frac{1}{\sigma}\right)}{\sigma^{1/4}} \right) \right]^2. \quad (7.39) \]
\[ v_F(\chi_0) = \frac{3}{4} \int_{\sigma_0}^{\sigma_0} \frac{1}{\sigma^{2} \sqrt{\sigma - 1}} d\sigma - \frac{1}{3\sigma_0} \int_{1}^{\sigma_0} \frac{1}{\sigma^{2} \sqrt{\sigma - 1}} d\sigma. \quad (7.40) \]

Corollaries 7 and 9 then give the supremum of this speed and the proper radius of \( M_\tau \) as,
\[ \lim_{\sigma_0 \to \infty} v(\sigma_0) = \frac{3}{4} \frac{\sqrt{\pi} \Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\rho_{M_\tau}}{\tau} \approx 1.31103. \quad (7.41) \]

This shows that the matter-dominated Universe supports superluminal Fermi velocities at proper distances away from the Fermi observer, sufficiently close to the radius, \( \rho_{M_\tau} \), of the Fermi space slice of \( \tau \)-simultaneous events.
Chapter 8

Conclusion

Theorems 2 and 3 of this thesis give transformation formulas for Fermi coordinates for observers comoving with the Hubble flow in expanding Robertson-Walker spacetimes, along with exact expressions for the metric tensors in those coordinates. We have shown that Fermi coordinates are global for non inflationary cosmologies, i.e., when the scale factor $a$ satisfies the condition, $\dot{a}(t) \leq 0$. Global Fermi coordinates may be useful for the purpose of studying the influence of global expansion on local dynamics and kinematics [5].

Our results also apply to cosmologies that include inflationary periods, though in such cases the Fermi charts are local. However, if the spacetime includes an early inflationary period, but $\dot{a}(t) \leq 0$ for all $t \geq t_0$, for some $t_0$, then by recalibrating the scale factor to $\tilde{a}(t) = a(t + t_0)$, a global Fermi coordinate chart and all of our results are immediately available for the submanifold of spacetime events with $t > t_0$.

In Chapter 6 we found exact expressions for the Fermi relative velocities of comoving (and necessarily receding) test particles. It was shown that superluminal relative Fermi velocities exist, and that those velocities increase with proper distance from the observer. Superluminal least upper bounds were given for cosmologies whose scale factors follow power laws. We note that although the overall qualitative behavior of the relative speeds $v_H$ and $v_F$ may be compared, it follows from Corollary 5 that at any given proper time of the Fermi observer, $v_H$ and $v_F$ measure speeds of the same comoving test particle (with fixed coordinate $\chi_0$) only when it is at different spacetime points. For a comoving test particle at a given spacetime point, $v_H$ and $v_F$ give the particle’s relative speeds at different times of the Fermi observer.

The existence of superluminal relative velocities bears on the question of whether space is expanding, c.f. [35, 5, 8, 9, 20] and the numerous references in those papers. On this matter, our results may be contrasted with arguments given in [8]. In that paper, the coordinate transformation for the Milne universe, repeated in our Eqs.(7.6) and (7.7), was used to compare the Hubble and Fermi (or Minkowski) relative speeds $v_H$ and $v_F$ of comoving test particles. The incongruity of superluminal Hubble speeds and necessarily subluminal Minkowski speeds in the Milne universe was discussed. It was argued that the analogous qualitative difference would also occur for cosmologies that include matter or radiation, through a comparison of Hubble speeds to speeds defined via coordinates with a “rigid” radial coordinate.\textsuperscript{1} However, if the latter class of coordinates includes Fermi coordinates — the coordinates used to deduce that conclusion for the Milne universe — our results in Sect. 5 for the radiation-dominated and matter-dominated cosmological models do not

\textsuperscript{1}c.f. Sect. V of [8], p. 63.
support that conjecture.

For cosmological models with a scale factor of the form $a(t) = t^\alpha$ for $0 < \alpha \leq 1$, the existence of superluminal relative velocities of comoving particles may be understood in terms of the geometry of the simultaneous space slices, $\{M_r\}$. In Chapter 7, it was shown through the use of specific examples that superluminal relative Fermi velocities exist provided “there is enough space” in the sense that the proper radius $\rho_{M_r}$ of $M_r$ satisfies the condition $\rho_{M_r} > \tau$.

In Chapter 7 we also showed that the relative Fermi speeds of comoving particles are bounded by one-half the speed of light in the de Sitter universe, a spacetime considered to be “expanding.” By way of contrast, superluminal relative Fermi velocities were proven to exist in the static Schwarzschild spacetime (with interior and exterior metric joined at the boundary of the interior fluid), a spacetime that is not usually regarded as “expanding.” Thus, it may be argued that existence of superluminal relative velocities, in general, is not the appropriate criterion for the purpose of defining what is meant by the expansion of space.

Does space expand? An affirmative answer may be given for the non inflationary Robertson-Walker cosmologies studied in this thesis, in the following sense. For any comoving geodesic observer, the Fermi space slices of $\tau$-simultaneous events, $\{M_r\}$, that foliate the spacetime have finite proper diameters that are increasing functions of the observer’s proper time. This is Theorem 4a. Theorem 4b explains how it is possible for the space slices to have only finite extent. What stops these hypersurfaces of constant Fermi time from continuing beyond their proper diameters? The theorem makes precise the way in which all spacetime events are simultaneous to synchronous time $t = 0$, the big bang in cosmological models admitting an initial singularity.
References


