Float-up picture of extended levels in the integer quantum Hall effect: A numerical study

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The fate of the integer quantum Hall effect (IQHE) at weak magnetic field is studied numerically in the presence of correlated disorders. We find a systematic float-up and merging picture for extended levels on the low-energy side which results in direct transitions from higher-plaquette IQHE states to the insulator. Such direct transitions are controlled by a quantum critical point with a universal scaling form of conductance. The phase diagram is in good agreement with recent experiments.

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How extended levels evolve with disorders and the magnetic field \( B \) is central to our understanding of the integer quantum Hall effect (IQHE). Earlier on, Khmel’nitzkii and Laughlin had argued that extended levels should continuously float up towards higher energy with reducing \( B \). And the assumption that extended levels never merge has led to a select rule of the global phase diagram for the IQHE, in which a direct transition from a higher-plaquette (\( \nu > 1 \)) state to the insulator is prohibited. But direct transitions have been observed in many recent experiments, which have renewed theoretical interest in reexamining the float-up picture in IQHE systems.

Previous numerical studies in the tight-binding model (TBM) with white-noise disorders have indicated the existence of direct transitions from higher IQHE plaquette states to the insulator. But a detailed analysis has also revealed that the lattice effect plays a central role there: such a direct transition first happens near the band center due to the presence of extended level carrying negative topological Chern number (a peculiar lattice effect) which forms a high-energy IQHE-insulator boundary and starts to “float down” towards the low-energy regime with increasing disorder or reducing \( B \). During the float-down process, the boundary keeps merging with lower extended levels such that the plateaus disappear in a one-by-one fashion.

However, such a float-down picture, due to the lattice effect, is not expected in the continuum limit. Recall that in a continuum model, there does not exist a high IQHE-insulator boundary as the band center is essentially located at infinite energy. In this case a levitation of extended levels by disorders is expected as discussed in Refs. 11–13. Since it has been generally believed that the experimental situation should be physically described by the continuum model due to the weakness of magnetic fields compared to the bandwidth and low density of charge carriers, it becomes especially interesting whether the float-up of extended levels alone can also lead to a direct transition in the low-energy regime.

A float-up of the lowest extended level actually has been seen in a numerical calculation based on the TBM but its journey has quickly ended by merging into the float-down IQHE-insulator boundary from the band center. In order to study how such a float-up feature near the band edge evolves, one has to somehow “delay” the floating-down process of the high IQHE-insulator boundary. Note that the inter-Landau-level mixing caused by uncorrelated (white-noise) disorders happens more strongly near the band center, which may enhance the tendency for extended levels near the band center to move down. So one can try to “smooth” the lattice effect by introducing short-range correlations among disorders. As a result to be shown below, the float-up process will then become a dominant effect for those lower extended levels as the float-down of the high IQHE-insulator boundary is significantly slowed, in contrast to the case in the white-noise limit at similar weak magnetic fields.

In this paper, we present a systematic floating-up and merging pattern revealed for extended levels near the band edge. Specifically, the lowest extended level starts to float upward at stronger disorder or weaker \( B \) and eventually emerges into the second lowest extended level to form a new IQHE-insulator boundary on the low-energy side, leading to a \( \nu = 2 \rightarrow 0 \) direct transition, while the aforementioned upper IQHE-insulator boundary still remains at high energy. And such a lower IQHE-insulator boundary keeps moving up to merge with higher-energy extended levels to result in 3-0, 4-0, . . . direct transitions with increasing disorders or reducing \( B \). The phase diagram is in good agreement with the experiments. Furthermore, direct transitions to the insulator at the lower boundary are found to be consistent with a quantum critical point picture, and in particular the conductance as a function of \( B - B_c \) (\( B_c \) denotes the critical magnetic field) is of the universal form for 1-0, 2-0, . . . up to 6-0 transitions within the numerical resolutions.

The TBM \( H = -\sum_{i,j} e^{i\phi_{i,j}} c_i^\dagger c_j + H.c. + \sum_{w} w_i c_i^\dagger c_i \), with the magnetic flux per plaquette \( \phi = \sum_{i,j} a_{i,j} = 2\pi m/M \) (\( m \) and \( M \) are integers). And we define \( B = m/M \). The correlated disorder \( w_i \) is generated by \( w_i = W/\pi \sum_{f} f e^{-|R_f|^2/\lambda_0^2} \). Here \( R_f \) denotes the spatial position of site \( i \). \( W \) and \( \lambda_0 \) are the strength and correlation length scale of disorders, respectively. \( f_i \) is a random number distributing uniformly between \((-1,1)\).

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To illustrate how extended levels near the band edge evolve with the disorder strength \( W \), the density of states \( f_i \) is generated by \( w_i = W/\pi \sum_{f} f e^{-|R_f|^2/\lambda_0^2} \). Here \( R_f \) denotes the spatial position of site \( i \). \( W \) and \( \lambda_0 \) are the strength and correlation length scale of disorders, respectively. \( f_i \) is a random number distributing uniformly between \((-1,1)\).

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$B = 1/64$, $\lambda_0 = 1$, and the sample size $32 \times 64$. We have checked that the positions of all the peaks of $\rho_{\text{ext}}$ shown in Fig. 1(a) are sample size independent at three different sample sizes: $N = 16 \times 64$, $N = 24 \times 64$, and $N = 32 \times 64$. These peaks denote the positions of extended levels while the nonpeak part of $\rho_{\text{ext}}$ should scale to zero presumably in the thermodynamic limit similar to the results reported before$^{8,14}$ with weak disorders. In Fig. 1(a), as $W$ is increased from 0.7 to 1.4, extended level positions (marked by diamonds) all start to float up slightly from the filling number $n_L = \nu + 0.5$ ($\nu = 0, 1, \ldots$).

Besides the general float-up feature, Fig. 1(a) also shows that the lowest extended level distinctly moves faster than the second lowest extended level, so that the first plateau between these two extended levels should be narrowed during this float-up process. If such a narrowing is to continue with the increase of the disorder strength, then it eventually can result in the vanishing of the first plateau, leading to a direct plateau-insulator transition. To investigate this possibility, we fix the Landau level filling number at $n_L = 2$, which is around the center of the second plateau at weak disorders, and then calculate the longitudinal conductance $\sigma_{xx}$ using the Landauer formula by calculating all the Lyapunov exponents from the transfer matrix method$^{15,16}$ with stripe samples$^{17}$ of $L_x = L$ and $L_y > 10^6$.

In Fig. 1(b), an example how $\sigma_{xx}$ changes with $W$ is shown at the fixed $n_L = 2$: At small $W$, $\sigma_{xx}$ remains small and decreases with sample length $L$, characterizing the insulating feature at the center of the plateau. But with the increase of $W$, $\sigma_{xx}$ first increases to signal that the extended levels at the lower filling number approach the plateau center at $n_L = 1.55$. Such a direct transition from $\nu = 2$ to the insulator persists into very weak magnetic fields from $B = 1/64$ to $B = 1/2304$. $W_c$, at which the merged lowest two extended levels pass $n_L = 2$, is shown in the inset of Fig. 1(c) as a function of $B$. We see that $W_c$ monotonically decreases with $B$ and is extrapolated to zero in a fashion of $B^{1/2}$ in the weak $B$ limit.

Similar direct transitions can also be induced by reducing the strength of the magnetic field $B$. By following the trace of extended levels in the $n$-$B$ plane ($n$ is the on-site electron density) at fixed $W = 1.4$ and $\lambda_0 = 1$, we determine the phase diagram in Fig. 1(c), where the solid circles represent the lower IQHE-insulator boundary, while the open circles are the positions of various extended levels between plateaus which merge into the boundary at weaker $B$. This phase diagram is very similar to a recent experimental phase diagram (Fig. 2 in Ref. 4) as well as the earlier ones obtained in Refs. 5 and 6. We have checked that $\sigma_{xy}$ indeed saturates to $\nu e^2/h$ in the plateau region while it approaches zero on insulating side. Both $\sigma_{xx}$ and $\sigma_{xy}$ at the $\nu \rightarrow 0$ transition are close to $\nu e^2/2h$ (Ref. 18) in accordance with experiments.$^6$

Apparently the above results critically depend on how reliably one can identify the positions of extended levels using finite-size calculations. Let us focus on the merged extended
levels as the lower IQHE-insulator boundary shown in Fig. 1(c). By fixing \( n_L = 2 \), marked by the arrow \( C \) in Fig. 1(c), we calculated the longitudinal conductance \( \sigma_{xx} \) with \( B \) changing continuously at fixed \( W = 1.4 \). We found a peak in \( \sigma_{xx} \) at \( B_c = 1/70 \) as a 2-0 transition. In Fig. 2(a), \( \sigma_{xx} \) as a function of \( B \) with \( n_L = 1, 2, \ldots, 6 \) at \( W = 1.4 \) are shown for sample width \( L = 96 \). The \( L \)-independent peak positions should correspond to \( \nu = 1 \rightarrow 0, \ 2 \rightarrow 0, \ldots, \ 6 \rightarrow 0 \) transitions in Fig. 1(c). Remarkably, all these data can be collapsed onto a universal curve

\[
\frac{\sigma_{xx}}{\sigma_c} = 2 \exp(s)[1 + \exp(2s)],
\]

(1) as shown in Fig. 2(b), if one defines a variable \( s = c_s(L)(B - B_c)/B_c \), which is different from the logarithmical form discussed in Ref. 13. Here the parameter \( 1/c_s(L) \) represents the relative width of \( \nu \rightarrow 0 \) transition at a finite \( L \). Furthermore, by collapsing the data at different \( L \)'s, we find a scaling curve for \( \nu = 2 \rightarrow 0 \) transition in Fig. 3(a), with

\[
c(L) \propto L^{1/\lambda_0}
\]

(2) from \( L = 32 \) to \( L = 160 \). The correlation length exponent is identified to be \( \lambda_0 = 4.6 \pm 0.5 \) in the inset of Fig. 3(a), about doubled from \( \lambda = 2.3 \) for the \( \nu = 1 \rightarrow 0 \) transition which has been similarly determined at \( n_L = 1 \). We emphasize that these scaling features provide very strong evidence for the existence of a new quantum critical point at the 2-0 transition. However, the exponent revealed here may not be accurate as the finite-size correction to the exponent is known to be important.\(^1\) The exponents for \( 3-0, \ldots, 6-0 \) transitions seem further increased but are more difficult to determine as larger sample sizes are needed. Furthermore, Eq. (1) still holds as we change \( \lambda_0 \) from 1 to 2 and 3 or \( n_L \) away from integer numbers. The details will be presented elsewhere.

Based on Eqs. (1) and (2), we conclude that the \( \nu = 2 \rightarrow 0 \) transition corresponds to a quantum critical point with measure zero in the \( L \rightarrow \infty \) limit. We note that the same scaling form has been previously obtained\(^2\) for the 1-0 transition, where \( \rho_{xx} = \exp(-s)h/e^2 \) and \( \rho_{xy} = h/e^2 \) leading to Eq. (1). Identifying such a simple scaling relation for \( \nu = 1, 2 \rightarrow 0 \) as well as higher plateaus to insulator transitions may be the most striking evidence for a single quantum critical point at each transition. The standard scaling method can be also
applied to the $\nu = 2 \rightarrow 0$ transition to independently verify the one-parameter scaling law.\textsuperscript{21} As shown in Fig. 3(b), by collapsing the same data as $\sigma_{xx}(L)/\sigma_c = f(\xi/L)$ by a correlation length $\xi$, we find $\xi \propto \left| (B-B_j)/B_j \right|^{-\varphi}$ with $\varphi = 4.5 \pm 0.5$ [the inset of Fig. 3(b)] in agreement with the above result.

We have also checked the case right before the lowest two extended levels merge together. By scanning $B$ at a fixed $n_L \approx 1.8$ nearby the scan $C$ shown in Fig. 1(c), $\sigma_{xx}$ exhibits two distinct peaks at $B_{c1} \approx 0.0151$ and $B_{c2} \approx 0.0169$ with $L = 128$ [see the middle inset of Fig. 4(a)]. The main panel of Fig. 4(a) shows the finite-size scaling curve of $\sigma_{xx}/\sigma_c$ as a function of $\xi/L$ obtained by collapsing the data of different sample sizes at $B < B_{c1}$. The right inset indicates that $\xi$ diverges at $B_{c1}$ with an exponent $\varphi = 2.4$ which is essentially the same as the standard one for the 1-0 transition. A similar finite-size scaling curve has been also obtained for the branch at $B > B_{c2}$, corresponding to the 2-1 transition. It is noted that the above analysis resembles the study\textsuperscript{22} of the spin-unresolved case at strong magnetic field where a small spin-orbit coupling is used to lift the spin degeneracy to create two separated but very close quantum critical points. Similar to the latter case, if one “mistakenly” treats the present case as a single critical point at $B_m$, a middle point between $B_{c1}$ and $B_{c2}$, and proceeds with a finite-size scaling analysis, then one gets Fig. 4(b) where the quality of data collapsing becomes markedly worse and in particular $\xi$ shows a saturation trend approaching $B_m$, contrary to the assumption of a critical point at $B_m$. Thus, the two separated extended levels with $|B_{c1} - B_{c2}|/B_m \approx 0.11$ are not inconsistent as a single critical point in our numerical analysis. Finally, we make a remark that even if the 2-0 transition that we observed is actually two transitions with very small $|B_{c1} - B_{c2}|$ indistinguishable numerically, the nice scaling properties [Eqs. (1) and (2)] that we found still indicate that there is a new quantum critical point. In this case, the splitting $|B_{c1} - B_{c2}| \neq 0$, if it exists, should be caused by a relevant operator of such a new critical point.

To summarize, we have identified a float-up and merging pattern for extended levels near the band edge at weak $B$ (down to $B = 1/2304$ where there are 1152 Landau levels between the band edge and center). The corresponding phase diagram with direct transitions is in excellent agreement with experiments where the essential features can be explained by the narrowing and destruction of each IQHE plateau due to the sequential merging of neighboring extended levels as the mobility gap in between collapses.

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\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig4.png}
\caption{The double peak $\sigma_{xx}$ before two lowest extended levels merge is shown in the middle panel of (a). (a) $\sigma_{xx}/\sigma_c$ as a function of $\xi/L$ by collapsing all the data at $B < B_{c1}$ with $\xi$ shown in the right inset. (b) By assuming a single critical point at $B_m$, the data collapsing shows worse quality and $\xi$ in the inset becomes saturated approaching $B_m$.}
\end{figure}

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