DIRICHLET SERIES ASSOCIATED TO $B$-MULTIPLICATIVE ARITHMETIC FUNCTIONS

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by

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We study the analytic properties of twisted Dirichlet series associated to B-multiplicative arithmetic functions. An example of such a function is the exponential of the sum of digits of an integer in the base. Twisting by additive characters, we can study the behavior of such functions in arithmetic progressions. These properties are reflected in analytic properties of the Dirichlet series.
Chapter 1
Introduction

A multiplicative arithmetic function is a function \( f : \mathbb{Z}_+ \rightarrow \mathbb{C} \) such that

\[ f(mn) = f(m)f(n), \quad (1.0.1) \]

for \((m,n) = 1\).

This property is reflected in the factorisation of the associated (formal) Dirichlet series. We have

\[ \sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p \left( \sum_{m=0}^{\infty} f(p^m)p^{-ms} \right). \quad (1.0.2) \]

Here the product is over the prime numbers.

The prime powers are building blocks for multiplicative arithmetic functions.

If \( a \) is completely multiplicative i.e. \((1.0.1)\) holds without the condition that \((m,n) = 1\), then

\[ \sum_{m=0}^{\infty} f(p^m)p^{-ms} = (1 - f(p)p^{-s})^{-1}. \quad (1.0.3) \]

Arithmetic functions appearing in Number Theory are often multiplicative. Well known examples of multiplicative arithmetic functions include Mobius function, Euler’s totient function and divisor sum function.

In this thesis, we study a certain analogue of multiplicative arithmetic functions called (after Gel’fond [6]) \(B\)-multiplicative arithmetic functions. Some of the original ideas can also be found in [1].

The notion is based on base \( B \) arithmetic. Here \( B \) is positive integer such that \( B \geq 2 \).

For a non-negative integer \( n \), let

\[ n = \sum_{i=0}^{r} n_iB^i \quad (1.0.4) \]
be the base $B$ expansion, where $0 \leq n_i \leq B - 1$.

For non-negative integers $n$ and $m$, let

$$ (n, m)_B = \sum_i \min\{n_i, m_i\} B^i \quad (1.0.5) $$

be their greatest common divisor in base $B$. We also define an analogue of being divisible in base $B$.

A $B$-multiplicative arithmetic function is a function $f : \mathbb{Z}_+ \to \mathbb{C}$ such that

$$ f(mn) = f(m)f(n), \quad (1.0.6) $$

for $(m, n)_B = 0$.

Often a multiplicative arithmetic function has an analogue as a $B$-multiplicative arithmetic function.

After proving basic properties of the above notions, we study polynomials and power series associated to $B$-multiplicative arithmetic functions. In the final chapter, we apply these results to study analytic properties of Dirichlet series associated to $B$-multiplicative arithmetic functions. We study meromorphic continuation of such Dirichlet series and determine explicit location of the poles.

It would be interesting to consider algebraicity of special values of the Dirichlet series at integers, possibly up to periods. We hope to consider this and related questions in the near future.
In this chapter, we consider certain aspects of arithmetic in base $B$ for an integer $B \geq 2$. For simplicity, we first consider the case of base 2. In §2.1, we describe the basic setup. In §2.2, we describe the notion of greatest common divisor in base 2. In §2.3, we study some properties of the notion. Here we also define the notion of divisibility in base 2. In §2.4, we consider the case of base $B$.

2.1 Setup

In this section, we describe the basic setup.

We start by recalling the following basic theorem.

**Theorem 2.1.1.** Any non-negative integer $a$ can be uniquely written as

$$a = \sum_{i=0}^{k} a_i 2^i,$$

where $a_i = 0$ or 1 and $a_k = 1$ for positive $a$.

In what follows, we let

$$a_2 = (a_k \ldots a_1 a_0).$$

The computation of the digits $a_i$’s can be done by the division algorithm.

Let us give some examples.

**Example 2.1.2.** Note that $7 = 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$. Thus, $7_2 = (111)$. Similarly, $8_2 = (1000)$ and $17_2 = (10001)$. 


2.2 Greatest common divisor

In this section, we describe the notion of greatest common divisor in base 2.

We start with the definition.

**Definition 2.2.1.** Let $a$ and $b$ be non-negative integers. Let $a_2 = (a_k...a_1a_0)$ and $b_2 = (b_l...b_1b_0)$, as above. Then, the greatest common divisor $(a, b)_2 \in \mathbb{Z}_{\geq 0}$ of $a$ and $b$ in base 2 is given by

$$(a, b)_2 = \sum_{r=0}^{\min\{k,l\}} \min\{a_r, b_r\} 2^r.$$ 

By definition,

$$((a, b)_2)_2 = (\min\{a_{\min\{k,l\}}, b_{\min\{k,l\}}\} \ldots \min\{a_1, b_1\} \min\{a_0, b_0\}). \quad (2.2.1)$$

The equality can be used to compute the greatest common divisor in base 2. Let us give some examples.

**Example 2.2.2.** Recall that $7_2 = (111)$, $8_2 = (1000)$ and $17_2 = (10001)$. Note that $((7, 8)_2)_2 = (000)$ and thus $(7, 8)_2 = 0$. Note that $((8, 17)_2)_2 = (0000)$ and thus $(8, 17)_2 = 0$. Note that $((7, 11)_2)_2 = (001)$ and thus $(7, 11)_2 = 1 \cdot 2^1 + 1 \cdot 2^0 = 3$.

We regard the notion analogous to the usual notion of greatest common divisor.

We now define the notion of being relatively prime in base 2.

**Definition 2.2.3.** We say that $a$ and $b$ are relatively prime in base 2 if $(a, b)_2 = 0$.

In view of Example 2.2.2, it follows that 7 and 8 are relatively prime in base 2. However, 7 and 11 are not relatively prime in base 2.

We have the following useful criterion.

**Lemma 2.2.4.** The non-negative integers $a$ and $b$ are relatively prime in base 2 if and only if $a_i b_i = 0$, for $0 \leq i \leq \min\{k, l\}$. 
Proof. By (2.2.1) and Theorem 2.1.1, it follows that \((a, b)_2 = 0\) if and only if \(\min\{a_i, b_i\} = 0\), for \(0 \leq i \leq \min\{k, l\}\). This finishes the proof.

\[ \square \]

We have the following immediate corollary.

**Corollary 2.2.5.** If \((a, b)_2 = 0\), then \((a + b)_i = \max\{a_i, b_i\}\) for \(0 \leq i \leq \max\{k, l\}\).

We now try to see whether there is a relation between the usual notion of greatest common divisor and the one in base 2.

It turns out that there is no direct relation. Here are examples.

**Example 2.2.6.** (1). Note that \(2_2 = (10)\) and \(3_2 = (11)\). In particular, \(((2, 3)_2)_2 = (10)\) and thus, \((2, 3)_2 = 1 \cdot 2^1 = 2\). However, \((2, 3) = 1\).

(2). Note that \(6_2 = (110)\) and \(9_2 = (1001)\). In particular, \(((6, 9)_2)_2 = (000)\) and thus, \((6, 9)_2 = 0\). However, \((6, 9) = 3\).

### 2.3 Properties

In this section, we study some properties of greatest common divisor in base 2.

Let us recall basic properties of the usual greatest common divisor.

**Proposition 2.3.1.** Let \(a, b\) and \(c\) be non-negative integers.

(1). We have \((a, 0) = 0\), \((a, a) = a\) and \((a, b) = (b, a)\).

(2). We have \((a, b) \leq \min\{a, b\}\).
(3) If \(a\) and \(b\) are even, then \((a, b) \neq 1\).

(4) If \(c|a\) and \(c|b\), then \(c|(a, b)\).

(5) If \((a, b) = 1\) and \(b|ac\), then \(b|c\).

We now consider analogous properties for greatest common divisor in base 2.

We start with analogue of the above properties (1)-(3).

**Proposition 2.3.2.** Let \(a\) and \(b\) be non-negative integers.

(1). We have \((a, 0)_2 = 0\), \((a, a)_2 = a\) and \((a, b)_2 = (b, a)_2\).

(2). We have \((a, b)_2 \leq \min\{a, b\}\).

(3). If \(a\) and \(b\) are odd, then \((a, b)_2 \neq 0\).

**Proof.** Note that

\[
((a, 0)_2)_2 = 0 \cdot 2^0 = 0
\]

and thus, \((a, 0)_2 = 0\).

Note that

\[
((a, a)_2)_2 = (\min\{a_k, a_k\} \ldots \min\{a_1, a_1\} \min\{a_0, a_0\}) = (a_k \ldots a_1 a_0)
\]

and thus \((a, a)_2 = \sum_{i=0}^{k} a_i 2^i = a\).

By definition, we have

\[
(a, b)_2 = \sum_{r=0}^{\min\{k, l\}} \min\{a_r, b_r\} 2^r = \sum_{r=0}^{\min\{l, k\}} \min\{b_r, a_r\} 2^r = (b, a)_2.
\]

Note that

\[
(a, b)_2 = \sum_{r=0}^{\min\{k, l\}} \min\{a_r, b_r\} 2^r \leq \sum_{i=0}^{k} a_i 2^i = a.
\]

Similarly, \((a, b)_2 \leq b\). Thus, \((a, b)_2 \leq \min\{a, b\}\).
If \(a\) and \(b\) are odd, then \(a_0 = b_0 = 1\). Thus,

\[
(a, b)_2 \geq 1 \cdot 2^0 = 1.
\]

\[
\square
\]

**Remark.** The direct analogue of property (3) of Proposition 2.3.1 does not hold in general. Here is an example. Note that \(2_2 = (10)\) and \(4_2 = (100)\). In particular, \(((2, 4)_2)_2 = (00)\) and thus \((2, 4)_2 = 0\). However, 2 and 4 are even.

To consider analogue of properties (4) and (5) of Proposition 2.3.1, we first need an analogue of the notion of divisibility. Our analogue is as follows.

**Definition 2.3.3.** We say that \(a\) divides \(b\) is base 2, if \(a_i \leq b_i\) for all \(0 \leq i \leq k\). We denote this by \(a \preceq b\).

Note that \(a \preceq b\) implies that \(k \leq l\).

Let us give some examples.

**Example 2.3.4.** Note that \(3_2 = (11)\) and \(11_2 = (1011)\). Thus, \(3 \preceq 11\).

The notion may seem analogous to the usual notion of less than or equal to. Here is a relation.

**Lemma 2.3.5.** Suppose that \(a \preceq b\). Then, \(a \leq b\).

**Proof.** By definition, \(k \leq l\) and \(a_i \leq b_i\) for all \(0 \leq i \leq k\). Thus,

\[
a = \sum_{i=0}^{k} a_i 2^i \leq \sum_{i=0}^{k} b_i 2^i \leq \sum_{i=0}^{l} b_i 2^i = b.
\]

\[
\square
\]

**Remark.** The converse of the lemma does not hold in general. Here is an example. Recall that \(3_2 = (11)\) and \(4_2 = (100)\). Thus 3 does not divide 4 is base 2. However, \(3 \leq 4\). This example also shows that neither \(a \preceq b\) nor \(b \preceq a\) may hold in general. This is another difference with the usual notion of less than or equal to.
The following lemma perhaps strengthens analogy with the usual notion of divisibility.

**Lemma 2.3.6.** Let \(a, b\) and \(c\) be non-negative integers.

1. We have \(0 \leq a\) and \(a \leq a\).
2. If \(a \leq b\) and \(b \leq c\), then \(a \leq c\).

We now prove an analogue of property (4) of Proposition 2.3.1.

**Proposition 2.3.7.** If \(c \leq a\) and \(c \leq b\), then \(c \leq (a, b)_2\).

**Proof.** Suppose that \(c \leq a\) and \(c \leq b\).

Let \(c_2 = (c_m \ldots c_1 c_0)\). By definition, \(m \leq k\) and \(m \leq l\). In particular, \(m \leq \min\{k, l\}\).

By definition, \(c_i \leq a_i\) and \(c_i \leq b_i\) for all \(0 \leq i \leq m\). In particular, \(c_i \leq \min\{a_i, b_i\}\) for all \(0 \leq i \leq m\).

In view of (2.2.1), this finishes the proof.

\(\square\)

We finally consider an analogue of property (5) of Proposition 2.3.1. Our analogue is as follows.

**Proposition 2.3.8.** If \((a, b)_2 = 0\), \(c \neq b\), \((a, c)_2 = 0\) and \(b \leq a + c\), then \(b \leq c\).

**Proof.** Suppose that \((a, b)_2 = 0\), \(c \neq b\) and \(b \leq a + c\).

We now suppose that \(b\) does not divide \(c\) in base 2. Thus, there exists an \(0 \leq j \leq l\) such that \(c_j \neq b_j\). Thus, \(b_j = 1\) and \(c_j = 0\).

As \(b \leq a + c\), we have \(b_j \leq (a + c)_j\). As \(b_j = 1\), we conclude that \((a + c)_j = 1\). As \((a, c)_2 = 0\), in view of Corollary 2.2.5 we conclude that \(a_j = 1\).
In particular, \( a_j = b_j = 1 \). A contradiction as \((a, b)_2 = 0\).

\[ \square \]

**Remark.** (1) The hypothesis \( c \neq b \) is necessary. Here is an example. Let \( c = b = 3 \) and \( a = 4 \). Recall that \( 3_2 = (11) \), \( 4_2 = (100) \) and \( 7_2 = (111) \). Thus, \((a, b)_2 = 0\) and \( b \preceq a + c \). However, \( b \) does not divide \( c \) in base 2.

(2) Also, the hypothesis \((a, c)_2 = 0\) is necessary. Here is an example. Let \( c = 9 \), \( b = 10 \) and \( a = 5 \). Note that \( 5_2 = (101) \), \( 9_2 = (1001) \), \( 10_2 = (1010) \) and \( 14_2 = (1110) \). Thus, \( b \preceq a + c \). However, \( b \) does not divide \( c \) in base 2.

### 2.4 Arbitrary base

In this section, we consider the case of base \( B \).

Recall that \( B \) is a positive integer such that \( B \geq 2 \).

We start by recalling the following basic theorem.

**Theorem 2.4.1.** Any non-negative integer \( a \) can be uniquely written as

\[
a = \sum_{i=0}^{k} a_i B^i,
\]

where \( 0 \leq a_i \leq B - 1 \) and \( a_k \neq 0 \) for positive \( a \).

Based on the theorem, we can analogously define the notions of greatest common divisor and divisibility in base \( B \). Most of the consideration in the previous sections remains true with similar proofs. We skip the details.
We just state what does not see to work directly. Namely, Corollary 2.2.5, property (3) of Proposition 2.3.2 and Proposition 2.3.8.
In this chapter, we describe certain generalities regarding $B$-multiplicative arithmetic functions. In §3.1, we start with a motivating example of the Thue-Morse sequence. In §3.2, we describe the definition. In §3.3, we describe certain examples. In §3.4, we describe certain general properties.

3.1 Thue-Morse sequence

In this section, we describe certain generalities regarding the Thue-Morse sequence.

We start with the following basic identity.

**Proposition 3.1.1.**

\[
\prod_{n=0}^{\infty} (1 + x^{2^n}) = \sum_{j=0}^{\infty} x^j.
\]

**Proof.** Note that

\[
\prod_{n=0}^{\infty} (1 + x^{2^n}) = \sum_{k=0}^{\infty} b_k x^{\sum_{j=0}^{k} a_j 2^j},
\]

where $a_j = 0$ or $1$ and $b_k$ denotes the number of ways of writing $\sum_{j=0}^{k} a_j 2^j$ in base 2.

In view of Theorem 2.1.1, this finishes the proof.

\[\square\]

**Remark.** By induction, we can show that

\[
\prod_{n=0}^{m-1} (1 + x^{2^n}) = \sum_{i=0}^{2^m-1} x^i.
\]

Here $m$ is a positive integer. This gives another proof of the proposition.

We now consider the following variant.
Let
\[
\prod_{n=0}^{\infty} (1 - x^{2^n}) = \sum_{j=0}^{\infty} a_j x^j.
\] (3.1.2)

Here are some examples.

**Example 3.1.2.** By direct computation, \(a_0 = 1, a_1 = a_2 = -1, a_3 = 1, a_4 = -1, a_5 = a_6 = 1\) and \(a_7 = -1\).

We may guess that \(a_j = \pm 1\), for all non-negative integers \(j\). We in fact have the following lemma.

**Lemma 3.1.3.** Let \(j\) be a positive integer and \(j_2 = (j_k \ldots j_1 j_0)\). We have

\[
a_j = (-1)^{\sum_{i=0}^{k} j_k}.
\]

**Proof.** By a similar argument as in the proof of Proposition 3.1.1,
\[
\prod_{n=0}^{\infty} (1 + x^{2^n}) = \sum_{l=0}^{\infty} (-1)^{\sum_{m=0}^{l} a_m} x^{\sum_{m=0}^{l} a_m 2^m},
\] (3.1.3)

where \(a_m = 0\) or \(1\).

In view of Theorem 2.1.1, this finishes the proof.

\[\square\]

**Remark.** Note that \(\sum_{i=0}^{k} j_k\) equals the number of \(1\)'s in the base 2 expansion of \(j\).

We now recall the definition of the Thue-Morse sequence.

**Definition 3.1.4.** Let \(n\) be a non-negative integers and \(n_2 = (n_k \ldots n_1 n_0)\). Let \(t_n = \sum_{i=0}^{k} n_i\) modulo 2. In particular, \(t_n \in \{0, 1\}\). We say that \(\{t_n\}_{n=0}^{\infty}\) is the Thue-Morse sequence.

Here are some examples.
Example 3.1.5. By direct computation or from Example 3.1.2, \( t_0 = 0, t_1 = t_2 = 1, t_3 = 0, t_4 = 1, t_5 = t_6 = 0 \) and \( t_7 = 1 \).

We have the following basic property.

**Proposition 3.1.6.** If \( (n,m)_2 = 0 \), then \( a_{n+m} = a_na_m \).

**Proof.** Suppose that \( (n,m)_2 = 0 \) and \( nm \neq 0 \).

In view of Corollary 1.2.5, it thus follows that the numbers of 1’s in the binary expansion of \( n + m \) equals the number of 1’s in the binary expansion of \( n \) plus the number of 1’s in the binary expansion of \( m \).

In view of the remark following Lemma 2.1.3, this finishes the proof.

\( \square \)

### 3.2 Generalities

In this section, we describe certain generalities regarding multiplicative functions in an arbitrary base.

Recall that \( B \) be a positive integer such that \( B \geq 2 \).

Let \( f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C} \) an arithmetic function.

Recall that \( f \) is said to be multiplicative if \( (n,m) = 1 \), then \( f(nm) = f(n)f(m) \).

Arithmetic functions arising in Number Theory are often multiplicative. For example, Euler’s \( \phi \)-function.

Keeping in mind analogies in Chapter 1, we now define the notion of being multiplicative in base \( B \).
**Definition 3.2.1.** An arithmetic function $f$ is said to be $B$-multiplicative (or BM) if $f(0) = 1$ and $(n, m)_B = 0$, then $f(n + m) = f(n)f(m)$.

Here is an example.

**Example 3.2.2.** By Proposition 3.1.6, the sequence $\{a_n\}_{n=0}^{\infty}$ is 2-multiplicative.

We also make the following definition.

**Definition 3.2.3.** An arithmetic function $f$ is said to be homogenously $B$-multiplicative (HBM) if $f$ is $B$-multiplicative and $f(mB^k) = f(m)$, for all non-negative integers $m$ and $k$.

Here is an example.

**Example 3.2.4.** If $m_2 = (m_1...m_1m_0)$, then $(m2^k)_2 = (m_1...m_1m_00...0)$. Here there are $k$-zeros in the end and the equality follows from the definition of base 2 expansion. Thus, the Thue-Morse sequence is homogenously 2-multiplicative.

We have the following useful Lemma regarding characterisation of BM functions.

**Lemma 3.2.5.** Let $m_B = (m_1...m_1m_0)$. An arithmetic function $f$ is BM if and only if

$$f(m) = \prod_{i=0}^{l} f(m_iB^i), \quad (3.2.1)$$

for all non-negative integers $m$.

**Proof.** We first suppose that $f$ is $BM$ and show the equality (3.2.1).

By definition of base $B$ expansion,

$$(m_iB^i)_B = (m_0...0).$$
Here there are _i_-zeros in the end.

Thus, for \(0 \leq i \neq j \leq l\) we have

\[(m_i B^i, m_j B^j)_B = 0.\]

By the definition of BM function, the equality (3.2.1) follows.

We now suppose that (3.2.1) holds and show \(f\) is BM.

We first note that \(f(0) = 1\).

Let \(n\) and \(m\) be non-negative integers such that \((n, m)_B = 0\), i.e. \(n_i m_i = 0\) for \(0 \leq i \leq \min\{k, l\}\), by Lemma 2.2.4.

It follows that

\[(n + m)_j = (\max\{n\max\{k, l\}, m\max\{k, l\}\}, \ldots, \max\{n_1, m_1\}, \max\{n_0, m_0\}).\]  

(3.2.2)

Thus, we have

\[f(n + m) = \prod_{t=0}^{\max\{k, l\}} f(\max\{n_t, m_t\} B^t).\]

As \(n_i m_i = 0\) for \(0 \leq i \leq \min\{k, l\}\), it follows that \(f(n + m) = f(n)f(m)\) and thus \(f\) is BM.

\[\square\]

**Remark.** It follows that an HBM arithmetic function \(f\) is determined by \(f(k)\), for \(0 \leq k \leq B - 1\). Moreover, these values can be arbitrary.

### 3.3 Generating polynomials

In this section, we consider generating polynomials of B-multiplicative arithmetic functions.

The generating polynomials give a basic invariant of an arithmetic function.

We have the following theorem regarding generating polynomials of BM and HBM functions.
Theorem 3.3.1. (1) An arithmetic function $f$ is BM if and only if

$$
\sum_{n=0}^{B^l-1} f(n)x^n = \prod_{k=0}^{l-1} \left( \sum_{m=0}^{B-1} f(mB^k)x^{mB^k} \right),
$$

(3.3.1)

for all positive integers $l$.

(2) An arithmetic function $f$ is HBM if and only if

$$
\sum_{n=0}^{B^l-1} f(n)x^n = \prod_{k=0}^{l-1} \left( \sum_{m=0}^{B-1} f(m)x^{mB^k} \right),
$$

(3.3.2)

for all positive integers $l$.

**Proof.** Suppose that $f$ is BM. We prove that the equality (3.3.1) holds by induction on $l$.

There is nothing to prove for $l = 1$. Suppose that

$$
\sum_{n=0}^{B^l-1} f(n)x^n = \prod_{k=0}^{l-1} \left( \sum_{m=0}^{B-1} f(mB^k)x^{mB^k} \right). \quad (3.3.3)
$$

We now consider

$$
\prod_{k=0}^{l} \left( \sum_{m=0}^{B-1} f(mB^k)x^{mB^k} \right) = \prod_{k=0}^{l-1} \left( \sum_{m=0}^{B-1} f(mB^k)x^{mB^k} \right) \cdot \sum_{m=0}^{B-1} f(mB^l)x^{mB^l}. \quad (3.3.4)
$$

In view of (3.3.3), this equals

$$
\sum_{n=0}^{B^l-1} f(n)x^n \cdot \sum_{m=0}^{B-1} f(mB^l)x^{mB^l} = \sum_{n=0}^{B^l-1} \sum_{m=0}^{B-1} f(n)f(mB^l)x^{n+mB^l}. \quad (3.3.5)
$$

We claim that

$$
f(n + mB^l) = f(n)f(mB^l). \quad (3.3.6)
$$

In view of the definition of BM functions, it suffices to show that for non-negative integers $m, n, l$ such that $0 \leq n \leq B^l - 1$, we have

$$
(n, mB^l)_B = 0. \quad (3.3.7)
$$
This can be shown as follows.

Recall that if \( m_B = (m_k \ldots m_1 m_0) \), then \( (mB^l)_B = (m_k \ldots m_1 m_0 0 \ldots 0) \). Here there are \( l \)-zeros in the end. As \( 0 \leq n \leq B^l - 1 \), it follows that there are at most \( l \)-digits in the base \( B \) expansion of \( n \). In view of the fact that the last \( l \) digits in the base \( B \) expansion of \( mB^l \) are zero, this finishes the proof of (3.3.7).

In view of (3.3.6), the expression (3.3.5) equals

\[
\sum_{n=0}^{B^l-1} \sum_{m=0}^{B-1} f(n + mB^l)x^{n+mB^l} = \sum_{k=0}^{B^{l+1}-1} f(k)x^k. \tag{3.3.8}
\]

Here the last equality follows from the fact that any integer \( k \) such that \( 0 \leq k \leq B^{l+1} - 1 \) can be uniquely written as \( n + mB^l \), where \( 0 \leq n \leq B^l - 1 \) and \( 0 \leq m \leq B - 1 \).

By induction, this finishes the proof of (3.3.1).

Suppose that the equality (3.3.1) holds. We prove that \( f \) is BM.

Note that

\[
\prod_{k=0}^{l-1} \left( \sum_{m=0}^{B-1} f(mB^k)x^{mB^k} \right) = \sum_{m=0}^{B^l-1} \prod_{m_i=0}^{i-1} f(m_iB^i)x^m. \tag{3.3.9}
\]

Indeed, this follows from an argument similar to the proof of (3.1.1).

In view of (3.3.1), it follows that

\[
f(m) = \prod_{i=0}^{k} f(m_iB^i). \tag{3.3.10}
\]

Thus, \( f \) is BM (cf. Lemma 3.2.5).

Suppose that \( f \) is HBM. We prove that the equality (3.3.2) holds.

This follows immediately from part (1) of the Theorem and the definition of HBM functions (cf. Definition 3.2.3).

Suppose that the equality (3.3.2) holds. We prove that \( f \) is HBM.
By a similar argument as in the proof of (3.3.10), we have

\[ f(m) = \prod_{i=0}^{k} f(m_i). \]  

(3.3.11)

We first note that \( f(0) = 1 \).

In view of the base \( B \) description of \( mB^l \) and (3.3.11), it thus follows that \( f(mB^k) = f(m) \) and \( f \) is HBM.

\[ \square \]

3.4 Base \( B \) analogue of classical arithmetic functions

In this section, we consider analogues of a class of classical arithmetic functions in base \( B \) and show that they are indeed multiplicative in the base.

We first recall certain classical arithmetic functions.

Recall that \( n \) is a non-negative integer.

**Definition 3.4.1.** (1). Let \( \mu \) be the Mobius function given by \( \mu(0) = 0, \mu(1) = 1, \mu(n) = (-1)^k \) if \( n \) is a product of \( k \)-distinct primes and \( \mu(n) = 0 \), otherwise.

(2). Let \( \lambda \) be the Carmichael function given by \( \lambda(0) = 1 \) and \( \lambda(n) \) is a smallest positive integer \( m \) such that \( a^m \equiv 1 \mod n \) for all integers \( a \) with \( (a, m) = 1 \), for positive \( n \). By Fermat’s little theorem, \( \lambda(p) = p - 1 \), for prime \( p \). We also have have \( \lambda(n) = lcm\{\lambda(p_1^{a_1}), ..., \lambda(p_r^{a_r})\} \), where \( n = \prod_{i=1}^{r} p_i^{a_i} \) is the prime decomposition.

(3). Let \( d \) be the arithmetic function given by \( d(0) = 1 \) and \( d(n) = \sum_{i|n} i \), otherwise.

(4). Let \( \nu \) be the function given by \( \nu(0) = 0 \) and \( \nu(n) \) equals the number of prime divisors of \( n \), otherwise. We consider the arithmetic function \( 2^{\nu(n)} \).
The above arithmetic functions except the Carmichael function are indeed multiplicative.

We now consider analogue of above classical arithmetic functions in base $B$.

Recall that $B$ is a positive integer such that $B \geq 2$ and $n_B = (n_k...n_1n_0)$.

Our analogue is the following.

**Definition 3.4.2.** (1). Let $\lambda_B$ be the arithmetic function given by

$$\lambda_B(n) = (-1)^{\sum_{i=0}^{k} n_i}.$$  

(2). Let $\mu_B$ be the arithmetic function given by

$$\mu_B(n) = \lambda_B(n),$$  

if $n_i \leq 1$, for $0 \leq i \leq k$ and $\mu_B(n) = 0$, otherwise.

(3). Let $d_B(n)$ be the arithmetic function given by

$$d_B(n) = \prod_{i=0}^{k} (1 + n_i).$$

(4). Let $\nu_B(n)$ be the function given by $\nu_B(n)$ equals the number of non-zero $n_i$'s in the base $B$ expansion of $n$. We consider the arithmetic function $2^{\nu_B(n)}$.

Note that the sequence $\{\lambda_2(n)\}_{n=0}^{\infty}$ is nothing but the Thue-Morse sequence.

We have the following theorem regarding the above arithmetic functions.

**Theorem 3.4.3.** The arithmetic functions $\lambda_B$, $\mu_B$, $d_B$ and $2^{\nu_B}$ are HBM.

**Proof.** Let $n$ and $m$ be non-negative integers such that $(n, m)_B = 0$. Recall that $(n + m)_i = n_i + m_i$, for $0 \leq i \leq \max\{k, l\}$. Also, we have $nB^r = (n_k...n_1n_0...0)$. Here $r$ is a non-negative integer and
there are $r$-zeros in the end.

We start with the arithmetic functions in the order (1)-(4).

In view of the definition of $\lambda_B$ and the above description of the base $B$ expansion of $n + m$ and $nB^r$, it follows that

$$\lambda_B(n + m) = \lambda_B(n)\lambda_B(m), \lambda_B(nB^r) = \lambda_B(n).$$  \hfill (3.4.1)

Thus, the arithmetic function $\lambda_B$ is HBM.

We now consider the arithmetic function $\mu_B$.

We first show that

$$\lambda_B(n + m) = \lambda_B(n)\lambda_B(m).$$  \hfill (3.4.2)

Let us first consider the case when $0 \leq (n + m)_i \leq 1$, for $0 \leq i \leq \max\{k, l\}$. In this case, $\mu_B(m + n) = \lambda_B(m + n)$, $\mu_B(n) = \lambda_B(n)$ and $\mu_B(m) = \lambda_B(m)$. As $\lambda_B$ is HBM, it follows that (3.4.2) holds.

We now consider the case when $(n + m)_j > 1$, for some $0 \leq j \leq \max\{k, l\}$. In particular, $n_j > 1$ or $m_j > 1$. Thus, $\mu_B(m + n) = 0$ and $\mu_B(n) = 0$ or $\mu_B(m) = 0$. In particular, the equality (3.4.2) holds.

In view of the base $B$ expansion of $nB^r$, we can similarly show that $\mu_B(nB^r) = \mu_B(n)$.

Thus, the arithmetic function $\mu_B$ is HBM.

We now consider the arithmetic function $d_B$.

We have

$$d_B(n + m) = \prod_{i=0}^{\max\{k, l\}} (1 + (n + m)_i) = \prod_{i=0}^{\max\{k, l\}} (1 + n_i + m_i)$$  \hfill (3.4.3)

Here by abuse of notation, we let $n_j = 0$ and $m_j = 0$, for $j > \min\{k, l\}$.

Note that

$$d_B(n)d_B(m) = \prod_{i=0}^{\max\{k, l\}} (1 + n_i + m_i + n_im_i) = \prod_{i=0}^{\max\{k, l\}} (1 + n_i + m_i).$$  \hfill (3.4.4)
The last equality follows from the fact that $n_i m_i = 0$, for $0 \leq i \leq \max\{k, l\}$.

In view of (3.4.3) and (3.4.4), it follows that $d_B(n + m) = d_B(n)d_B(m)$.

In view of the base $B$ expansion of $nB^r$, we can similarly show that $d_B(nB^r) = d_B(n)$.

Thus, the arithmetic function $d_B$ is HBM.

We now consider the arithmetic function $2^\nu$.

In view of the base $B$ expansion of $n + m$ and the hypothesis that $(n, m)_B = 0$, it follows that

$$\nu(n + m) = \nu(n) + \nu(m).$$

(3.4.5)

In particular, $2^{\nu(n+m)} = 2^{\nu(n)}2^{\nu(m)}$.

In view of the base $B$ expansion of $nB^r$, we can similarly show that $2^{\nu(nB^r)} = 2^{\nu(n)}$.

Thus, the arithmetic function $2^\nu$ is HBM.

□

Remark. (1). The condition that $(n, m)_B = 0$ is necessary for multiplicativity to hold in general. Here is an example. Let $B = 2$ and $n = m = 1$. Then, $\lambda_2(n)\lambda_2(m) = (-1) \cdot (-1) \neq 1 = \lambda_2(n + m)$.

(2). By the identical argument, the arithmetic function $A^\nu$ is HBM, for any positive integer $A$.

We have the following proposition regarding the generating polynomial of the above arithmetic functions, except the arithmetic function $d_B$.

**Proposition 3.4.4.** (1). Generating polynomial of the arithmetic function $\lambda_B$ is given by

$$\sum_{n=0}^{B^l-1} \lambda_B(n)x^n = \prod_{k=0}^{l-1} \left( \sum_{m=0}^{B-1} (-1)^m x^{mB^k} \right) = \prod_{k=0}^{l-1} \left( 1 - (-x^{B^k})^B \right)(1 + x^{B^k})^{-1}.$$
(2). Generating polynomial of the arithmetic function $\mu_B$ is given by
\[ \sum_{n=0}^{B^l-1} \mu_B(n) x^n = \prod_{k=0}^{l-1} (1 - x^{B^k}). \]

(3). Generating polynomial of the arithmetic function $2^\nu$ is given by
\[ \sum_{n=0}^{B^l-1} 2^{\nu(n)} x^n = \prod_{k=0}^{l-1} (1 + \sum_{m=1}^{B-1} 2x^{mB^k}). \]

**Proof.** This follows immediately from Theorem 3.3.1, Theorem 3.4.3 and the definition of the arithmetic functions.
\[ \square \]
Chapter 4

Power series associated to $B$-multiplicative arithmetic functions

In this chapter, we consider polynomials and power series associated to $B$-multiplicative arithmetic functions.
In §4.1, we consider the associated summatory function. In §4.2, we consider the associated power series.

4.1 Summatory function

In this section, we consider the summatory function associated to a $B$-multiplicative arithmetic function.

Recall that $n$ is a non-negative integer and $B$ a positive integer such that $B \geq 2$.

Let $f$ be a BM arithmetic function.

We define the summatory function of $f$ as follows.

**Definition 4.1.1.** Let $S_f$ be the summatory function of $f$ given by

$$S_f(n) = \sum_{i=0}^{n-1} f(i).$$

We set $S_f(0) = 0$.

The summatory function accounts for average behaviour of the arithmetic function.

We have the following basic lemma regarding the summatory functions.

**Lemma 4.1.2.** Let $f$ be HBM arithmetic function. For a non-negative integer $m$ such that $0 \leq m < B$, we have

$$S_f(m + nB) = f(n)S_f(m) + S_f(n)S_f(B).$$  (4.1.1)
Proof. By definition, we have

\[ S_f(m + nB) = \sum_{k=0}^{nB-1} f(k) + \sum_{j=nB}^{nB+m-1} f(j) \]

\[ = \sum_{h=0}^{n} \sum_{j=0}^{B-1} f(j + hB) + \sum_{j=0}^{m-1} f(j + nB) \]

\[ = \sum_{j=0}^{B-1} f(j) \sum_{h=0}^{n} f(h) + f(n) \sum_{j=0}^{m-1} f(j) \]

\[ = S_f(n)S_f(B) + f(n)S_f(m). \]

(4.1.2)

Here the second equality follows from the remainder theorem and the second last from (3.3.6).

□

4.2 Power series

In this section, we consider certain power series associated to \( B \)-multiplicative arithmetic functions.

Let the notation be as in the previous section.

We set \( S(n) = S_f(n) \).

Let \( S = S(B) \) and \( M = \max_{0 \leq n \leq B-1} |f(n)| \).

As \( f(0) = 1 \), we have

\[ M \geq 1. \] (4.2.1)

Note that the numbers of digits \( l \) in the base \( B \) expansion of \( m \) equals \( \left\lfloor \frac{\ln(m)}{\ln(B)} \right\rfloor \). In view of Lemma 3.2.5, it thus follows that

\[ f(m) \leq M^l = M \left\lfloor \frac{\ln(m)}{\ln(B)} \right\rfloor. \] (4.2.2)

We consider the hypothesis

\[ |S| > M \] (4.2.3)

Note that this holds if \( f(n) \geq 0 \), for all non-negative integers \( n \) and \( f \) is not identically zero on positive integers.
In what follows, we suppose this possibly stronger hypothesis.

Let \( x \in \mathbb{R} \) such that \( 0 \leq x < B \), let
\[
x = \sum_{j \geq 0} \beta_j B^{-j}
\]
be the base \( B \) expansion of \( x \), where \( 0 \leq \beta_j < B \).

For \( k \in \mathbb{Z}_{\geq 0} \), let
\[
\alpha_k = \lfloor B^k x \rfloor = \sum_{j=0}^{k} \beta_j B^{k-j}.
\]
(4.2.5)

For \( k \in \mathbb{Z}_{\geq 1} \), note that
\[
\alpha_k = \beta_k + \alpha_{k-1}B.
\]
(4.2.6)

We now consider the following definition.

**Definition 4.2.1.** Let \( \rho : [0, B) \to \mathbb{R} \) be the function given by
\[
\rho(x) = \sum_{j=1}^{\infty} f(\alpha_{j-1})S_f(\beta_j)S^{-j},
\]
where \( x \in \mathbb{R}_{\geq 0} \).

(4.2.7)

This can be considered as an analogue of expansion with respect to \( B \)-multiplicative arithmetic functions.

We have the following proposition regarding the convergence.

**Proposition 4.2.2.** The power series defining \( \rho \) is convergent on the domain.

**Proof.** As \( 0 \leq \beta_j < B \), we have
\[
S_f(\beta_j) \leq S.
\]
(4.2.8)

Thus,
\[
S_f(\beta_j)S^{-j} \leq S^{1-j}.
\]
(4.2.9)
In view of (4.2.2), we have
\[ f(\alpha_k) \leq M \frac{\ln(\alpha_k)}{\ln(B)}. \]  
(4.2.10)

By definition (cf. (4.2.5)), we have
\[ \alpha_k \leq B^k x. \]  
(4.2.11)

Thus,
\[ \frac{\ln(\alpha_k)}{\ln(B)} \leq k + \frac{\ln(x)}{\ln(B)}. \]  
(4.2.12)

In view of these inequalities, it follows that
\[ \rho(x) \leq \sum_{j=1}^{\infty} M^{j-1} \frac{\ln(x)}{\ln(B)} S^{1-j} = M^{\frac{\ln(x)}{\ln(B)}} \sum_{j=1}^{\infty} (MS^{-1})^{j-1}. \]  
(4.2.13)

Finally, from (4.2.3) the above geometric series thus converges.

This finishes the proof.
\[ \square \]

We have the following lemma regarding certain values of \( \rho. \)

**Lemma 4.2.3.** If \( n \) is an integer such that \( 0 \leq n < B, \) then \( \rho(n) = 0. \)

**Proof.** This follows immediately as \( \beta_0 = n \) and \( \beta_j = 0, \) for \( j \in \mathbb{Z}_{>0}. \)
\[ \square \]

We now consider the following definitions.

**Definition 4.2.4.** For \( k \in \mathbb{Z}_{\geq 0}, \) let \( \psi_k : [0, B) \to \mathbb{R} \) be the function given by
\[ \psi_k(x) = S^{\frac{\ln(x)}{\ln(B)}} \cdot S_f(\alpha_k). \]  
(4.2.14)

Let \( \psi : [0, B) \to \mathbb{R} \) be the function given by
\[ \psi(x) = S^{\frac{\ln(x)}{\ln(B)}} \cdot (S_f(\lfloor x \rfloor) + \rho(x)). \]  
(4.2.15)

In what follows, we consider properties of these functions.
We have the following alternate description of $\psi_k$.

**Lemma 4.2.5.** For $0 \leq x < B$,

$$
\psi_k(x) = S^{\ln(x) \over \ln(B)} \cdot (S_f([x]) + \sum_{j=1}^{k} f(\alpha_{j-1})S_f(\beta_j)S^{-j}). \quad (4.2.16)
$$

**Proof.** We first show that

$$
S_f(\alpha_k) = S^k \cdot (S_f([x]) + \sum_{j=1}^{k} f(\alpha_{j-1})S_f(\beta_j)S^{-j}). \quad (4.2.17)
$$

In view of (4.2.6) and Lemma 4.1.2, it follows that

$$
S_f(\alpha_k) = f(\alpha_{k-1})S_f(\beta_k) + S_f(\alpha_{k-1})S. \quad (4.2.18)
$$

Repeating the procedure $k$-times, we obtain (4.2.17).

By definition of $\psi_k$, the previous equality implies the identity in the lemma.

We have the following consequence.

**Proposition 4.2.6.** For $0 \leq x < B$,

$$
\psi(x) = \lim_{k \to \infty} \psi_k(x). \quad (4.2.19)
$$

**Proof.** This immediately follows from Lemma 4.2.5 and definitions.

We have the following property of $\psi$.

**Proposition 4.2.7.** For $0 \leq x < 1$,

$$
\psi(Bx) = \psi(x). \quad (4.2.20)
$$
In particular, the function $\psi(B^x)$ is periodic with period 1 on its domain.

**Proof.** By definition, note that

$$\psi_k(B^x) = \psi_{k+1}(x).$$  \hspace{1cm} (4.2.21)

We now let $k \to \infty$.

The previous proposition finishes the proof. $\square$

We now consider the rate of convergence of $\psi_k \to \psi$, as $k \to \infty$.

Let $\eta = \frac{M}{S}$. Note that $\eta < 1$ (cf. (4.2.3)).

**Proposition 4.2.8.** For $0 \leq x < B$,

$$|\psi_k(x) - \psi(x)| \ll S^{-\ln(x)/\ln(M)} \eta^k.$$  \hspace{1cm} (4.2.22)

**Proof.** In view of Lemma 4.2.5 and definitions, it follows that

$$|\psi_k(x) - \psi(x)| = \left| S^{-\ln(x)/\ln(M)} \sum_{j=k+1}^{\infty} f(\alpha_{j-1}) S_f(\beta_j) S^{-j} \right|.  \hspace{1cm} (4.2.23)$$

By a similar argument as in the proof of (4.2.13), it follows that

$$|\psi_k(x) - \psi(x)| \leq S^{-\ln(x)/\ln(M)} \eta^k \cdot \frac{M}{1 - \eta}.  \hspace{1cm} (4.2.24)$$

This finishes the proof. $\square$

We now consider the continuity of $\psi$.

**Theorem 4.2.9.** The function $\psi$ is continuous on $(0, B)$.

**Proof.** We fix $x_0 \in (0, B)$.  

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Let $x \in (0, B)$ such that $|x - x_0| < \frac{1}{10^k}.

For sufficiently large $k$, we first show that

$$|\psi_k(x) - \psi_k(x_0)| \leq x_0 \eta^k. \tag{4.2.25}$$

Let us first consider the case $x \geq x_0$.

In this case,

$$|\psi_k(x) - \psi_k(x_0)| \leq S \left( \frac{\ln(B^k x_0)}{\ln(B)} \right) \left( S_f([B^k x]) - S_f([B^k x_0]) \right) \leq S \left( \frac{\ln(B^k x_0)}{\ln(B)} \right) f([B^k x]). \tag{4.2.26}$$

Here the last inequality follows from the fact that $0 \leq B^k x - B^k x_0 < 1$.

In view of (4.2.3) and arguments similar to those in the proof of Proposition 4.2.2, we have

$$S^{- \frac{\ln(B^k x_0)}{\ln(B)}} f([B^k x]) \leq S^{- \frac{\ln(B^k x_0)}{\ln(B)}} M^{- \frac{\ln(B^k x_0)}{\ln(B)}}. \tag{4.2.27}$$

Note that

$$S^{- \frac{\ln(B^k x_0)}{\ln(B)}} M^{- \frac{\ln(B^k x_0)}{\ln(B)}} \leq S^{- \frac{\ln(B^k x_0)}{\ln(B)}} M^{- \frac{\ln(B^k x_0)}{\ln(B)}} M^{\frac{1}{\ln(B)}} = \eta^k \eta \frac{\ln(x_0)}{\ln(B)} M^{\frac{1}{\ln(B)}}. \tag{4.2.28}$$

Here the first inequality follows from the inequality

$$\ln(B^k x) \leq \ln(B^k x_0) + 1, \tag{4.2.29}$$

for sufficiently large $k$.

Combining the above inequalities, we obtain (4.2.25).

The case when $x < x_0$ is similar and we skip the details.

We now prove the continuity.
From the triangle inequality, we have

\[ |\psi(x) - \psi(x_0)| \leq |\psi(x) - \psi_k(x)| + |\psi_k(x) - \psi_k(x_0)| + |\psi_k(x_0) - \psi(x_0)|. \tag{4.2.30} \]

In view of (4.2.22), (4.2.25) and the fact that \( \eta < 1 \), it follows that for given \( \epsilon > 0 \) there exists \( k \) possibly dependent on \( x_0 \) such that

\[ |\psi(x) - \psi(x_0)| < \epsilon. \tag{4.2.31} \]

This finishes the proof.

\[ \square \]

We have the following immediate consequence.

**Proposition 4.2.10.** Let \( \sigma \) be the function given by

\[ \sigma(x) = S^{-x}(S(\lfloor B^x \rfloor) + \rho(B^x)), \tag{4.2.32} \]

where \( x < 1 \).

Then, \( \sigma \) is continuous on its domain.

**Proof.** Note that

\[ \sigma(x) = \psi(B^x). \tag{4.2.33} \]

As the function \( x \mapsto B^x \) is continuous, the corollary immediately follows from the previous theorem.

\[ \square \]

For \( n \in \mathbb{Z}_{\geq 1} \) such that \( n \leq B \), we have

\[ S(n) = S^{x_n} \sigma(x_n), \tag{4.2.34} \]

where \( x_n = \frac{\ln(n)}{\ln(B)} \).

Let

\[ \mu_0 = \min_{0 \leq x < 1} \sigma(x), \mu_1 = \max_{0 \leq x < 1} \sigma(x) \tag{4.2.35} \]
and
\[ \delta = \frac{\ln(S)}{\ln(B)}. \]  \hspace{1cm} (4.2.36)

We have the following corollary.

**Corollary 4.2.11.** The sequence \( \{n^{-\delta}S(n)\} \) is a dense subset of \([\mu_0, \mu_1]\).

**Proof.** Note that
\[ n^{-\delta}S(n) = \sigma(x_n). \]  \hspace{1cm} (4.2.37)

As the sequence \( \{x_n\} \) modulo 1 is dense in \([0, 1)\), the corollary follows from the previous proposition.
\[ \square \]
Chapter 5

Dirichlet series of $B$-multiplicative arithmetic functions

In this chapter, we consider Dirichlet series of $B$-multiplicative arithmetic functions. In §5.1, we first consider Kloosterman sum. In §5.2, we consider the Dirichlet series.

5.1 Kloosterman sum of $B$-multiplicative arithmetic functions

In this section, we consider Kloosterman sum of $B$-multiplicative arithmetic functions.

Let $B$ be a positive integer such that $B \geq 2$ and $f$ a HBM arithmetic function.

We now define corresponding Kloosterman sum.

**Definition 5.1.1.** Let $a$ be an integer and $l, q$ positive integers. The Kloosterman sum $S_f(l, \frac{a}{q})$ is given by

$$S_f(l, \frac{a}{q}) = \sum_{n=0}^{B^l-1} f(n) e(n \cdot \frac{a}{q}),$$

where $e(z) = \exp(2\pi iz)$.

We have the following description of the Kloosterman sum.

**Proposition 5.1.2.**

$$S_f(l, \frac{a}{q}) = \prod_{k=0}^{l-1} \left( \sum_{m=0}^{B-1} f(m) e(m \frac{aB^k}{q}) \right).$$

**Proof.** This follows immediately from part (2) of Theorem 3.3.1, with $x = e(\frac{a}{q})$.

□
5.2 Dirichlet series of $B$-multiplicative arithmetic functions

In this section, we consider Dirichlet series of $B$-multiplicative arithmetic functions. This section is partly motivated by [1].

We first recall certain generalities regarding the Riemann Zeta function.

**Definition 5.2.1.** The Riemann Zeta function $\zeta$ is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s \in \mathbb{C}$.

We have the following fundamental theorem regarding the Riemann Zeta function.

**Theorem 5.2.2.** (1). The sum defining the Riemann Zeta function $\zeta(s)$ converges absolutely for $\Re(s) > 1$.

(2). The Riemann Zeta converges can be analytically continued to the entire complex plane, except a simple pole at $s = 1$.

The Riemann zeta function admits an Euler product decomposition and also satisfies a functional equation.

Suppose that $f$ is a HBM arithmetic function such that

$$f(n) \ll n^c,$$  \hspace{1cm} (5.2.1)

for some constant $c \in \mathbb{R}_{>0}$.

Let $a$ be an integer and $q$ a positive integer.

We now define the Dirichlet series zeta function of $f$. It is often also referred as the zeta func-
Definition 5.2.3. The Dirichlet series $\varphi_f(s, \frac{a}{q})$ of $f$ is given by

$$\varphi_f(s, \frac{a}{q}) = \sum_{n=0}^{\infty} f(n)e(n \cdot \frac{a}{q})(n+1)^{-s},$$

where $s \in \mathbb{C}$.

Note that when $f$ is the constant function $1$ and $a = q = 1$, the Dirichlet series equals the Riemann zeta function.

We have the following analogue of part (1) of Theorem 5.2.2.

Proposition 5.2.4. The Dirichlet series $\varphi_f(s, \frac{a}{q})$ converges absolutely for $\Re(s) > c + 1$.

Proof. Note that

$$\left| \sum_{n=M}^{m} f(n)e(n \cdot \frac{a}{q})(n+1)^{-s} \right| \leq \sum_{n=M}^{m} |f(n)(n+1)^{-s}| < \sum_{n=M}^{m} (n+1)^{c-\Re(s)}. \quad (5.2.2)$$

Here the former inequality follows from the triangle inequality and the fact that $|e(n \cdot \frac{a}{q})| = 1$. The later inequality follows from the inequality (5.2.1).

In view of part (1) of Theorem 5.2.2, it follows that the right hand side in the above expression converges as $m \to \infty$ for $\Re(s) - c > 1$ i.e. $\Re(s) > c + 1$.

This finishes the proof.

\[\square\]

The analogue of part (2) of Theorem 5.2.2 is somewhat delicate and is as follows.

Theorem 5.2.5. Let $f$ be HBM and $(B, q) = 1$. Then, the Dirichlet series $\varphi_f(s, \frac{a}{q})$ can be analyti-
cally continued to the entire complex plane except simple poles at
\[ s = \frac{1}{l \log B}(\log |S_f(l, \frac{a}{q})| + i \arg(S_f(l, \frac{a}{q})) + 2\pi i j) - k, \]  \hspace{1cm} (5.2.3)

where \( j \in \mathbb{Z}, \, k \in \mathbb{Z}_{\geq 0} \) and \( l \in \mathbb{Z}_{>0} \) such that \( B^l \equiv 1 \mod q. \)

**Proof.** The proof is divided into a few steps.

Let \( l \) be as above and \( n \in \mathbb{Z} \) such that \( 0 \leq n < B^l. \)

As \( f \) is HBM, it follows that
\[ f(n + mB^l) = f(n)f(m). \]  \hspace{1cm} (5.2.4)

Here \( m \) is a non-negative integer.

For \( s \in \mathbb{C} \) such that \( \Re(s) > 1 + c, \) we now have
\[
\varphi_f(s, \frac{a}{q}) = \sum_{n=0}^{B^l-1} \sum_{m=0}^{\infty} f(n + mB^l)e(n \cdot \frac{a}{q})e(m \cdot \frac{a}{q})(n + mB^l + 1)^{-s} \\
= \sum_{n=0}^{B^l-1} f(n)e(n \cdot \frac{a}{q}) \sum_{m=0}^{\infty} f(m)e(m \cdot \frac{a}{q})(B^l(m + 1) - (B^l - n - 1))^{-s} \\
= \sum_{n=0}^{B^l-1} f(n)e(n \cdot \frac{a}{q}) \sum_{m=0}^{\infty} f(m)e(m \cdot \frac{a}{q})(m + 1)^{-s}B^{-ts}\left(1 - \frac{B^l - n - 1}{B^l(m + 1)}\right)^{-s} \\
= \sum_{n=0}^{B^l-1} f(n)e(n \cdot \frac{a}{q}) \sum_{m=0}^{\infty} f(m)e(m \cdot \frac{a}{q})(m + 1)^{-s}B^{-ts}\left(1 + \sum_{k=1}^{\infty} \binom{s + k - 1}{k} (\frac{B^l - n - 1)^k}{(m + 1)^k B^l k}) \right). \]  \hspace{1cm} (5.2.5)

Here the first and third equality follows from the definition. The second follows from (5.2.4) and the fact that \( e(mB^l \cdot \frac{a}{q}) = e(m \cdot \frac{a}{q}), \) as \( B^l \equiv 1 \mod q. \) Finally, the last equality follows from the Binomial theorem.
We thus have

\[ \varphi_f(s, \frac{a}{q}) \cdot B^l s = \sum_{n=0}^{B_l - 1} f(n) e(n \cdot \frac{a}{q}) \sum_{m=0}^{\infty} f(m) e(m \cdot \frac{a}{q})(m + 1)^{-s} \left( 1 + \sum_{k=1}^{\infty} \binom{s + k - 1}{k} \frac{(B_l - n - 1)^k}{(m + 1)^k B^l k} \right) \]

\[ = \varphi_f(s, \frac{a}{q}) \cdot S_f(l, \frac{a}{q}) + \sum_{k=1}^{\infty} \binom{s + k - 1}{k} \frac{(B_l - n - 1)^k}{(m + 1)^k B^l k} \sum_{n=0}^{B_l - 1} f(n) e(n \cdot \frac{a}{q}) \sum_{m=0}^{\infty} f(m) e(m \cdot \frac{a}{q})(m + 1)^{-s}. \]

Here the last equality follows from the definitions.

It follows that

\[ \varphi_f(s, \frac{a}{q}) \cdot (B^l s - S_f(l, \frac{a}{q})) = \sum_{k=1}^{\infty} \binom{s + k - 1}{k} \frac{(B_l - n - 1)^k}{B^l k} \sum_{n=0}^{B_l - 1} f(n) e(n \cdot \frac{a}{q}) \sum_{m=0}^{\infty} f(m) e(m \cdot \frac{a}{q})(m + 1)^{-s-k} \]

\[ = \sum_{k=1}^{\infty} \binom{s + k - 1}{k} \varphi_f(s + k, \frac{a}{q}) \sum_{n=0}^{B_l - 1} f(n) e(n \cdot \frac{a}{q}) \left( 1 - \frac{n + 1}{B^l} \right)^k. \]

\[(5.2.6)\]

In particular, we obtain an expression for \( \varphi_f(s, \frac{a}{q}) \) in terms of \( \varphi_f(s + k, \frac{a}{q}) \), where \( k \in \mathbb{Z}_{\geq 0} \). Thus, it suffices to define \( \varphi_f(s, \frac{a}{q}) \), for \( s \in \mathbb{C} \) such that \( c < \Re(s) \leq c + 1 \) and then repeat the procedure.

By definition of \( \varphi_f(s, \frac{a}{q}) \), it follows that

\[ \lim_{k \to \infty} \varphi_f(s, \frac{a}{q}) = 1. \]

\[(5.2.7)\]

We also note that

\[ \sum_{n=0}^{B_l - 1} f(n) e(n \cdot \frac{a}{q}) \left( 1 - \frac{n + 1}{B^l} \right)^k \leq \left( \frac{B_l - 1}{B^l} \right)^k, \]

\[(5.2.8)\]

as \( k \to \infty \).

Let

\[ A(s, n) = \frac{s(s + 1) \cdots (s + n)}{n!} = \frac{\Gamma(s + n + 1)}{\Gamma(s) \Gamma(n + 1)}. \]

Here \( \Gamma \) is the usual Gamma function.
By [3, p. 47], we have

$$A(s, n) = \frac{1}{\Gamma(s)} ((n + 1)^s \cdot (1 + \frac{1}{2}(n + 1)^{-1} \frac{s}{s - 1} + O((n + 1)^{-2}))).$$

Thus,

$$|A(s, n)| \ll_s (n + 1)^{\Re(s)}.$$

For $a \in \mathbb{C}$ such that $|a| < 1$ and $s$ in a compact subset, we know that $\sum_{n=1}^{\infty} (n + 1)^{\Re(s)} a^n$ converges absolutely. We thus have the following convergence.

**(C)** For $a \in \mathbb{C}$ such that $|a| < 1$ and $s$ in a compact subset, the summation $\sum_{n=1}^{\infty} A(s, n) a^n$ converges absolutely.

In view of (5.2.7), (5.2.8) and (C), it follows that the summation in the expression (5.2.6) converges absolutely. In particular, we can extend the Dirichlet series meromorphically to the indicated region.

We now consider a slightly different formulation.

From now, we let

$$\Phi(s, \frac{a}{q}) = \Gamma(s) \varphi_f(s, \frac{a}{q}).$$

be the completed Dirichlet series.

From (5.2.6), we have

$$\Phi(s, \frac{a}{q})(B^l s - S_f(l, \frac{a}{q})) = \sum_{k=1}^{\infty} \Gamma(s) \binom{s + k - 1}{k} \varphi_f(s + k, \frac{a}{q}) \sum_{n=0}^{B^l - 1} f(n) e(n \cdot \frac{a}{q}) \left(1 - \frac{n + 1}{B^l}\right)^k$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} \Phi_f(s + k, \frac{a}{q}) \sum_{n=0}^{B^l - 1} f(n) e(n \cdot \frac{a}{q}) \left(1 - \frac{n + 1}{B^l}\right)^k$$

(5.2.10)

In view of (5.2.7) and (5.2.8), it follows that the above infinite sum converges uniformly on compact subsets which exclude poles of the Gamma function.

Going back to the convergence consideration before the completed Dirichlet series, we can define
\( \varphi(s, \frac{a}{q}) \) for \( s \in \mathbb{C} \) except when \( B^{ls} - S_f(l, \frac{a}{q}) = 0 \).

The last equality holds if and only if

\[
\hat{s} = \frac{1}{l \log B} (\log |S_f(l, \frac{a}{q})| + i \arg(S_f(l, \frac{a}{q})) + 2\pi i j), \tag{5.2.11}
\]

where \( j \in \mathbb{Z} \).

As we are defining the meromorphic continuation via translation by positive integers, it follows that the Dirichlet series admits a meromorphic continuation on the indicated region in the theorem.

This finishes the proof. □

**Remark.** (1). We have a canonical choice for \( l \), namely the order of \( B \) modulo \( q \).

(2). Perhaps, a natural question is whether the set of poles in the Theorem is optimal. We are tempted to think that it is in fact smaller in certain cases.

(3). We can ask whether the Dirichlet series \( \varphi_f(s, \frac{a}{q}) \) satisfies a functional equation as in the case of Riemann Zeta function.

(4). We can also ask whether special values of the Dirichlet series are algebraic up to periods as in the case of Riemann Zeta function. It would be interesting to explore arithmetic aspects of the Dirichlet series.
References


