Quantum interference and Coulomb interaction in arrays of tunnel junctions

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We study the electronic properties of an array of small metallic grains connected by tunnel junctions. Such an array serves as a model for a granular metal. Previous theoretical studies of junction arrays were based on models of quantum dissipation which did not take into account the diffusive motion of electrons within the grains. We demonstrate that these models break down at sufficiently low temperatures: for a correct description of the screening properties of a granular metal at low energies, the diffusive nature of the electronic motion within the grains is crucial. We present both a diagrammatic and a functional integral approach to analyze the properties of junction arrays. In particular, an effective action is obtained which enables us to describe the array at an arbitrary temperature. In the low-temperature limit, our theory yields the correct, dynamically screened Coulomb interaction of a normal metal, whereas at high temperatures the standard description in terms of quantum dissipation is recovered.

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I. INTRODUCTION

During the past two decades much attention has been devoted to the study of disordered bulk metals with electron-electron interactions, 1–3 on the one hand and arrays of normal metallic grains connected by tunnel junctions, 4–6 on the other hand. Coulomb interaction plays a very important role in both types of systems and many nontrivial physical effects occur due to this. In disordered metals, e.g., the Coulomb interaction results in quantum corrections to conductivity, and reduces the density of states, a phenomenon known as zero bias anomaly. Qualitatively similar effects were predicted 6 and studied experimentally 7,9,8,10 for arrays of tunnel junctions in the normal state, where a sufficiently strong Coulomb interaction can suppress the conductivity (Coulomb blockade of tunneling).

In spite of the similarities in the physical behavior of disordered metals and normal arrays of tunnel junctions, completely different approaches are used for a theoretical description of these respective systems. As concerns disordered metals with interaction, either diagrammatic methods 1 or nonlinear σ models 11–13 are employed. The starting point of these theories is a model of a weakly disordered bulk metal. On the other hand, for a description of Coulomb blockade effects in a junction array or a granular material, a completely different approach based on the theory of Ambegaokar, Eckern and Schön 14 (AES) is commonly used. The free-energy functional proposed by AES does not contain any disorder: the finite conductance obtained within this model is due to tunneling of electrons between grains, and can be incorporated into the theory through a term describing the so-called quantum dissipation, first introduced phenomenologically in Ref. 15.

Although both approaches were introduced almost 20 years ago, only a few indirect attempts were made to reconcile them. We mention the work by Nazarov 16 and Levitov and Shytov, 17 that treated the diffusive zero-bias anomaly and the Coulomb blockade of tunneling on the same footing. More recently, Nazarov 18 and Kamenev 19 showed how Coulomb blockade phenomena survive in diffusive systems. Indeed, the properties of a granular metal without electron-electron interactions are qualitatively similar to those of a disordered bulk metal (see, e.g., Ref. 20), and it would be natural to expect that this qualitative similarity persists even in the presence of interactions. In fact, one may conjecture that bulk disordered metals, on the one hand, and granular metals or arrays of tunnel junctions from the other hand, can be described within one unifying scheme.

In order to develop such a scheme, in this paper we consider a granular normal metal with Coulomb interaction between electrons. The consideration is simplified by taking into account only the long-range part of the interaction that leads to Coulomb blockade effects. We assume that the macroscopic dimensionless conductance $g_T$ of the granular metal is large, which enables us to develop a perturbative theory with $g_T^{-1}$ as the small parameter. Within this theory, physical quantities of interest can be calculated at arbitrary temperatures. The diagrammatic analysis is supplemented by the derivation of a σ model that can be considerably simplified, provided one does not take into account weak-localization effects.

We demonstrate below that granular metals and networks of tunnel junctions can always be described using the standard techniques developed in the theory of disordered metals (expansions in cooperons and diffusons, σ-model calculations). With the help of these techniques one can in principle calculate any physical quantity without using the notion of quantum dissipation. Disorder is inevitably present in the system even if electrons move ballistically within the grains. In the latter case any small irregularity in the shape of the grains causes the electron motion within the grains to be chaotic, and this assumption is sufficient for our theory to be applicable. An integrable shape of the grains does not seem to be realistic, and even if it occurred for isolated grains,
tunneling from grain to grain would add an additional chaoticity. These ideas about intrinsic disorder in granular metals have been used in previous studies.\textsuperscript{20–22}

We find that, at not too low a temperature, the AES free-energy functional gives correct results. However, in the limit $T \rightarrow 0$, AES theory is no longer applicable. The region of validity of the AES free-energy functional is determined by the inequality $T \gg g \tau \delta$, where $\delta = (\nu V)^{-1}$ is the mean level spacing, $\nu$ is the density of states at the Fermi surface, and $V$ is the volume of a single grain.

We conclude in particular that the AES notion of quantum dissipation at $T = 0$ can only be applied to tunnel junctions connecting infinitely large conductors. If the volume of the conductors is finite, the AES picture can be applied at finite temperatures only. Apparently, the notion of quantum dissipation should be treated with care. A qualitative discussion of the problem “dissipation versus dynamical screening” in the case of superconducting grains can be found in a recent lecture by Altshuler.\textsuperscript{23}

The remainder of the paper is organized as follows. In Sec. II we formulate the model and write physical quantities in terms of functional integrals over anticommuting fields. We decouple the Coulomb interaction by an integration over auxiliary fields. In Sec. III we consider the AES action, and show that it does not correspond to the physics of normal metals with a screened Coulomb interaction. In Sec. IV we develop a diagrammatic technique. We show what kind of diagrams correspond to the AES action, and demonstrate that, at low temperatures, additional contributions arise. Summing up these additional diagrams we arrive at expressions corresponding to the dynamically screened Coulomb interaction of a normal metal. Section V contains a derivation of an action for a granular metal with electron-electron interaction which is applicable at arbitrary temperatures. The results are compared with the results obtained diagrammatically and with those obtained on the basis of the AES action. Our results are summarized in Sec. VI.

**II. CHOICE OF THE MODEL**

We consider an array of normal metallic grains coupled to each other. The dimensionality of the array may be arbitrary. The grains are assumed to contain imperfections: there can be impurities inside them as well as on their surfaces. This implies that the electron motion in the grains is chaotic. The mean level spacing in a single grain is equal to $\delta = (\nu V)^{-1}$. However, the shapes of the grains need not differ considerably from each other, and we assume for simplicity that the grains are arranged in a cubic lattice. The electrons can hop from grain to grain; moreover, they interact with each other. We are interested only in the long-range part of the Coulomb interaction, and write it in a simplified form describing charging of the grains. In such a formulation the spin of the electrons is not important, and can be taken into account at the end of the calculations when writing proper densities.

Under these assumptions the electron motion can be conveniently described by a functional integral of the type

$$\int \exp(-S[\psi]) D\psi,$$

with an action $S[\psi]$ that can be written in the form

$$S[\psi] = S_g[\psi] + S_i[\psi] + S_c[\psi].$$

The action $S_i[\psi]$ in Eq. (2.1) describes the electron motion within the grains, and we write it in the form

$$S_i[\psi] = \sum_i \int_0^\beta d\tau \psi_i^*(\tau,r)(\partial_\tau - \mu + \hat{H}_i)\psi_i(\tau,r),$$

where $\psi_i(\tau,r)$ is a fermion field in grain $i$ at an imaginary time $\tau$ and a coordinate $r$ ($\psi^*$ is its complex conjugate), $\mu$ is the chemical potential, and $\beta = T^{-1}$ is the inverse temperature. The fermion fields $\psi_i(\tau)$ satisfy the condition

$$\psi_i(\tau) = -\psi_i(\tau + \beta).$$

The operator $\hat{H}_i$ in Eq. (2.2) includes scattering by impurities, and can be written as

$$\hat{H}_i = -\frac{1}{2m} \nabla_i^2 + u_i(r),$$

where $u_i(r)$ is the impurity potential. We assume that it is Gaussian distributed, with the correlation

$$\langle u_i(r) u_j(r') \rangle = \frac{1}{2 \pi \nu \delta_{r r'}} \delta(r - r') \delta_{ij}.$$

The action $S_i[\psi]$ stands for tunneling between the grains,

$$S_i[\psi] = \sum_{ij} \int_0^\beta d\tau \psi_i^*(\tau) t_{ij} \psi_j(\tau),$$

and $S_c[\psi]$ describes the charging of the grains,

$$S_c[\psi] = \frac{e^2}{2} \sum_{ij} \int_0^\beta d\tau \eta_{ij}(\tau) C^{-1}_{ij} n_j(\tau),$$

where

$$n_j(\tau) = \int d\tau' \psi_j^*(\tau,r) \psi_j(\tau,r)$$

is the density field in grain $i$, $e$ is the electron charge, and $C_{ij}$ is the capacitance matrix. Equations (2.1)–(2.7) describe the model completely, and one can start explicit calculations.

Due to the Coulomb interaction, the action $S[\psi]$ is not quadratic in $\psi$ and the integration over $\psi$ cannot be performed immediately. A convenient way to proceed in such a case is to decouple the term $S_i[\psi]$ by a Gaussian integration over auxiliary fields. This transformation can be written as

$$\int D(\psi^*,\psi) e^{-S[\psi]} = N \int DV e^{-S_2[V]} \int D(\psi^*,\psi) e^{-S_1[\psi,V]},$$

where $N$ is a normalization factor and $V_j(\tau)$ is the decoupling field. The action $S_2[V]$ in Eq. (2.8) has the form

$$S_2[V] = \frac{1}{2} \int_0^\beta d\tau \sum_{ij} V_j(\tau) C_{ij} V_j(\tau).$$
which shows that the variable $V_i(\tau)$ has the meaning of a voltage on grain $i$ at the time $\tau$. The new effective action $S_{\text{eff}}[\psi, V]$ reads

$$S[\psi, V] = S_{\text{eff}}[\psi, V] + S_0[\psi],$$

with

$$S_{\text{eff}}[\psi, V] = \sum_i \int_0^\beta d\tau \psi_i^\dagger(\tau)[\partial_\tau - \mu + \hat{H}_i + ieV_i(\tau)]\psi_i(\tau).$$

(2.10)

One more transformation can be performed exactly. Following the procedure of Ref. 24, where a single grain was considered, we represent $V_i(\tau)$ in the form

$$V_i(\tau) = V_i^0 + \tilde{V}_i(\tau),$$

(2.11)

where $V_i^0$ is the static part and $\int_0^\beta \tilde{V}_i(\tau)d\tau = 0$. The replacement

$$\psi_i(\tau) \rightarrow \psi_i(\tau)\exp[-i\phi_i(\tau)],$$

(2.12)

with

$$\phi(\tau) = e^{\int_0^\tau \tilde{V}_i(\tau')d\tau'},$$

(2.13)

does not violate the condition given by Eq. (2.3), and we can remove the variable $\tilde{V}_i(\tau)$ from the action $S_{\text{eff}}[\psi, V]$ [Eq. (2.10)]. However, this variable appears in the action $S_0[\psi]$ as an additional phase of the field $\phi(\tau)$. As concerns the static part $V_i^0$, its fluctuations can be neglected even in a single isolated grain, provided the temperature is high, $T \gg \delta_{24}$. This restriction is not necessary for the system of the coupled grains in the limit of large conductance $g_T \gg 1$ considered here. We note here that Eq. (2.12) is a typical gauge transformation.

Using Eqs. (2.1)–(2.13) we finally reduce the calculation of physical quantities to the computation of a functional integral of the form

$$\int \exp(-S_0[\psi] - S_{\text{eff}}[\psi, \phi] - S_2[\phi])D\psi D\phi,$$

(2.14)

where

$$S_0[\psi] = \sum_i \int_0^\beta d\tau \psi_i^\dagger(\tau)[\partial_\tau - \mu + \hat{H}_i]\psi_i(\tau),$$

(2.15)

$$S_{\text{eff}}[\psi, \phi] = \sum_{ij} \int_0^\beta d\tau \psi_i^\dagger(\tau)\psi_j(\tau)\exp[i\phi_{ij}(\tau)],$$

(2.16)

$$S_2[\phi] = \int_0^\beta d\tau \sum_{ij} \frac{C_{ij}}{2e^2} \frac{d\phi_i(\tau)}{d\tau} \frac{d\phi_j(\tau)}{d\tau},$$

(2.17)

and $\phi_{ij}(\tau) = \phi_i(\tau) - \phi_j(\tau)$.

Equations (2.14)–(2.17) completely specify the model that will be studied in the subsequent sections. Disorder is still present in the operators $\hat{H}_i$ [Eq. (2.4)], but the mean free path within the grains is assumed to be large, $lk_0 \gg 1$, where $k_0$ is the Fermi momentum. In this limit, the macroscopic conductivity is determined mainly by the tunneling conductance $g_T$,

$$g_T = \pi/(2e^2R_T) = 2\pi^2t^2\nu^2 \approx 6.45k\Omega/R_T,$$

(2.18)

where $R_T$ is the tunneling resistance.

III. QUANTUM DISSIPATION DESCRIPTION OF GRANULAR METALS WITH COULOMB INTERACTION

A functional integral formulation was used in Ref. 14 to treat the quantum dynamics of a Josephson junction. The action derived in that work (AES action) was used later in the context of a normal tunnel junction,45 as well as in connection with arrays of tunnel junctions in a number of papers.6 The AES action can be obtained from Eqs. (2.14)–(2.17). First, one should average the Green function $G_0$ corresponding to the action $S_0[\psi]$, [Eq. (2.15)], over impurities in the first Born approximation, which gives (in Fourier representation)

$$G_{0e}(\mathbf{p}) = \frac{1}{4E_c} \sum_{ij} \int_0^\beta d\tau \left(\frac{d\phi_{ij}}{d\tau}\right)^2$$

$$+ \frac{g_T}{2} \sum_{ij} \int_0^\beta d\tau d\tau' \alpha(\tau - \tau')\{1 - \cos[\phi_{ij}(\tau) - \phi_{ij}(\tau')],$$

(3.1)

where

$$\alpha(\tau - \tau') = \frac{T^2}{\sin^2\pi T(\tau - \tau')}.$$  

(3.2)

In Eq. (3.2), $E_c$ is the charging energy, $E_c = e^2/2C_{ij}$. The second term in Eq. (3.2) contains the sum over neighboring grains, and $t$ is the tunneling energy between them. In the limit of small phase fluctuations we can expand the second term in Eq. (3.2) with respect to $\phi_{ij}$, and obtain

$$S[\phi] = \frac{1}{4E_c} \sum_{ij} \int_0^\beta d\tau \left(\frac{d\phi_{ij}(\tau)}{d\tau}\right)^2$$

$$+ \frac{g_T}{4} \sum_{i,j} \int_0^\beta d\tau d\tau' \alpha(\tau - \tau')\{\phi_{ij}(\tau) - \phi_{ij}(\tau')\}^2.$$  

(3.4)

Using the periodicity of all functions in $\beta$, we write the Fourier expansion as
\[ \phi_{ij}(\tau) = T \sum_{\omega_n} \phi_{ij}(i\omega_n) \exp(-i\omega_n\tau), \quad \alpha(\tau) \]
\[ = T \sum_{\omega_n} \alpha(i\omega_n) \exp(-i\omega_n\tau), \quad (3.5) \]

where \( \omega_n = 2\pi nT \) are Matsubara frequencies. Hence for the action we obtain

\[ S[\phi] = \frac{1}{4E_c} \sum_i T \sum_{\omega_n} \omega_n^2 |\phi_i(\omega_n)|^2 \]
\[ + \frac{g_T}{\pi} \sum_{i,j} T \sum_{\omega_n} |\omega_n||\phi_{ij}(i\omega_n)|^2. \quad (3.6) \]

The second term in Eq. (3.6), describing the tunneling, is linear in frequency and keeps its form down to \( T \to 0 \). Therefore, it was attributed to quantum dissipation.\(^{14}\) Performing a Fourier transformation in space,

\[ \phi_i(\omega_n) = \sum_k \phi_{i\omega}(k) \exp(ik\mathbf{R}_i), \quad (3.7) \]

we rewrite the action \( S[\phi] \) as

\[ S[\phi] = \frac{1}{2} \sum_k T \sum_{\omega_n} \left( \frac{\omega_n^2}{2E_c} + \frac{4g_T}{\pi} |\omega_n| \sum_a \right) \]
\[ \left( 1 - \cos kd_{ij} \right) |\phi_{i\omega}(k)|^2, \quad (3.8) \]

where \( d_{ij}, a=x, y, z, \) are vectors connecting the centers of the neighboring grains. The phase-phase correlation function

\[ \Pi_{\text{AES}}(\omega_n, \mathbf{k}) = T(\phi_{i\omega}(k)\phi_{i\omega}^*(k)) \]

can be immediately calculated using Eq. (3.8), and we obtain

\[ \Pi_{\text{AES}}(\omega_n, \mathbf{k}) = \frac{4E_c}{\omega_n^2 + \frac{8}{\pi g_T E_c} |\omega_n| \sum_a (1 - \cos kd_{ij})}. \quad (3.9) \]

Suppose that one assumes, following Refs. 5 and 6, that Eq. (3.2) is applicable to a system of tunnel junctions down to \( T \to 0 \). This would imply in particular that Eq. (3.9) is also correct for \( T \to 0 \), too. However, the derivative \( d\phi_i/d\tau \) proportional to the voltage on the grains, which in turn is linearly related to the charge density. Multiplying Eq. (3.9) by \( \omega_n^2 \), one thus obtains the density-density correlation function \( K(\omega_n, \mathbf{k}) \). Let us consider the limit of small frequencies and wave vectors \( k \). In this limit, the network should correspond to a bulk normal metal, where one should have a propagator corresponding to a screened Coulomb interaction. However, the limit \( \omega, k \to 0 \) in Eq. (3.9) does not reproduce the screened Coulomb interaction.

How to resolve this discrepancy? In Sec. IV we carry out an expansion in \( t_{ij} \), and demonstrate that keeping only the second order, which led us to Eq. (3.2), is not sufficient and that, in the limit of small frequencies or temperatures, one should sum up an infinite class of diagrams. In Sec. V we will then show how these processes can be included into an effective action formulation structurally similar to the AES approach.

\[ \text{FIG. 1. Polarization operator for a granular metal in the lowest order in tunneling.} \]

\[ \text{IV. SCREENED COULOMB INTERACTION IN GRANULAR METALS} \]

In this section we present a detailed derivation of the density-density correlation function \( K(\omega_n, \mathbf{k}) \) summing an infinite series of diagrams. For the computation we use the standard diagram technique\(^{25}\) with a modification suggested in Refs. 21 and 22 to include tunneling between the grains. Within this technique one should make an expansion in the tunneling \( t_{ij} \), in the phases \( \phi_i \), and in the impurity potential \( u_i(r) \). Below we denote the electron Green functions by solid lines, the phases \( \phi_i \) by wavy lines, and the tunneling elements \( t_{ij} \) by crossed circles. Impurity propagators corresponding to Eq. (2.4) are denoted by dashed lines.

Before turning to the quantitative discussion, let us briefly outline the skeleton of the analysis. To lowest order in the tunneling matrix elements, the polarization operator of the system is represented by the process depicted in Fig. 1. This diagram describes the (absolute square of) amplitude for tunneling from one grain into the next, and the subsequent relaxation inside the "target grain." The quantitative evaluation of this process leads to the coupling constant \( g_T \) of the AES approach.

On this level, coherence effects associated to multiple tunneling and/or impurity scattering are completely neglected; the tunneled electron "forgets" about its phase memory, implying that the dissipative picture of the AES approach obtains. Indeed, the two Green functions depicted in Fig. 1 "live" in different grains, which means that no phase correlation is possible. How can this picture change in principle? Including one more order in the tunneling, leads to corrections of the structure shown in Fig. 3. What makes these processes qualitatively different from those discussed above is that they include the possibility of phase-coherent multiple impurity scattering: The two Green functions labeled \( i \) or \( j \) scatter off the same impurities and/or chaotic potential where the associated scattering phases may cancel due to the fact that one of the Green functions is advanced, the other retarded. The net two-particle modes emanating from such processes, "diffusons" in the context of disordered systems, are represented through the hatched regions of Fig. 4, right. The key question now is, are such processes relevant or not? A crude estimate can be given as follows: It is known that for low temperatures (for a precise definition of "low," see below), each diffuson mode diverges as \( \delta T \). The fact that we are considering processes of higher-order in...
the tunneling amplitude introduces one more power of $g_T$. Thus, for $T \sim \delta g_T$, higher-order processes become as important as the first-order contribution. For lower temperatures, these processes must be taken into account to obtain a physically correct picture. In the following we will put this discussion onto a firm basis, and discuss how the coherent tunneling series can be summed in a controlled way.

Expanding the functional integral [Eq. (2.14)] in the tunneling term $S_1$ [Eq. (2.16)], to second order in $t_{ij}$ and in $\phi_i$ we obtain the diagrams represented in Fig. 1.

The two electron lines relate to different grains. As the impurities in the different grains are not correlated, impurity lines connecting two Green functions do not exist (although the diagram represents a polarization loop). The second and third diagrams in Fig. 1 are equal to each other and have an opposite sign with respect to the first diagram. The analytical result for the sum of the three diagrams reads

$$P_0(\omega_n, \mathbf{k}) = -|\omega_n|(2/\pi) g_T \sum_{a=1}^{3} (1 - \cos k_a d),$$

(4.1)

where the dimensionless tunneling conductance $g_T$ is given by Eq. (2.18). We see from Eq. (4.1) that, although the Green functions $G_{0\nu}(\mathbf{p})$ determined by Eq. (3.1) contain $\tau_{\text{imp}}$, all information about the disorder drops out of the polarization loop. To calculate the phase-phase correlation function, one should note that the bare propagator corresponding to $S_2$, [Eq. (2.17)], has the form

$$\Pi_0(\omega_n, \mathbf{k}) = \frac{4E_c}{\omega_n^2},$$

(4.2)

and diverges at $\omega \to 0$. Therefore, one should sum up the infinite geometrical series represented in Fig. 2.

The summation can be easily performed and we arrive at the propagator $\Pi_{\text{AES}}(\omega, \mathbf{k})$ [Eq. (3.9)]. Summarizing, we conclude that, in order to reproduce the results obtained in the quadratic approximation of the AES action, [Eqs. (3.4), (3.6), and (3.8)], one should sum up the diagrams of Fig. 2. On the other hand, the form of the propagator $\Pi_{\text{AES}}(\omega_n, \mathbf{k})$ does not correspond to the propagator of a normal metal with a screened Coulomb interaction (see, e.g., Ref. 1), which signals that something is missing in this analysis.

To resolve this paradox one should consider diagrams of higher order in tunneling, which complicates the calculations. First of all, let us note that in order to consider higher-order diagrams one should specify tunneling more explicitly. We assume that the radius of the area at which the grains contact each other much exceeds atomic distances (a large number of conducting channels). At the same time, the potential barrier between the grains can be arbitrarily large, which can lead to an arbitrary tunneling conductance. Such a situation can be conveniently modeled by random tunneling matrix elements $t_{pq}$ correlated as

$$\langle t_{pq} t_{p'q'} \rangle = t^2 (\delta_{pp'} \delta_{qq'} + \delta_{pq} \delta_{p'q'}),$$

(4.3)

where $t_{pq}$ are written for the same contact. Correlations of the matrix elements of different contacts are assumed to be zero. To simplify notations we do not write subscripts numerating the contacts. Taking Eq. (4.3) into account is indeed very important. In order to illustrate this fact, we consider two diagrams represented in Fig. 3(a). We immediately see the difference between them: the first diagram is not averaged over $t_{pq}$ and can contain four different momenta. After averaging over $t_{pq}$ one obtains the second diagram containing only three different momenta. As has been mentioned, we will expand not only in the tunneling amplitude $t$ but also in the phases $\phi$. Performing the expansion in the phases and denoting them by wavy lines we have three different possibilities as depicted in Fig. 3(b).

Now we can start averaging over impurities. As usual,25 we first average the Green functions and obtain Eq. (3.1) for the average Green function $G_{0\nu}(\mathbf{p})$. Then we can consider averages with impurity lines connecting the Green functions on opposite sides of the diagrams in Fig. 3(b). Diagrams with nonintersecting impurity lines are most important, and we recognize the well-known diffusons.

In the present paper we consider the limit of sufficiently small grains, such that all relevant energies like temperature $T$, frequency $\omega$, tunneling energy $t$, etc. are much smaller
than the Thouless energy $E_T = \pi^2 D_0 / R^2$, where $D_0$ is the diffusion coefficient of a single grain, and $R$ is the radius of the grain. In this limit the spectrum of the diffusion in a single grain is zero dimensional (no dependence on momenta inside the grain). Coupling between the grains leads to a dependence on quasimomentum $k$. Diagrams that should be summed in order to give the complete form of the diffusion are represented in Fig. 4(a).

Now let us present some details of the calculation of the diagrams in Fig. 3. The different classes of diagrams contributing to the polarization operator of the granular metals before impurity averaging are shown in Fig. 3(b). Before impurity averaging, the sum over Matsubara frequencies for the first diagram in Fig. 3(b) is

$$T \sum \limits_{\varepsilon_n} G(i\varepsilon_n + i\omega_n, p_1) G(i\varepsilon_n, p_2).$$

(4.4)

Writing this expression, we used Eq. (4.3). After impurity averaging of the first diagram in Fig. 3(b), we obtain the diagram represented in Fig. 4(a).

Now the disorder averaging can be done and we obtain, in a standard way, the average of the product of two Green functions for a granular metal, $\langle G^i G^j \rangle = D(\omega_n, q)$. Here

$$D = D^{(0)} + D^{(0)} \Sigma D,$$

(4.5)

$D^{(0)}(\omega_n, k) = 2\pi \nu / |\omega_n|$ being the diffusion for a single isolated grain. The equation for the complete diffusionootnote{Note that the second diagram in Fig. 3(b) is independent of the frequency $\omega_n$, and yields a constant. The diagram in Fig. 4(a) contains only one diffusion. More complicated diagrams can be drawn and one of them is depicted in Fig. 5.}

However, all such many-diffuson contributions can be neglected, provided the tunneling conductance $g_T$ [Eq. (2.18)], is large. A direct calculation of this diagram shows that it involves the small parameter $1/g_T$. This is similar to expansions in the diffusion modes for a bulk metal, where the inverse conductivity is the expansion parameter. The condition $g_T \gg 1$ means that we are far from the Anderson metal-insulator transition.

The result of averaging and summation of the diagrams represented in Fig. 3 can be written as

$$P(\omega_n, k) = -\frac{|\omega_n|^2 (2/\pi) g_T \sum_{a} (1 - \cos k_a d)}{|\omega_n| + (2/\pi) g_T \delta \sum_{a} (1 - \cos k_a d)}.$$ 

(4.8)

We note here that the functions $P_0$ and $P_1$ have different signs. Comparing the function $P_1(\omega_n, k)$ [Eq. (4.7)] with the quantum dissipation part $P_0(\omega_n, k)$ [Eq. (4.1)], represented in Fig. 1, we see immediately that the contribution $P_1(\omega_n, k)$ [Eq. (4.7)], can be neglected only if the temperature $T$ is sufficiently high, $T \ll |\omega_n| \gg g_T \delta$ [we recall that the static component of the phase $\phi$ has been neglected, and therefore $\omega_n \neq 0$ in Eq. (4.7)]. So the notion of quantum dissipation is applicable essentially at nonzero temperatures only for a granular material. However, it can be used for a contact connecting two bulk metals with an infinite volume.

At low temperatures $T \ll g_T \delta$, there is no reason to neglect the contribution $P_1(\omega_n, k)$ [Eq. (4.7)], and its presence changes completely the form of the propagator. Adding the functions $P_0(\omega_n, k)$ [Eq. (4.1)], and $P_1(\omega_n, k)$ [Eq. (4.7)], we obtain the total polarization $P(\omega_n, k)$:

$$P(\omega_n, k) = -\frac{|\omega_n|^2 (2/\pi) g_T \sum_{a} (1 - \cos k_a d)}{|\omega_n| + (2/\pi) g_T \delta \sum_{a} (1 - \cos k_a d)}.$$ 

(4.8)

Proper diagrams contributing to this function are represented in Fig. 6.

Now we can write the final expression for the phase correlation function $\Pi(\omega_n, k)$ that should be written instead of $\Pi_{\text{AES}}(\omega_n, k)$ [Eq. (3.9)]. Using the bare propagator $\Pi_0(\omega_n, k)$, [Eq. (4.2)], and the self-energy part $P(\omega_n, k)$ [Eq. (4.8)], we write the phase propagator $\Pi(\omega_n, k)$ in the form

$$\Pi(\omega_n, k) = \left\{ \begin{array}{ll} 1 & \text{if } \omega_n = 0 \\
\frac{1}{\omega_n} & \text{if } \omega_n \neq 0 \end{array} \right.$$ 

(4.9)

FIG. 5. Example of a high-order correction to the polarization of granular metals.

FIG. 6. Relevant polarization loops for a granular metal.
\[
\Pi(\omega_n, \mathbf{k}) = \frac{\Pi_0}{1 - \Pi_0 \rho} \left( \frac{4E_c}{\omega_n^2} \right) \left( \frac{(2/\pi)g_T \sum_a (1 - \cos k_ud)}{1 + 4E_c} \frac{|\omega_n| + (2/\pi)g_T \sum_a (1 - \cos k_ud)}{\omega_n + Dk^2} \right)
\]

Using the relation between the phase \( \phi \) and the voltage \( V \), deriving a simplified version of a replica model and Eq. (2.13), we derive an equation for the effective Coulomb interaction

\[
V_{\text{eff}}(\omega_n, \mathbf{k}) = \omega_n^2 \Pi(\omega_n, \mathbf{k}).
\]

On the other hand, the dynamically screened Coulomb interaction in a disordered bulk metal can be written in the form

\[
\tilde{V}_{\text{eff}}(\omega_n, \mathbf{k}) = \frac{\tilde{V}_0(\mathbf{k})}{1 + \tilde{V}_0(\mathbf{k})} \frac{\nu Dk^2}{|\omega_n| + Dk^2},
\]

where \( \tilde{V}_0 \) is the bare Coulomb potential. We see that Eqs. (4.9) and (4.10), written in the continuum limit, correspond to Eq. (4.11).

Thus we conclude that, when calculating physical quantities, the main contribution is due to small \( k \) only in the limit \( T \ll g_T \delta \). Provided this inequality is satisfied, the properties of the tunnel junction array are equivalent to the properties of a weakly disordered bulk metal with Coulomb interaction. In the opposite limit \( T \gg g_T \delta \), one can describe the system in the “quantum dissipation” approximation.

Calculations were performed in this section using a diagrammatic expansion in the lowest order in the field \( \phi \). On the other hand, the AES free-energy functional [Eq. (3.2)] is written for arbitrary values of \( \phi \). Can one generalize Eq. (4.10) to arbitrary values of \( \phi \)? We will try to do this in Sec. V, deriving a simplified version of a replica \( \sigma \) model and using a nontrivial saddle point.

V. EFFECTIVE ACTION FOR GRANULAR METALS WITH COULOMB INTERACTION

In the preceding sections we analyzed effects of the Coulomb interaction on the electron motion in granular materials using a diagrammatic expansion. This approach works well when fluctuations of the phase \( \phi \) or, in other words, fluctuations of the voltages on the grains, are small. However, one can imagine situations when the fluctuations are large. For example, one can try to consider nonperturbative excitations like instantons,\(^{26}\) corresponding to an integer charge transfer between grains. In principle, the AES action [Eq. (3.2)], can describe such excitations very well but, as we have seen previously, it can only be used at sufficiently high temperatures \( T \gg g_T \delta \). So, our task now is to derive an action that will be applicable at lower temperatures.

The derivation at temperatures \( T \ll g_T \delta \) is more complicated, because now we should explicitly take disorder into account, which was not necessary for the derivation of the AES action (strictly speaking, disorder determines the mean free time \( \tau_{\text{imp}} \) in the Green functions used for derivation of the AES functional, but this is a trivial contribution). On the other hand, the problem is not as complicated as the problem of Anderson localization in the presence of interactions considered in Refs. 3 and 11. In our diagrammatic expansions we considered the limit of large conductances \( g_T \gg 1 \), and neglected all weak-localization corrections.

Nevertheless, we start the derivation of the action using the replica \( \sigma \)-model approach for interacting systems suggested by Finkelstein.\(^{3,11}\) Necessary simplifications will be made in the \( \sigma \) model. Although there are difficulties in using the replica \( \sigma \) models for nonperturbative calculations, our goals are more modest, and the final results obtained in the limit \( g_T \gg 1 \), neglecting all weak-localization corrections, will not depend on replica indices at all.

The model under consideration was formulated in Sec. II. The derivation of the \( \sigma \) model can be carried out starting from Eqs. (2.1)–(2.7). Instead of the fields \( \psi \) and \( V \) one should write fields \( \psi = \{\psi^s\} \) and \( \{V^s\} \) carrying the replica index \( a = 1, \ldots, r \). Upon calculating physical quantities for an arbitrary \( r \), one should put \( r = 0 \).

A proper \( \sigma \) model for bulk systems with interaction was derived in Refs. 3 and 11. A generalization to granular metals can be made without difficulties following Ref. 20, where a \( \sigma \) model was written for a granular metal without interaction. Although one can proceed along the lines of Refs. 3 and 11, i.e., expanding the action obtained after averaging over impurities by a Gaussian integration over \( \psi \) in the field \( V \), a more economic way is to use the substitution given by Eqs. (2.12) and (2.13), as well as the subsequent equations (2.14)–(2.17). Of course, this is already an approximation, because we neglect the static component of the phase \( \phi \). However, for our purposes this is not an essential restriction because we do not want to consider effects of localization. Moreover, working in the limit \( T \gg \delta \) or \( g_T \gg 1 \) allows us to ignore the static component of \( \phi \).

The derivation of the \( \sigma \) model starting from Eqs. (2.14)–(2.17) is practically the same as the one presented in Ref. 20. As usual, one decouples the “effective interaction” of the type \( \psi^s \) that appears after averaging over impurities by a Gaussian integration over matrices \( \tilde{Q} \). These matrices contain as elements both the replica indices and Matsubara frequencies (or two imaginary times \( \tau \) and \( \tau' \)). Neglecting the tunneling term [Eq. (2.16)], one would obtain a zero-dimensional \( \sigma \) model (we assume that all relevant energies are smaller than the Thouless energy \( E_T \) of a single grain). The relevant tunneling term in the \( \sigma \) model is obtained by a cumulant expansion in the tunneling. The only difference with respect to the model without interaction of Ref. 20 is the presence of the phases \( \phi \) in Eq. (2.16), which leads to additional phase factors in the term describing the coupling between the grains.

The final results for the free energy \( F[\tilde{Q}, \phi] \) can be written as

\[
F[\tilde{Q}, \phi] = F_2[\phi] + F_0 + F_1[\tilde{Q}, \phi],
\]

\[
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\]

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The matrix $L$ satisfying the constraints $@$. In a frequency representation this operator is equal to a vector $U$ over both replica indices. The field $\Phi$ is important for us with the constraints $f$. The field $\Phi$ is diagonal in the replica indices, while the matrix $Q(\tau,\tau')$ is a $2r \times 2r$ matrix with the constraints

$$
\int Q(\tau,\tau')Q(\tau'',\tau')d\tau'' = \delta(\tau-\tau'), \quad \int \text{tr} Q(\tau,\tau)d\tau = 0.
$$

(5.5)

The action of the operator $\hat{\omega}$ in Eq. (5.3) on an arbitrary function $f(\tau,\tau)$ is given by the relation

$$
\hat{\omega}f(\tau,\tau') = -i \lim_{\tau' \to \tau} \frac{\partial}{\partial \tau} f(\tau,\tau').
$$

(5.6)

In a frequency representation this operator is equal to a vector of fermionic Matsubara frequencies. The matrix $Q(\tau,\tau')$, satisfying the constraints [Eq. (5.5)], can be conveniently parametrized as

$$
Q(\tau,\tau') = \int U(\tau,\tau'')\Lambda_{\tau',\tau''} \tilde{U}(\tau'',\tau')d\tau'' d\tau''',
$$

(5.7)

where

$$
\Lambda_{\tau',\tau} = \frac{iT}{\sin T\pi(\tau-\tau')}
$$

(5.8)

and $U$ and $\tilde{U}$ are unitary matrices:

$$
\int U(\tau,\tau'')\tilde{U}(\tau'',\tau')d\tau'' = \delta(\tau-\tau').
$$

(5.9)

The matrix $\Lambda$ in frequency representation has the form

$$
\Lambda_{mn} = \delta_{mn} \text{sgn}(e_n).
$$

The limiting cases of the free energy $F[Q,\Phi]$, [Eqs. (5.1)–(5.6)] are rather simple. Without electron-electron interaction the phase difference between neighboring grains is equal to zero, $\Phi_{ij}=0$. Such an action for granular metals corresponds to that derived within the supersymmetry scheme by one of the authors.\textsuperscript{20} The second limiting case is achieved at sufficiently high temperatures. In this limit disorder is not important and the matrix $\Phi$ does not fluctuate, being equal to the value $\Lambda$ [Eq. (5.8)]. Inserting $Q=\Lambda$ into Eqs. (5.1)–(5.4), we immediately reproduce the AES action [Eq. (3.2)] (in fact we obtain the $\tau$ times replicated AES functional, but only one replica field is important for us).

At not too high a temperature, Eqs. (5.7)–(5.9) lead to a nontrivial behavior even if the disorder is weak and all effects related to weak localization can be neglected. The limit of weak disorder enables us to neglect fluctuations of $\Phi$ at given $\Phi$, and take its value from a saddle-point equation. We thus reduce the computation of the functional integral with the free energy functional to the solution of a saddle-point equation, substituting the solution to the free energy $F[\Phi,\Phi]$ [Eqs. (5.1)–(5.4)], calculating a functional integral with the reduced free energy $F[\Phi]$, $\tilde{F}[\Phi]=F[\tilde{Q},\Phi]$, where $\tilde{Q}$ is the solution of the saddle-point equation for a given $\Phi$.

Minimizing the functional $F[Q,\Phi]$ [Eqs. (5.1)–(5.4)], with the constraints [Eqs. (5.5)], we obtain the equation

$$
\frac{gT}{2} \sum_j \int [e^{i\Phi_{ij}}\tilde{Q}(\tau,\tau')e^{-i\Phi_{ij}}\tilde{Q}(\tau',\tau'')] d\tau'' + \frac{\pi}{\delta}[\tilde{Q}(\tau,\tau),\hat{\omega}]=0,
$$

(5.11)

where the summation over $j$ is performed over the nearest neighbors of $i$. In the commutator $[\ldots,\ldots]$, one should properly exchange the times when changing the order of the functions.

In general, this equation can be solved only numerically. However, for $gT \gg 1$, drastic simplifications arise. For a large intergranular coupling, the phase $\Phi$ fluctuates only weakly from grain to grain, which means that $\Phi_{ij}$ is small. Under these conditions, a solution of the diagonal form

$$
\tilde{Q}_{ij} = e^{iK_{ij}(\tau)}\Lambda_{\tau\tau'} e^{-iK_{ij}(\tau')}
$$

(5.12)

can be sought for. Equation (5.11) represents an integral equation that enables us to find, in principle, the solution $K_{ij}(\tau)$ for arbitrary values of the parameter $gT/\delta/T$. As the solution is assumed to have a diagonal form, all replica indices decouple and the equations can be solved for each replica separately. Therefore we can drop the replica indices, and consider Eq. (5.11) as an equation for a single function $K_{ij}(\tau)$ for a given function $\Phi_{ij}(\tau)$.

In the high-temperature limit $T \gg gT/\delta$, the second term in Eq. (5.11) is much larger than the first one, and we arrive at the solution

$$
K_{ij}(\tau) = 0,
$$

(5.13)

which leads to the AES free-energy functional [Eq. (3.2)]. In the opposite limit of low temperatures $T \ll gT/\delta$, Eq. (5.11) can be simplified using the assumption that $K_{ij}(\tau)$ varies slowly in space. Then one may expand the exponentials in $K_{ij}(\tau) = K_{ij}(\tau) - K_{ij}(\tau')$, which leads to the equation

$$
gT \sum_j \Lambda[(K_{ij} + \Phi_{ij}),\Lambda] + \frac{i\pi}{\delta} \frac{\partial}{\partial K_{ij}^{\tau\tau'}} = 0.
$$

(5.14)

Equation (5.14) is presented without writing the integration over imaginary times $\tau$ explicitly, but of course this
integration is implied. Transforming this equation to frequency and momentum space, we obtain
\[ 2g_T \sum_{j=1}^{3} (1 - \cos q_j d) \left[ 1 - \text{sgn}(\omega_n) \text{sgn}(\omega_m) \right] \]
\[ + \frac{\pi}{\delta} (\omega_n - \omega_m) \left[ \text{sgn}(\omega_n) - \text{sgn}(\omega_m) \right] K_{q,n-m} \]
\[ = - \frac{i\pi}{\delta} V_{q,n-m} \left[ \text{sgn}(\omega_n) - \text{sgn}(\omega_m) \right], \tag{5.15} \]
which gives the solution
\[ K_{k,n} = -V_{k,n} i \frac{\text{sgn} \omega_n}{(2\pi)g_T \delta \sum_{a=1}^{3} (1 - \cos kd_a) + |\omega_n|}, \tag{5.16} \]
where \( K_{k,n} \) and \( V_{k,n} = (i/e) \omega_n \phi_n(k) \) are the Fourier transforms of the functions \( K(\tau) \) and \( V(\tau) \), respectively. Substituting this ansatz for \( K \) back into the action, we obtain
\[ F[V] = \frac{1}{4E_c} \sum_n \sum_k V_{k,n} \]
\[ \times \left( 1 + 4E_c \left[ |\omega_n| + (2\pi)g_T \delta \sum_a (1 - \cos k_d d_a) \right] \right) \]
\[ \times V_{-k,-n} \tag{5.17} \]
where \( E_c \) is the charging energy. In the limit of small characteristic momentum, \( k \ll d^{-1} \), we recover the diffusively screened Coulomb interaction in a disordered environment.

From here one can proceed to a calculation of physical observables such as the conductance, the density of states, or others. To this end one should introduce configurations \( Q \) fluctuating around the mean-field configuration discussed above. Next one would follow the standard algorithm for doing calculations within the \( \sigma \)-model approach: (i) express the quantity of interest in terms of \( Q \), and (ii) compute the functional average over the effective action \( F[Q] = F[V, T] \) as well as possible.

Let us outline how the connection between the enlarged AES-type formulation [Eqs. (5.1)–(5.4)] and the Finkelstein approach for interacting disordered media can be made explicit. To do so, we subject our \( Q \)'s to a gauge transformation
\[ Q_{T \to e^{i\phi} Q_T e^{-i\phi}}, \tag{5.18} \]
as a result, (i) the hopping part of the action, \( F_T \) becomes \( \phi \) independent. However, (ii) in the frequency part, \( \hat{\omega}_T \to \hat{\omega}_T + \partial_T \phi \). Gaussian integration over \( \phi \) then produces an effective action \( F[Q] \) which, after taking a continuum limit, is identified as the action of the Finkelstein approach.

**VI. CONCLUSION**

In this paper we constructed a theoretical framework to describe electronic transport in arrays of tunnel junctions or granular metals, in the limit of large intergranule conductance \( g_T \). Both disorder and the electron-electron interaction were taken into account. The prime objective of this enterprise was to unify two large, and seemingly nonoverlapping, theories of interacting metallic compounds: the AES approach, focusing on charging phenomena in individual grains and their impact on large-scale transport behavior; and the Finkelstein theory of disordered interacting metals with its emphasis on the interplay interaction/disorder. The key to the reconciliation of these two approaches was to observe that for low enough temperatures, \( T < g_T \delta \), the effective action underlying the AES theory becomes incomplete. The physical reason for this is that for low temperatures the electrons and holes participating in the tunneling processes between individual grains maintain their quantum phase memory for a long time, largely in excess of the average tunneling time. This means that the particle tunneling is not only accompanied by charging and dissipation (as in the AES approach), but also by other, more long-ranged physical processes. Specifically, we identified the quantum dissipation contained in the AES action as the high-temperature limit of the long-ranged screened electron-electron interaction, an observation first made in Ref. 23. We re-emphasize that these phenomena were not bound to the presence of a significant disorder concentration; indeed, none of the results discussed above displayed dependence on some “disorder concentration.” The only thing that mattered was the chaoticity of the electron motion on scales set by the phase coherence length, a condition that is practically always met in real life systems.

Technically, two different routes for including these processes into the theory were proposed. One consisted of a perturbative summation of diagram classes associated with multiple chaotic scattering. The other, based on an effective action formulation, indentified what is ‘missing’ in the AES action. Remarkably, this second approach readily led to a unification of the AES approach and the Finkelstein nonlinear \( \sigma \) model for weakly disordered interacting metals. With the benefit of hindsight, this fact is easy to understand: It is a well-known fact in mesoscopic physics that long processes resulting from multiple scattering can be interpreted as a certain type of Goldstone modes. Within an effective action approach, these modes must be described by a known degree of freedom, the \( Q \) matrices of the nonlinear \( \sigma \) model. Thus it is no surprise that we obtained an actions of like Eqs. (5.1)–(5.4), involving \( Q \) fields and the Coulomb phase fields of the AES approach, as effective descriptions of the low-energy phase of the system. For large temperatures, the fluctuations of the \( Q \) matrices became inessential, and the AES action was retrieved. In contrast, the continuum limit of this action was identified as the Finkelstein model and the \( Q \) matrices smoothly integrated into the AES approach, without leading to conceptual complications.

Let us make some remarks on potential experimental ramifications of our findings. It was predicted in Refs. 6 and 7 that two-dimensional normal arrays of tunnel junctions...
should undergo a Kosterlitz-Thouless-Berezinskii (KTB) phase transition at a temperature $T_c$ of the order of the charging energy $E_c$. At low temperatures $T < T_c$, the array should be in an insulating state (Coulomb blockade), whereas for $T > T_c$, the array is conducting. In particular, at $T = T_c$, the conductivity $\sigma$ of the array is predicted to increase with temperature according to a square-root cusp dependence, $\sigma(T) \sim \exp(-2b/\sqrt{T/T_c - 1})$, where $b$ is a constant of order unity. The experiments mentioned in Sec. I searched for this KTB transition. In view of our results, the applicability of Refs. 6 and 7, and the KTB scenario should be governed by the parameter $g_T \delta/T_c$. The experiments of Refs. 7, 9 and 10 were done on arrays with relatively large grains, such that $g_T \delta/T_c \ll 1$. Indeed, the results obtained in Refs. 7 and 10 were in agreement with the KTB scenario. However, Ref. 9 found a thermally activated behavior of $\sigma(T)$, rather than the predicted square-root cusp dependence on temperature. On the other hand, the experiment by Yamada et al. was done on arrays consisting of relatively small Cu grains with a size of about 40 Å, such that $g_T \delta/T_c \approx 20$. Clearly this is beyond the range of applicability of Refs. 6 and 7, and the KTB scenario should be treated with care in this case. Remarkably, Ref. 8 concluded upon good agreement of the results with the theory. Given these discrepancies, we conclude that a careful analysis of the temperature-dependent conductivity of junction arrays in the framework of the theory presented in this paper would be of interest.

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