EXISTENCE AND UNIQUENESS RESULTS FOR EULER AND

NAVIER-STOKES EQUATIONS

A thesis submitted in partial fulfillment of the requirements
for the degree of Master of Science
in Mathematics

by

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Dedication

To my wife Lyudmila.
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I would like to thank all the instructors I have had the pleasure of taking a course with at CSUN.
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We prove existence and uniqueness of a smooth solution to each of the initial value problems for the Euler and the Navier-Stokes equations in 3-D case on some time interval $[0, T]$ applying classical energy methods. We use an approximation scheme with mollification and various a priori estimates on Sobolev norms of velocity $v$. At first, we justify global-in-time existence and uniqueness of solutions to regularizations of the Euler and the Navier-Stokes equations. Then we show that there exists a time $T$ such that a unique solution exists on the time interval $[0, T]$, and subsequence of regularized solutions converges to this solution. In our work, we follow the book ”Vorticity and incompressible flow” by A. J. Majda and A. L. Bertozzi. Numerous details are explained and several justifications are added.

We consider some properties of the Euler and the Navier-Stokes equations and a reformulation of these equations due to Leray, which excludes the pressure term from the equations. Also, we prove some properties of the Fourier transform, the Poisson equation, Sobolev spaces, mollifiers, Leray’s projection operator into divergence-free vector space, and Hodge’s decomposition of vector fields, all of which we use in this work.
Chapter 1

An Introduction to Vortex Dynamics for Incompressible Fluid Flows.

Studying vorticity, the curl of the velocity field, is an important aspect of the mathematical theory of incompressible flows. In this chapter we introduce the Euler and the Navier-Stokes equations for incompressible flows, inviscid and viscous respectively, present elementary properties of these equations, and consider some elementary examples. At the end of the chapter, we derive a reformulation of the equations due to Leray, which excludes the pressure term from the equations. Our exposition of this material follows chapter 1 of [6].

1.1 The Euler and the Navier-Stokes Equations.

Incompressible flows of homogeneous fluids in \( \mathbb{R}^N \), \( N = 2, 3 \), are determined by the fluid velocity field \( v(x, t) = (v^1(x, t), v^2(x, t), \ldots, v^N(x, t))^t \) on \( \mathbb{R}^N \times [0, \infty) \). The mathematical model of such flows, that we study, constitutes the initial value problem including the Navier-Stokes equation (1.1), the incompressibility condition (1.2), and the initial velocity field (1.3),

\[
\frac{Dv}{Dt} = -\nabla p + \nu \Delta v, \tag{1.1}
\]

\[
\text{div } v = 0, \quad (x, t) \in \mathbb{R}^N \times [0, \infty), \tag{1.2}
\]

\[
v(x, 0) = v_0, \quad x \in \mathbb{R}^N, \tag{1.3}
\]

where scalar \( p(x, t) \) is the pressure, \( D/Dt \) is the convective derivative,

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} dx_j = \frac{\partial}{\partial t} + \sum_{j=1}^{N} v_j \frac{\partial}{\partial x_j}, \tag{1.4}
\]

the gradient operator \( \nabla \) is

\[
\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_N} \right)^t, \tag{1.5}
\]

the Laplace operator \( \Delta \) is

\[
\Delta = \nabla^2 = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}, \tag{1.6}
\]

and the divergence of a velocity field \( \text{div } v \) is

\[
\text{div } v = \sum_{j=1}^{N} \frac{\partial v_j}{\partial x_j}. \tag{1.7}
\]

The constant kinematic viscosity \( \nu \geq 0 \) and \( \nu \) is inversely proportional to Reynolds number \( Re \). For \( \nu = 0 \) (inviscid flow) the equation (1.1) is called the Euler equation. The Euler
equation is used in many applications to approximate viscous solution for high Reynolds numbers (very small $\nu$).

The Navier-Stokes equation (1.1) follows from the law of conservation of momentum [7]. Note that the equations (1.1), (1.2) contain the time derivative of velocity $v$ only. Later in this chapter we show that the pressure $p$ could be excluded being determined by the velocity $v(x,t)$.

1.2 Symmetry Groups for the Euler and the Navier-Stokes Equations.

Next proposition reviews some elementary symmetry groups for solutions to the Euler and the Navier-Stokes equations.

**Proposition 1.1** (Symmetry groups for the Euler and the Navier-Stokes equations.) Let $v, p$ be a solution to the Euler or the Navier-Stokes equations. Then the following transformations also yield solutions:

(i) **Galilean invariance:** For any constant velocity vector $c \in \mathbb{R}^N$,

$$v_c(x,t) = v(x - ct, t) + c, \quad p_c(x,t) = p(x - ct, t)$$  \hspace{1cm} (1.8)

is also a solution pair.

(ii) **Rotation symmetry:** for any rotation matrix $Q$ ($Q^T = Q^{-1}$),

$$v_Q(x,t) = Q^T v(Qx,t), \quad p_Q(x,t) = p(Qx,t)$$  \hspace{1cm} (1.9)

is also a solution pair.

(iii) **Scale invariance:** for any $\lambda, \tau \in \mathbb{R}$,

$$v_{\lambda, \tau}(x,t) = \frac{\lambda^2}{\tau} v\left(\frac{x}{\lambda}, \frac{t}{\tau}\right), \quad p_{\lambda, \tau}(x,t) = \frac{\lambda^2}{\tau^2} p\left(\frac{x}{\lambda}, \frac{t}{\tau}\right),$$  \hspace{1cm} (1.10)

is a solution pair to the Euler equation, and for any $\tau \in \mathbb{R}^+$,

$$v_\tau(x,t) = \tau^{-1/2} v\left(\frac{x}{\tau^{1/2}}, \frac{t}{\tau}\right), \quad p_\tau(x,t) = \tau^{-1} p\left(\frac{x}{\tau^{1/2}}, \frac{t}{\tau}\right)$$  \hspace{1cm} (1.11)

is also a solution pair to the Navier-Stokes equation.

**Proof.** Let $v(x,t)$ and $p(x,t)$ be a solution pair to the Euler ($\nu = 0$) or to the Navier-Stokes ($\nu > 0$) equations.

(i) Show that $v_c = v(x - ct, t) + c$ and $p_c = p(x - ct, t)$ are solutions of these equations as well $\forall c \in \mathbb{R}^N$. For the i-th component of $v(x,t)$ and $\nabla_x p(x,t)$ by the equation (1.1), we have

$$\frac{\partial v^i}{\partial t} + \sum_{j=1}^N v^j \frac{\partial v^i}{\partial x_j} = - \frac{\partial p}{\partial x_j} + \nu \sum_{j=1}^N \frac{\partial^2 v^i}{\partial x_j^2}$$  \hspace{1cm} (*)
Substitute \( v^i \) and \( p \) by \( v_c^i \) and \( p_c \) respectively. Since

\[
\frac{\partial v_c^i}{\partial t} = \sum_{j=1}^{N} \frac{\partial v^i}{\partial x_j} (-c^j) + \frac{\partial v_i^c}{\partial t}, \quad \frac{\partial^k v_c^i}{\partial x_j^k} = \frac{\partial^k v^i}{\partial x_j^k}, \quad k = 1, 2, \quad \frac{\partial p_c}{\partial x_i} = \frac{\partial p}{\partial x_i},
\]

then we get

\[
\frac{\partial v_c^i}{\partial t} + \sum_{j=1}^{N} \frac{\partial v_c^i}{\partial x_j} (-c^j) + \sum_{j=1}^{N} (v_j^c + c^j) \frac{\partial v_i^c}{\partial x_j} = -\frac{\partial p_c}{\partial x_i} + \nu \sum_{j=1}^{N} \frac{\partial^2 v_i^c}{\partial x_j^2}.
\]

After cancelation in the left hand side, the last equation coincides with the identity (\(*\)). So, \( v_c(x,t) \) and \( p_c(x,t) \) satisfy the Navier-Stokes equation (1.1).

Also, since \( \text{div} \ v_c = \sum_{i=1}^{N} \frac{\partial v_c^i}{\partial x_i} = \sum_{i=1}^{N} \frac{\partial v^i}{\partial x_i} = \text{div} \ v = 0 \), then \( v_c \) satisfies the incompressibility condition (1.2) too.

(ii) Show that \( v_Q(x,t) = Q^{-1}v(Qx,t), \quad p_Q(x,t) = p(Qx,t) \) is also a solution pair to the Euler and the Navier-Stokes equations.

Because \( Q \) is a rotation matrix, \( Q^{-1} = Q^t = Q \) and \( Q^2 = I \), the identity matrix.

Substitute \( v_Q \) and \( p_Q \) in the equations (1.1) and (1.2) instead of \( v \) and \( p \) respectively. We have

\[
Q^{-1} \frac{\partial v}{\partial t} + \sum_{j=1}^{N} Q^{-1} v^j Q^{-1} \frac{\partial v}{\partial x_j} Q = -\nabla_x p \ Q + \nu Q^{-1} \Delta v Q^2
\]

and \( Q^{-1} \text{div} \ v \ Q = 0 \) or \( \text{div} \ v = 0 \).

Multiplying the first of these equations by \( \tau \) and the second one by \( \tau \), we get the Euler equation and the incompressibility condition correspondingly which are identities for the solution pair \( v \) and \( p \). So, \( v_{\lambda,\tau} \) and \( p_{\lambda,\tau} \) is a solution pair to the Euler equation.

(iii) Now substitute into the Euler equation and into the equation (1.2), \( v_{\lambda,\tau} = \frac{\lambda}{\tau} v \left( \frac{x}{\lambda}, \frac{t}{\tau} \right) \) and \( p_{\lambda,\tau} = \frac{\lambda^2}{\tau^2} p \left( \frac{x}{\lambda}, \frac{t}{\tau} \right) \). We have

\[
\frac{\lambda}{\tau^2} \frac{\partial v}{\partial t} + \sum_{j=1}^{N} \frac{\lambda}{\tau} v_j^\lambda \frac{\lambda}{\tau} \frac{\partial v}{\partial x_j} = -\frac{\lambda^2}{\tau^2} \nabla_x p \frac{1}{\lambda} \quad \text{and} \quad \frac{\lambda}{\tau} \text{div} v \frac{1}{\lambda} = 0.
\]

Multiplying the first of these equations by \( \frac{\tau^2}{\lambda} \) and the second one by \( \tau \), we get the Euler equation and the incompressibility condition correspondingly which are identities for the solution pair \( v \) and \( p \). So, \( v_{\lambda,\tau} \) and \( p_{\lambda,\tau} \) is a solution pair to the Euler equation.
Finally, substitute into the equations (1.1) and (1.2) \( v = \frac{1}{\sqrt{\tau}} v \left( \frac{x}{\sqrt{\tau}}, \frac{t}{\tau} \right) \) and \( p = \frac{1}{\tau} p \left( \frac{x}{\sqrt{\tau}}, \frac{t}{\tau} \right) \). Then we have

\[
\frac{1}{\sqrt{\tau}} \frac{\partial v}{\partial t} - \frac{1}{\sqrt{\tau}} \sum_{j=1}^{N} v_j \frac{\partial v_j}{\partial x_j} - \frac{1}{\tau} \nabla x p - \nu \left( \frac{1}{\sqrt{\tau}} \right)^3 \Delta v
\]

and

\[
\frac{1}{\sqrt{\tau}} \text{div} \frac{1}{\sqrt{\tau}} = 0.
\]

Multiplying the first of these equations by \( \tau^{3/2} \) and the second one by \( \tau \), we get the Navier-Stokes equation and the incompressibility condition correspondingly which are again identities for the solution pair \( v \) and \( p \). Thus, \( v_\tau \) and \( p_\tau \) is a solution pair to the Navier-Stokes equation.

\[
\text{Notice that inviscid flows require a two-parameter symmetry group for scaling transformations while flows with viscosity need only a one parameter group for that.}
\]

### 1.3 Particle Trajectories.

**Definition 1.1.** Define the particle-trajectory mapping \( X(\cdot, t) : \alpha \in \mathbb{R}^N \mapsto X(\alpha, t) \in \mathbb{R}^N \) where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)^t \) is the initial point of fluid particle location at time \( t = 0 \) and \( X(\alpha, t) = (X_1, X_2, \ldots, X_N)^t \) is the location at time \( t \). The parameter \( \alpha \) is called the Lagrangian particle marker. For given fluid velocity \( v(x, t) \), the particle-trajectory mapping is determined by the following initial value problem:

\[
\frac{dX(\alpha, t)}{dt} = v(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha \tag{1.12}
\]

Note that in general the ODE (1.12) is nonlinear.

**Definition 1.2.** Consider an initial domain \( \Omega \subset \mathbb{R}^N \) in fluid at time \( t = 0 \). In time \( t \), fluid particles move from the domain \( \Omega \) to \( X(\Omega, t) = \{ X(\alpha, t) : \alpha \in \Omega \} \). Define the Jacobian of this transformation by

\[
J(\alpha, t) = \det \left( \nabla_\alpha X(\alpha, t) \right), \tag{1.13}
\]

where \( \nabla_\alpha = \left[ \frac{\partial}{\partial \alpha_1}, \ldots, \frac{\partial}{\partial \alpha_N} \right] \)

Next we consider a relation between rate of change of \( J \) and fluid velocity \( v \). To do this, we need the following lemma about derivative of determinant.
Lemma 1.1. Let $A(t) = [a_{ij}(t)]$ with $i, j = 1, 2, \ldots, N$ be a matrix of order $N$ and its elements $a_{ij} \in C^1((t_1, t_2))$. Then $\forall t \in (t_1, t_2)$, $\frac{d}{dt} \det(A) = \sum_{i,j=1}^{N} \frac{d}{dt} a_{ij} A_{ij}$ where $A_{ij}$ are cofactors of elements $a_{ij}$.

Proof. Consider the set of columns $c_j = [a_{ij}]_{i=1}^{N}, j = 1, 2, \ldots, N$. Then $A = [c_1 c_2 \ldots c_N]$. Hence

$$
\frac{d}{dt} \det(A(t)) = \lim_{h \to 0} \frac{1}{h} [\det(A(t + h)) - \det(A(t))] = 
\lim_{h \to 0} \frac{1}{h} \{\det([c_1(t + h) c_2(t + h) \ldots c_N(t + h)]) - \det([c_1(t) c_2(t) \ldots c_N(t)])\} = 
\lim_{h \to 0} \frac{1}{h} \{\det([c_1(t + h) c_2(t + h) \ldots c_N(t + h)]) - \det([c_1(t) c_2(t + h) \ldots c_N(t + h)]) - 
\det([c_1(t) c_2(t) c_3(t + h) \ldots c_N(t + h)]) - \ldots 
- \det([c_1(t) c_2(t) \ldots c_{N-1}(t) c_N(t + h)]) + 
\det([c_1(t) c_2(t) \ldots c_{N-1}(t) c_N(t + h)]) - \det([c_1(t) c_2(t) \ldots c_{N-1}(t) c_N(t)])\} = 
\lim_{h \to 0} \det \left( \left[ \frac{c_1(t+h)-c_1(t)}{h} \ c_2(t+h) \ldots c_N(t+h) \right] \right) + 
\lim_{h \to 0} \det \left( \left[ c_1(t) \frac{c_2(t+h)-c_2(t)}{h} \ c_3(t+h) \ldots c_N(t+h) \right] \right) + \ldots 
+ \lim_{h \to 0} \det \left( \left[ c_1(t) c_2(t) \ldots c_{N-1}(t) \frac{c_N(t+h)-c_N(t)}{h} \right] \right) = 
\det \left( \left[ \frac{dc_1}{dt} \ c_2 \ldots c_N \right] \right) + \det \left( \left[ c_1 \frac{dc_2}{dt} \ c_3 \ldots c_N \right] \right) + \ldots + \det \left( \left[ c_1 \ c_2 \ldots c_{N-1} \frac{dc_N}{dt} \right] \right),
$$

using the multilinearity of the determinant.

Decompose each determinant in this sum by column consisting of derivatives. Then we get

$$
\frac{d}{dt} \det[A(t)] = \sum_{i=1}^{N} \frac{d}{dt} [a_{i1}(t)] A_{i1}(t) + \sum_{i=1}^{N} \frac{d}{dt} [a_{i2}(t)] A_{i2}(t) + \ldots 
+ \sum_{i=1}^{N} \frac{d}{dt} [a_{iN}(t)] A_{iN}(t) = \sum_{i,j=1}^{N} \frac{d}{dt} [a_{ij}(t)] A_{ij}(t).
$$

Proposition 1.2 is an application of this lemma.

Proposition 1.2. Let $X(\cdot, t)$ be a particle-trajectory mapping of a smooth velocity field $v \in \mathbb{R}^N$. Then

$$
\frac{\partial J}{\partial t} = (\text{div}_x v)|_{(X(\alpha, t), t)} J(\alpha, t).
$$

(1.14)
Proof. Consider

\[ \nabla_\alpha X(\alpha, t) = \begin{bmatrix} \nabla_\alpha X_1(\alpha, t) & \nabla_\alpha X_2(\alpha, t) & \ldots & \nabla_\alpha X_N(\alpha, t) \end{bmatrix} = \begin{bmatrix} \frac{\partial X_1}{\partial \alpha_1} & \frac{\partial X_1}{\partial \alpha_2} & \ldots & \frac{\partial X_1}{\partial \alpha_N} \\ \frac{\partial X_2}{\partial \alpha_1} & \frac{\partial X_2}{\partial \alpha_2} & \ldots & \frac{\partial X_2}{\partial \alpha_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial X_N}{\partial \alpha_1} & \frac{\partial X_N}{\partial \alpha_2} & \ldots & \frac{\partial X_N}{\partial \alpha_N} \end{bmatrix}. \]

Let \( A_{ij} \) be cofactors of the elements \( \frac{\partial X_i}{\partial \alpha_j} \). Then decomposing \( \det(\nabla_\alpha X(\alpha, t)) \) by the \( k \)-th column \((1 \leq k \leq N)\) and using well-known properties of determinants, we have

\[ \sum_{j=1}^{N} \frac{\partial X_k}{\partial \alpha_j} A_{ij} = \delta_{ik} \det(\nabla_\alpha X(\alpha, t)) = \delta_{ik} J, \]

where Kronecker’s symbol \( \delta_{ik} = \begin{cases} 1, & \text{if } k = i \\ 0, & \text{if } k \neq i \end{cases} \).

Now by the definition of \( J \) (see (1.13)) and the Lemma 1.1, we obtain

\[ \frac{\partial J}{\partial t} = \frac{\partial}{\partial t} \det \left( \frac{\partial X_i(\alpha, t)}{\partial \alpha_j} \right) = \sum_{i,j=1}^{N} \frac{\partial}{\partial t} \left( \frac{\partial X_i(\alpha, t)}{\partial \alpha_j} \right) A_{ij} = \sum_{i,j=1}^{N} \frac{\partial}{\partial t} \left( \frac{\partial X_i(\alpha, t)}{\partial \alpha_j} \right) \frac{\partial v^i}{\partial x_k} \frac{\partial X_k}{\partial \alpha_j} = \sum_{i,k=1}^{N} \sum_{j=1}^{N} A_{ij} \frac{\partial}{\partial \alpha_j} X_k = \sum_{i,k=1}^{N} v^i X_k \frac{\partial}{\partial \alpha_j} \sum_{j=1}^{N} A_{ij} = \left( \sum_{i=1}^{N} v^i X_i \right) J = \text{div}_X \left[ v \left( X(\alpha, t), t \right) \right] J. \]

As an immediate consequence of this proposition, we get that the incompressibility condition (1.2) implies \( J(\alpha, t) \) is constant in time. Moreover, since at the time \( t = 0 \), \( X(\alpha, 0) = \alpha \), then \( J(\alpha, 0) = \det(\nabla_\alpha \alpha) = \det(I) = 1 \), so \( J(\alpha, t) = J(\alpha, 0) = 1, \forall t > 0. \)

In its turn, the fact that the Jacobian of the transformation is 1 implies that the flow preserves the volume of a domain.

**Definition 1.3.** A flow \( X(\cdot, t) \) is incompressible if for all subdomains \( \Omega \) with smooth boundaries and any \( t > 0 \), the flow is volume preserving:

\[ \text{vol} \ X(\Omega, t) = \text{vol} \ \Omega. \]

Next, a calculus formula determines the rate of change of any smooth function in domain \( X(\Omega, t) \) moving with the fluid.

**Proposition 1.3** (The Transport Formula). Let \( \Omega \subset \mathbb{R}^N \) be an open, bounded domain with
a smooth boundary, and let $X$ be a given particle-trajectory mapping of a smooth velocity field $v$. Then for any smooth function $f(x, t)$,

$$\frac{d}{dt} \int_{X(\Omega,t)} f \, dx = \int_{X(\Omega,t)} [f_t + \text{div}_x(fv)] \, dx. \quad (1.15)$$

**Proof.** Use the change of variables, at first returning from $X(\Omega, t)$ to the initial domain $\Omega$ and then moving back to $X(\Omega, t)$.

$$\int_{X(\Omega,t)} f(x,t) \, dx = \int_{\Omega} f(X(\alpha,t), t) \, J(\alpha,t) \, d\alpha.$$ 

Then applying Proposition 1.2, we get

$$\frac{d}{dt} \left[ \int_{X(\Omega,t)} f(x,t) \, dx \right] = \int_{\Omega} \left[ \left( \frac{\partial f}{\partial t} + \sum_{j=1}^{N} \frac{\partial f}{\partial X_j} \frac{\partial X_j}{\partial t} \right) J(\alpha,t)+f(X(\alpha,t), t) \frac{\partial J}{\partial t} \right] \, dx =$$

$$\int_{\Omega} \left[ \left( \frac{\partial f}{\partial t} + \sum_{j=1}^{N} \frac{\partial f}{\partial X_j} v^j \right) J + f \frac{\partial J}{\partial t} \right] \, dx = \int_{\Omega} \left( \frac{\partial f}{\partial t} + \nabla_X f \cdot v + f \text{div}_x v \right) J \, dx =$$

$$\int_{\Omega} \left[ \frac{\partial f}{\partial t} + \text{div}_x(fv) \right] J \, dx = \int_{X(\Omega,t)} \left[ \frac{\partial f}{\partial t} + \text{div}_x(fv) \right] \, dx. \quad \square$$

Note that for $f \equiv 1$, the Transport formula gives

$$\frac{d}{dt} \text{vol} X(\Omega,t) = \int_{X(\Omega,t)} \text{div}_x f \, dx,$$

which explains the equivalence of volume preservation and the incompressibility condition $\text{div}_x v = 0$.

We summarize the results in the following proposition.

**Proposition 1.4.** For the smooth flow the following three conditions are equivalent:

(i) a flow is incompressible, i.e., $\forall \Omega \subset \mathbb{R}^N$, $t \geq 0$, $\text{vol} X(\Omega,t) = \text{vol} \Omega$,

(ii) $\text{div} v = 0$,

(iii) $J(\alpha,t) = 1, \forall t \geq 0$.

1.4 The Vorticity, a Deformation Matrix, and Some Elementary Exact Solutions.

In this section we consider some models of typical local behavior of incompressible flow.
Definition 1.4. Consider $3 \times 3$ matrix $\nabla v = [v_{xj}]$. Then $\nabla v$ has the decomposition

$$\nabla v = D + \Omega,$$

(1.16)

where $D$ is a symmetric matrix, and $\Omega$ is an antisymmetric matrix, given respectively by

$$D = \frac{1}{2} [\nabla v + (\nabla v)^t],$$

(1.17)

$$\Omega = \frac{1}{2} [\nabla v - (\nabla v)^t].$$

(1.18)

$I$ is called the deformation or rate-of-strain matrix and $\Omega$ is called the rotation matrix.

Note that $\text{div} \ v = \sum_{i=1}^{3} v_{xi} = \text{tr} \ D = \text{tr}(\nabla v)$. So, for the incompressible flow, $\text{tr} \ D = 0$.

Next, recall the definition of the vorticity.

Definition 1.5. Let $v$ be a smooth vector field. The vorticity or rotor of the vector field $v$ is

$$\omega = \text{curl} \ v \equiv \nabla \times v = \left( \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right).$$

(1.19)

Remark. The rotation matrix $\Omega$ and the vorticity $\omega$ of the vector field $v$ satisfy the following relation:

$$\Omega h = \frac{1}{2} \omega \times h, \quad \forall h \in \mathbb{R}^3.$$

(1.20)

Proof. Let $\omega = (\omega^1, \omega^2, \omega^3)^t$ be defined by the equation (1.19). Then $\Omega = \frac{1}{2} [\nabla v - (\nabla v)^t]$ is given by

$$\Omega = \frac{1}{2} \begin{pmatrix} 0 & -\omega^3 & \omega^2 \\ -\omega^3 & 0 & -\omega^1 \\ \omega^2 & \omega^1 & 0 \end{pmatrix}.$$

Let $h = (h_1, h_2, h_3)^t \in \mathbb{R}^N$. Consider $\Omega h = \frac{1}{2} \omega \times h$.

The following lemma provides a simple local description for an incompressible fluid flow.

Lemma 1.2. To linear order in $|x - x_0|$, every smooth incompressible velocity field $v(x, t)$ is the unique sum of three terms:

$$v(x, t_0) = v(x_0, t_0) + \frac{1}{2} \omega \times (x - x_0) + D(x - x_0),$$

(1.21)

where $D$ is the symmetric deformation matrix with $\text{tr} D = 0$ and $\omega$ is the vorticity.
Proof. Let \( x_0, x \in \mathbb{R}^N \) and \( t_0 \geq 0 \). Using a Taylor series expansion for a smooth velocity field \( v(x, t) \) at a fixed point \((x_0, t_0)\), we get

\[
v(x, t_0) = v(x_0, t_0) + \nabla_x [v(x_0, t_0)] (x - x_0) + \mathcal{O} \left( |x - x_0|^2 \right).
\]

Then applying the definition (1.16) and the remark (1.20), we have

\[
v(x, t_0) = v(x_0, t_0) + \Omega(x - x_0) + \mathcal{D}(x - x_0) + \mathcal{O} \left( |x - x_0|^2 \right)
= v(x_0, t_0) + \frac{1}{2} \omega \times (x - x_0) + \mathcal{D}(x - x_0) + \mathcal{O} \left( |x - x_0|^2 \right).
\]

So, the equation (1.21) holds to linear order in \(|x - x_0|\).

The next corollary gives a natural physical interpretation to each of three terms in the equation (1.21).

**Corollary 1.1.** To linear order in \(|x - x_0|\), every smooth incompressible velocity field \( v(x, t) \) is the sum of infinitesimal translation, rotation, and deformation velocities.

**Proof.** Consider the initial value problem (1.12) for a particle-trajectory mapping corresponding to a fluid velocity field \( v(x, t) \).

1) Retaining the term \( v(x_0, t_0) \) only in the equation (1.21), we have

\[
\frac{dX}{dt} = v(x_0, t_0), \quad X\big|_{t=t_0} = \alpha.
\]

Integrating from \( t_0 \) to \( t > t_0 \), we get

\[
X(\alpha, t) = \alpha + v(x_0, t_0)(t - t_0).
\]

This equation describes an infinitesimal translation.

2) Now retain the second term \( \frac{1}{2} \omega \times (x - x_0) \) only in the equation (1.21). We obtain

\[
\frac{dX}{dt} = \frac{1}{2} \omega \times (X - X_0), \quad X\big|_{t=t_0} = \alpha.
\]

Decompose vorticity \( \omega = (\omega^1, \omega^2, \omega^3)^t = (\omega^1, 0, 0)^t + (0, \omega^2, 0)^t + (0, 0, \omega^3)^t \) and consider a solution of the ODE above as the superposition of solutions of this ODE with \( \omega \) replaced by each of its components separately. So for the third component of \( \omega \), we have

\[
\frac{dX}{dt} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \omega^3 \end{bmatrix} \times \begin{bmatrix} X^1 - X^1_0 \\ X^2 - X^2_0 \\ X^3 - X^3_0 \end{bmatrix} = \frac{\omega^3}{2} \begin{bmatrix} -(X^2 - X^2_0) \\ (X^1 - X^1_0) \\ 0 \end{bmatrix}.
\]
Consider the ODEs for each row of the matrices.

\[
\begin{align*}
\frac{dX^1}{dt} &= -\frac{\omega^3}{2} (X^2 - X^2_0) \\
\frac{dX^2}{dt} &= \frac{\omega^3}{2} (X^1 - X^1_0) \\
\frac{dX^3}{dt} &= 0,
\end{align*}
\]

Differentiating the first of these equations and applying the second one to the result, we get

\[
\frac{d^2}{dt^2} (X^1 - X^1_0) = -\frac{(\omega^3)^2}{4} (X^1 - X^1_0).
\]

Solve this equation. Then substituting the solution we get to the first equation from the system above, we obtain

\[
\begin{align*}
X^1(t) - X^1_0 &= C_1 \cos \left(\frac{\omega^3}{2} t\right) + C_2 \sin \left(\frac{\omega^3}{2} t\right), \quad X^1|_{t=t_0} = \alpha^1, \\
X^2(t) - X^2_0 &= C_1 \sin \left(\frac{\omega^3}{2} t\right) - C_2 \cos \left(\frac{\omega^3}{2} t\right), \quad X^2|_{t=t_0} = \alpha^2, \\
X^3(t) &= \alpha^3,
\end{align*}
\]

where \(C_1, C_2 \in \mathbb{R}\).

Introduce the following two-row matrices:

\[
X' = \begin{bmatrix} X^1 \\ X^2 \end{bmatrix}, \quad X'_0 = \begin{bmatrix} X^1_0 \\ X^2_0 \end{bmatrix}, \quad \alpha' = \begin{bmatrix} \alpha^1 \\ \alpha^2 \end{bmatrix}, \quad C' = \begin{bmatrix} C^1 \\ C^2 \end{bmatrix},
\]

and the rotation in the plane \(X^1OX^2\) matrix \(Q(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}\), where \(\phi\) is an angle of rotation about axis parallel to the \(X^3\) axis.

Then we have:

\[
X'(t) - X'_0 = Q(\frac{\omega^3}{2} t) C', \quad \alpha' - X'_0 = Q(\frac{\omega^3}{2} t) C'.
\]

Exclude \(C'\) and take in count that \(Q^{-1} = Q'\) and that

\[
\begin{align*}
Q(\frac{\omega^3}{2} t) Q(\frac{\omega^3 t_0}{2}) &= \begin{bmatrix} \cos \left(\frac{\omega^3}{2} t\right) & -\sin \left(\frac{\omega^3}{2} t\right) \\ \sin \left(\frac{\omega^3}{2} t\right) & \cos \left(\frac{\omega^3}{2} t\right) \end{bmatrix} \begin{bmatrix} \cos \left(\frac{\omega^3 t_0}{2}\right) & \sin \left(\frac{\omega^3 t_0}{2}\right) \\ -\sin \left(\frac{\omega^3 t_0}{2}\right) & \cos \left(\frac{\omega^3 t_0}{2}\right) \end{bmatrix} \\
&= \begin{bmatrix} \cos \left(\frac{\omega^3 (t-t_0)}{2}\right) & -\sin \left(\frac{\omega^3 (t-t_0)}{2}\right) \\ \sin \left(\frac{\omega^3 (t-t_0)}{2}\right) & \cos \left(\frac{\omega^3 (t-t_0)}{2}\right) \end{bmatrix} = Q \left(\frac{\omega^3 (t-t_0)}{2}\right).
\end{align*}
\]

Hence \(C' = Q^t \left(\frac{\omega^3 t_0}{2}\right) (\alpha' - X'_0)\) and \(X'(t) - X'_0 = Q \left(\frac{\omega^3 t}{2}\right) Q^t \left(\frac{\omega^3 t_0}{2}\right) (\alpha' - X'_0)\) or \(X'(t) - X'_0 = Q \left(\frac{\omega^3 (t-t_0)}{2}\right) (\alpha' - X'_0)\).

Note that \(\det(Q) = 1\) and \(\|X'(t) - X'_0\| = \|\alpha' - X'_0\|, \forall t > 0\). Therefore these
trajectories are circles centered at \( X_0 \) on the plane \( X^3 = \alpha^3 \) parallel to the coordinate plane \( X^1OX^2 \). So, replacing \( \omega \) by its third component, we get a rotation in the direction of \((0, \ 0, \ \omega^3)^t\) with angular velocity \( \frac{1}{2} |\omega^3| \).

Similarly, for the first and second components of \( \omega \), we get rotations in the directions of \((\omega^1, \ 0, \ 0)^t\) and \((0, \ \omega^2, \ 0)^t\) with angular velocities \( \frac{1}{2} |\omega^1| \) and \( \frac{1}{2} |\omega^2| \) respectively. Superposing these solutions for each components of \( \omega \), we conclude that the second term \( \frac{1}{2} \omega \times (X - X_0) \) in the equation (1.21) is an infinitesimal rotation in the direction of \( \omega \) with angular velocity \( \frac{1}{2} |\omega| \).

3) Finally, retain the term \( D(X - X_0) \) only in the particle-trajectory equation (1.21). Since \( D \) is a symmetric matrix, then there exists a rotation matrix \( Q \) such that \( Q^t D Q = \text{diag}(\gamma_1, \ \gamma_2, \ \gamma_3) \). Then \( \text{tr}(D) = \text{tr}(Q^t D Q) = \gamma_1 + \gamma_2 + \gamma_3 = 0 \) because traces are invariant under similarity transformations. So, \( \gamma_3 = -(\gamma_1 + \gamma_2) \). By the rotation symmetry (the Proposition 1.1(ii)), we can replace \( v = D(X - X_0) \) by \( v_Q = Q^t D Q (X - X_0) = \text{diag}(\gamma_1, \ \gamma_2, \ -(\gamma_1 + \gamma_2))(X - X_0) \).

Thus, we have the following initial value problem:

\[
\frac{dX}{dt} = \text{diag}(\gamma_1, \ \gamma_2, \ -(\gamma_1 + \gamma_2))(X - X_0), \quad X|_{t = t_0} = \alpha.
\]

After integration, we get

\[
X(t) - X_0 = \text{diag} \left( e^{\gamma_1 t}, \ e^{\gamma_2 t}, \ e^{-(\gamma_1 + \gamma_2) t} \right) C, \quad \text{where } C \in \mathbb{R}^3.
\]

Using the initial value \( X|_{t = t_0} = \alpha \) and the fact that \( \text{diag}^{-1}(a_1, \ a_2, \ a_3) = \text{diag}(a_1^{-1}, \ a_2^{-1}, \ a_3^{-1}) \) with \( a_1, \ a_2, \ a_3 \in \mathbb{R} \) and \( a_1^2 + a_2^2 + a_3^2 \neq 0 \), we have

\[
\begin{align*}
C & = \text{diag} \left( e^{-\gamma_1 t_0}, \ e^{-\gamma_2 t_0}, \ e^{-(\gamma_1 + \gamma_2) t_0} \right) (\alpha - X_0) .
\end{align*}
\]

Therefore

\[
X(t) = X_0 + \text{diag} \left( e^{\gamma_1 (t - t_0)}, \ e^{\gamma_2 (t - t_0)}, \ e^{-(\gamma_1 + \gamma_2) (t - t_0)} \right) (\alpha - X_0).
\]

This solution describes a deformation of the fluid. For example, let \( X_0 = 0 \) and \( \gamma_1, \ \gamma_2 < 0 \). Then the fluid particles create a jet along \( X_3 \) axis (see Ex. 1.1).

In general, the third term \( D(X - X_0) \) in the equation (1.21) represents an infinitesimal deformation velocity in the direction \((X - X_0)\).

So, we conclude that up to linear order in \( |X - X_0| \), incompressible velocity field is the sum of infinitesimal translation, rotation, and deformation velocities.

\[
\square
\]

In the next proposition, we consider a large class of exact solutions to the Euler and the Navier-Stokes equations. To prove this proposition, we will need the following lemma.
Lemma 1.3. Let $A$ be a real, symmetric, $3 \times 3$ matrix with $\text{tr}A = 0$, and $b, c \in \mathbb{R}^3$. Then

$$-A(b \times c) = b \times (Ac) + (Ab) \times c.$$

(1.22)

Proof. Here we provide a straightforward computational proof.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ with $a_{11} + a_{22} + a_{33} = 0$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, and $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$.

1) Compute L.H.S. of the equation (1.22).

$$b \times c = (b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1)^t,$$

$$-A(b \times c) = -\left[\begin{array}{c}
[a_{11}(b_2c_3 - b_3c_2) + a_{12}(b_3c_1 - b_1c_3) + a_{13}(b_1c_2 - b_2c_1)] \\
[a_{12}(b_2c_3 - b_3c_2) + a_{22}(b_3c_1 - b_1c_3) + a_{23}(b_1c_2 - b_2c_1)] \\
[a_{13}(b_2c_3 - b_3c_2) + a_{23}(b_3c_1 - b_1c_3) + a_{33}(b_1c_2 - b_2c_1)]
\end{array}\right]$$

$$= \begin{bmatrix}
(b_1(a_{12}c_3 - a_{13}c_2) + b_2(a_{13}c_1 - a_{11}c_3) + b_3(a_{11}c_2 - a_{12}c_1)) \\
(b_1(a_{22}c_3 - a_{23}c_2) + b_2(a_{23}c_1 - a_{12}c_3) + b_3(a_{12}c_2 - a_{22}c_1)) \\
(b_1(a_{23}c_3 - a_{33}c_2) + b_2(a_{33}c_1 - a_{13}c_3) + b_3(a_{13}c_2 - a_{23}c_1))
\end{bmatrix}.$$

2) Now compute the R.H.S. of the equation (1.22).

$$b \times (Ac) = \begin{bmatrix}
b_2(a_{13}c_1 + a_{23}c_2 + a_{33}c_3) - b_3(a_{12}c_1 + a_{22}c_2 + a_{23}c_3) \\
b_3(a_{11}c_1 + a_{12}c_2 + a_{13}c_3) - b_1(a_{13}c_1 + a_{23}c_2 + a_{33}c_3) \\
b_1(a_{12}c_1 + a_{22}c_2 + a_{23}c_3) - b_2(a_{11}c_1 + a_{12}c_2 + a_{13}c_3)
\end{bmatrix}$$

$$(Ab) \times c = \begin{bmatrix}
(a_{12}b_1 + a_{22}b_2 + a_{33}b_3)c_3 - (a_{13}b_1 + a_{23}b_2 + a_{33}b_3)c_2 \\
[a_{13}b_1 + a_{23}b_2 + a_{33}b_3)c_1 - (a_{11}b_1 + a_{12}b_2 + a_{13}b_3)c_3] \\
[a_{11}b_1 + a_{12}b_2 + a_{13}b_3)c_2 - (a_{12}b_1 + a_{22}b_2 + a_{33}b_3)c_1]
\end{bmatrix}$$

$$= \begin{bmatrix}
(b_1(a_{12}c_3 - a_{13}c_2) + b_2(a_{22}c_3 - a_{23}c_2) + b_3(a_{23}c_3 - a_{33}c_2)) \\
(b_1(a_{13}c_1 - a_{11}c_3) + b_2(a_{23}c_1 - a_{12}c_3) + b_3(a_{33}c_1 - a_{13}c_3)) \\
(b_1(a_{12}c_2 - a_{13}c_1) + b_2(a_{22}c_2 - a_{23}c_1) + b_3(a_{13}c_2 - a_{23}c_1))
\end{bmatrix}.$$

Hence the sum $b \times (Ac) + (Ab) \times c = \begin{bmatrix}
[b_1(a_{12}c_3 - a_{13}c_2) + b_2(a_{13}c_1 + a_{33}c_3 + a_{22}c_3) + b_3(-a_{12}c_1 - a_{22}c_2 - a_{33}c_2)] \\
([b_1(-a_{23}c_2 - a_{33}c_3 - a_{11}c_3) + b_2(a_{23}c_1 - a_{12}c_3) + b_3(a_{11}c_1 + a_{12}c_2 + a_{33}c_1)] \\
[b_1(a_{22}c_2 + a_{23}c_3 + a_{11}c_2) + b_2(-a_{11}c_1 - a_{13}c_3 - a_{22}c_1) + b_3(a_{13}c_2 - a_{23}c_1)]
\end{bmatrix}$

Finally, using that $\text{tr}(A) = 0$, we get $b \times (Ac) + (Ab) \times c = \begin{bmatrix}
[b_1(a_{12}c_3 - a_{13}c_2) + b_2(a_{13}c_1 - a_{11}c_3) + b_3(a_{11}c_2 - a_{12}c_1)] \\
[b_1(a_{22}c_3 - a_{23}c_2) + b_2(a_{23}c_1 - a_{12}c_3) + b_3(a_{12}c_2 - a_{22}c_1)] \\
[b_1(a_{23}c_3 - a_{33}c_2) + b_2(a_{33}c_1 - a_{13}c_3) + b_3(a_{13}c_2 - a_{23}c_1)]
\end{bmatrix} = -A(b \times c)$. 

□
Consider now the following important proposition.

**Proposition 1.5.** Let $\mathcal{D}(t)$ be a real, symmetric, $3 \times 3$ matrix with $\text{tr} \mathcal{D}(t) = 0$. Determine the vorticity $\omega(t)$ from the ODE in $\mathbb{R}^3$,

$$\frac{d\omega}{dt} = \mathcal{D}(t)\omega, \quad \omega|_{t=0} = \omega_0 \in \mathbb{R}^3,$$

(1.23)

and the antisymmetric matrix $\Omega$ by means of the formula $\Omega h = \frac{1}{2} \omega \times h, \quad h \in \mathbb{R}^3$. Then

$$v(x, t) = \frac{1}{2} \omega(t) \times h + \mathcal{D}(t)x,$$

(1.24a)

$$p(x, t) = -\frac{1}{2} \left[ \mathcal{D}(t) + \mathcal{D}^2(t) + \Omega^2(t) \right] x \cdot x$$

(1.24b)

are exact solutions to the three-dimensional Euler and the Navier-Stokes equations.

**Proof.**

1) Consider componentwise the Navier-Stokes equation (1.1).

$$v^i_t + \sum_{j=1}^{3} v^j v^i_{x_j} = -p_{x_i} + \nu \Delta_x v^i, \quad i = 1, 2, 3.$$

Differentiating this equation by $x_k, \quad k = 1, 2, 3$, we have

$$(v^i_{x_k})_t + \sum_{j=1}^{3} v^j (v^i_{x_k})_{x_j} + \sum_{j=1}^{3} v^j_{x_k} v^i_{x_j} = -p_{x_i x_k} + \nu \Delta_x (v^i_{x_k}).$$

or

$$\frac{D}{Dt}(v^i_{x_k}) + \sum_{j=1}^{3} v^j_{x_k} v^i_{x_j} = -p_{x_i x_k} + \nu \Delta_x (v^i_{x_k}).$$

Introduce the notation $V \equiv \Delta_x v = [v^j_{x_k}]$ and $P \equiv [p_{x_i x_k}]$, Hessian matrix of the pressure $p$.

Then from the last equation we get the following matrix equation for $V$.

$$\frac{DV}{Dt} + V^2 = -P + \nu \Delta V.$$  

(1.25)

2) Represent $V$ again as a sum of its symmetric part $\mathcal{D}$ and antisymmetric part $\Omega$. Hence $V^2 = (\mathcal{D} + \Omega)^2 = (\mathcal{D}^2 + \Omega^2) + (\mathcal{D} \Omega + \Omega \mathcal{D})$, where the first term is symmetric and the second term is antisymmetric.

Note that $P$ is a symmetric matrix.

So, the symmetric part of the equation (1.25) is

$$\frac{D\mathcal{D}}{Dt} + \mathcal{D}^2 + \Omega^2 = -P + \nu \Delta \mathcal{D},$$  

(1.26)
and the antisymmetric part of the equation (1.25) is
\[ \frac{D\Omega}{Dt} + \mathcal{D}\Omega + \Omega \mathcal{D} = \nu \Delta \Omega. \]  
(1.27)

To get an equation in terms of vorticity, use that
\[ \Omega h = \frac{1}{2} \omega \times h, \quad \forall h \in \mathbb{R}^3. \]

Multiplying the matrix equation (1.27) by \( h \) from the right, we have
\[ \frac{D(\Omega h)}{Dt} + \mathcal{D}(\Omega h) + \Omega(Dh) = \nu \Delta (\Omega h) \]
or
\[ \frac{D}{Dt} (\omega \times h) + \mathcal{D}(\omega \times h) + \omega \times (\mathcal{D}h) = \nu \Delta (\omega \times h). \]

Applying the lemma 1.3, we get
\[ \mathcal{D}(\omega \times h) = -\omega \times (\mathcal{D}h) - (\mathcal{D}\omega) \times h. \]

Then we obtain
\[ \frac{D\omega}{Dt} \times h - (\mathcal{D}\omega) \times h = \nu (\Delta \omega \times h) \]
or
\[ \left( \frac{D\omega}{Dt} - \mathcal{D}\omega - \nu \Delta \omega \right) \times h = 0, \quad \forall h \in \mathbb{R}^3, \]
that gives
\[ \frac{D\omega}{Dt} = \mathcal{D}\omega + \nu \Delta \omega. \]  
(1.28)

The equations (1.26) and (1.28) have fundamental meaning in the incompressible flow theory.

3) Assume that both \( \mathcal{D} \) and \( \omega \) do not depend on the spatial variable \( x \) and that
\[ v(x, t) = \frac{1}{2} \omega(t) \times x + \mathcal{D}(t)x. \]

Then \( \nabla_x \omega, \Delta_x \omega = 0, \frac{D\omega}{Dt} = \frac{d\omega}{dt} + v \cdot \nabla_x \omega = \frac{d\omega}{dt}. \)

So, the equation for vorticity (1.28) reduces to the ODE \( \frac{d\omega}{dt} = \mathcal{D}\omega. \)

4) It remains to find the pressure \( p \) by solving the equation (1.26), the symmetric part of the Navier-Stokes equation.

Since under our assumption \( \Delta_x \mathcal{D}(t) = 0 \), and \( \Omega \) does not depend on \( x \) because \( \Omega h = \frac{1}{2} \omega(t)h \), then the equation (1.26) gives \( -P(t) = \frac{dp}{dt} + \mathcal{D}^2 + \Omega^2. \)

Note that the right-hand side of this equation is a known symmetric matrix.

Because second order derivatives of \( p \) by spatial variable, elements of the Hessian matrix \( P \), do not depend on \( x \), it is reasonable to represent \( p(x, t) \) as a quadratic function of \( x \).
Let \( p(x, t) = ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x_2 + 2ex_1x_3 + 2fx_2x_3 \), where \( a, b, c, d, e, f \) are some functions of \( t \).

Then \( a = \frac{1}{2}p_{x_1x_1}, b = \frac{1}{2}p_{x_2x_2}, c = \frac{1}{2}p_{x_3x_3}, d = \frac{1}{2}p_{x_1x_2}, e = \frac{1}{2}p_{x_1x_3}, \) and \( f = \frac{1}{2}p_{x_2x_3} \).

So \( p(x, t) = \frac{1}{2} \left( p_{x_1x_1}x_1^2 + p_{x_1x_2}x_1x_2 + p_{x_1x_3}x_1x_3 + p_{x_2x_1}x_1x_2 + p_{x_2x_2}x_2^2 + p_{x_2x_3}x_2x_3 + p_{x_3x_1}x_1x_3 + p_{x_3x_2}x_2x_3 + p_{x_3x_3}x_3^2 \right) = \frac{1}{2} \left( p_{x_1x_1} + p_{x_2x_2} + p_{x_3x_3} \right) \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2} P \cdot x.

Thus, \( v(x, t) \) and \( p(x, t) \) determined by the equations (1.24), satisfy the Navier-Stokes equation by the construction above. Also, under our assumptions \( (v(x, t) \) is linear in \( x \) and \( \omega = \omega(t) \) \( \Delta_x v, \nabla_x \omega = 0 \), that gives an independence on viscosity. So solutions to the equations (1.24) \( v(x, t) \) and \( p(x, t) \) satisfy the Euler equation as well.

At the end of this section, we consider some simple examples of the exact solutions to the equations (1.24) to illustrate the interaction between a rotation and a deformation.

Note that Corollary 1.1 is applicable up to the linear order in \( x \) while the pressure \( p \) in the equations (1.24) has a quadratic behavior. So, solutions to the equations (1.24) have direct physical meaning described in Corollary 1.1 locally in space and time only.

**Example 1.1 (Jet Flows).**

In the equations (1.23), (1.24), take \( D = \det(\gamma_1, \gamma_2, \gamma_1 + \gamma_2), \gamma_i > 0, i = 1, 2 \) and \( \omega_0 = 0 \). Then \( \omega = \det(e^{-\gamma_1t}, e^{-\gamma_2t}, e^{(\gamma_1+\gamma_2)t})C, C \in \mathbb{R}^3, \omega|_{t=0} = IC = 0 \). So, \( C = 0 \) and \( \omega = 0 \), i.e., this flow is irrotational. Hence for the particle-trajectory mapping we have the following initial value problem:

\[
\frac{dX}{dt} = v(x, t) = DX, \quad X|_{t=0} = \alpha, \quad \alpha \in \mathbb{R}^3.
\]

After integration, we get \( X(t) = \det(e^{-\gamma_1t}, e^{-\gamma_2t}, e^{(\gamma_1+\gamma_2)t})C_1, C_1 \in \mathbb{R}^3 \).

So, \( X(0) = IC_1 = \alpha \) and \( X(t) = \det(e^{-\gamma_1t}, e^{-\gamma_2t}, e^{(\gamma_1+\gamma_2)t})\alpha. \)

Note that \( X_1^2(t) + X_2^2(t) = e^{-2(\gamma_1+\gamma_2)t}(|\alpha_1|^2 + |\alpha_2|^2) \), i.e., the distance of a given fluid particle to the \( X_3 \) axis decreases exponentially in time. So, this axisymmetric flow without swirl forms two jets along positive and negative directions of the \( X_3 \) axis.

**Example 1.2 (Strain Flows).**

Take again \( \omega_0 = 0 \) and \( D = \det(-\gamma, \gamma, 0), \gamma > 0 \). Then again \( \omega = 0 \), i.e., flow is irrotational, and \( X(t) = \det(e^{-\gamma t}, e^{\gamma t}, 1)\alpha. \) Hence \( X_1(t)X_2(t) = \alpha_1\alpha_2 \) and \( X_3(t) = \alpha_3 \). So, this flow is independent on \( X_3 \), and the fluid particle trajectories are hyperbolas in the plane \( X_3 = \alpha_3 \). It forms a strain flow.
Example 1.3 (A Vortex).
Take now \( \omega_0 = (0, 0, k)^t \), \( k > 0 \) and \( D = 0 \) in the equations (1.23) and (1.24). Then \( \omega (t) = \omega_0 \) and \( v(X, t) = \frac{1}{2} \omega_0 \times X = \frac{1}{2} (-kX_2, kX_1, 0)^t \). We have already considered these flows in details in the corollary 1.1(2). We got

\[
\begin{bmatrix}
X_1(t) \\
X_2(t)
\end{bmatrix} = Q \left( \frac{1}{2} kt \right) \begin{bmatrix}
\alpha_1 \\
\alpha_2
\end{bmatrix}, \quad X_3(t) = \alpha_3,
\]

where \( Q \) is \( 2 \times 2 \) rotation matrix, i. e., \( Q \left( \frac{1}{2} kt \right) = \begin{bmatrix}
\cos \left( \frac{1}{2} kt \right) & \sin \left( \frac{1}{2} kt \right) \\
\sin \left( \frac{1}{2} kt \right) & \cos \left( \frac{1}{2} kt \right)
\end{bmatrix} \).

So, this flow forms a two-dimensional vortex in the plane \( X_3 = \alpha_3 \).

Example 1.4 (A Rotating Jet).
Here we will consider the superposition of a jet and a vortex taking \( \omega_0 = (0, 0, k)^t \), \( D = \text{diag}(-\gamma_1, -\gamma_2, \gamma_1 + \gamma_2) \), \( \gamma_1, \gamma_2, k > 0 \) in the equations (1.23) and (1.24).

Then \( \omega (t) = (0, 0, ke^{(\gamma_1+\gamma_2)t})^t \). So, the rotation is around \( X_3 \) axis, and the vorticity increases exponentially in time.

From the equation (1.24a), we get the velocity

\[
v(x, t) = (-\gamma_1 X_1 - \frac{k}{2} e^{(\gamma_1+\gamma_2)t} X_2, -\gamma_2 X_2 + \frac{k}{2} e^{(\gamma_1+\gamma_2)t} X_1, (\gamma_1 + \gamma_2)X_3)^t.
\]

Then coordinates of the particle-trajectories satisfy the following initial value problem:

\[
\begin{cases}
\frac{dX_1}{dt} = -\gamma_1 X_1 - \frac{k}{2} e^{(\gamma_1+\gamma_2)t} X_2, \\
\frac{dX_2}{dt} = -\gamma_2 X_2 + \frac{k}{2} e^{(\gamma_1+\gamma_2)t} X_1, \\
\frac{dX_3}{dt} = (\gamma_1 + \gamma_2)X_3, \\
X_i|_{t=0} = \alpha_i, \quad i = 1, 2, 3.
\end{cases}
\]

So, \( X_3(t) = \alpha_3 e^{(\gamma_1+\gamma_2)t} \) as for the jet flows in the example 1.1.

Now multiply the first from the equations above by \( X_1 \), the second one by \( X_2 \), and add the results. We have

\[
\frac{1}{2} \frac{d}{dt} \left( X_1^2 + X_2^2 \right) = - (\gamma_1 X_1^2 + \gamma_2 X_2^2).
\]

Hence

\[
-4 \max (\gamma_1, \gamma_2) dt \leq \frac{d(X_1^2 + X_2^2)}{X_1^2 + X_2^2} \leq -4 \min (\gamma_1, \gamma_2) dt.
\]

After integration we get

\[
C_1 e^{-4 \max (\gamma_1, \gamma_2) t} \leq X_1^2 + X_2^2 \leq C_2 e^{-4 \min (\gamma_1, \gamma_2) t} \quad \text{with} \quad C_1, C_2 > 0.
\]

Hence for \( t = 0 \) we have \( C_1 \leq \alpha_1^2 + \alpha_2^2 \leq C_2 \).

To sharpen our estimate, choose \( C_1 = \alpha_1^2 + \alpha_2^2 = C_2 \). So we obtain

\[
(\alpha_1^2 + \alpha_2^2) e^{-4 \max (\gamma_1, \gamma_2) t} \leq X_1^2 + X_2^2 \leq (\alpha_1^2 + \alpha_2^2) e^{-4 \min (\gamma_1, \gamma_2) t}
\]
that shows for a given fluid particle the same exponential decreasing in time of minimal and maximal distances to the $X_3$ axis as those for the jet without rotation in the example 1.1.

Thus, the flow forms a jet in the negative and positive directions of the $X_3$ axis, and the particle trajectories spiral around the $X_3$ axis with increasing angular velocity. This flow is a rotating jet, an axisymmetric flow with swirl.

1.5 Some Remarkable Properties of the Vorticity in Ideal Fluid Flows.

In this section we review some important properties of vorticity in inviscid flows ($\nu = 0$).

In the case $\nu = 0$, the vorticity equation (1.28) reduces to

$$\frac{D\omega}{Dt} = \mathcal{D}\omega,$$

(1.29)

where $\frac{D}{Dt}$ is a convective derivative, and $\mathcal{D} = \nabla v - \Omega$.

Then

$$\mathcal{D}\omega = \nabla v \omega,$$

since $\Omega \omega = \frac{1}{2} \omega \times \omega = 0$.

One of the properties we will consider, the Vorticity-Transport Formula, is based on the following lemma.

**Lemma 1.4.** Let $v(x, t)$ be any smooth velocity field (not necessary divergence free) with associated particle-trajectory mapping $X(\alpha, t)$ satifying

$$\frac{dX}{dt} = v(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha.$$

(1.30)

Let $h(x, t)$ be a smooth vector field. Then $h$ satisfies the initial value problem

$$\frac{Dh}{Dt} = (\nabla v)h, \quad h(\alpha, 0) = h_0(\alpha)$$

(1.31)

if and only if

$$h(X(\alpha, t), t) = \nabla_{\alpha}X(\alpha, t)h_0(\alpha).$$

(1.32)

**Proof.** Take $\nabla_\alpha$ of both sides of the equation (1.30). We have

$$\frac{d}{dt} \nabla_{\alpha}X(\alpha, t) = \nabla_x v(X(\alpha, t), t) \nabla_{\alpha}X(\alpha, t), \quad \nabla_{\alpha}X(\alpha, 0) = \nabla_{\alpha}\alpha = I.$$

Recall that both $\nabla_x v$ and $\nabla_{\alpha}X$ are $3 \times 3$ matrices. Now on this equation and the initial
value multiply from the right by \( h_0(\alpha) \in \mathbb{R}^3 \).

\[
\frac{d}{dt} \left[ \nabla_\alpha X(\alpha, t) h_0(\alpha) \right] = \nabla_x v(X(\alpha, t), t) \left[ \nabla_\alpha X(\alpha, t) h_0(\alpha) \right], \quad \nabla_\alpha X(\alpha, 0) h_0(\alpha) = h_0(\alpha).
\]

This ODE and initial condition are the same as in the initial value problem (1.31). So, by the uniqueness of solutions to initial value problems, we conclude that

\[
h(X(\alpha, t), t) = \nabla_\alpha X(\alpha, t) h_0(\alpha).
\]

The following proposition is an immediate application of the last lemma with \( h = \omega \).

**Proposition 1.6 (Vorticity-Transport Formula).** Let \( X(\alpha, t) \) be the smooth particle trajectories corresponding to a divergence-free velocity field \( v(x, t) \). Then the solution \( \omega \) to the inviscid vorticity equation (1.29), \( \frac{D\omega}{Dt} = \nabla v \omega \), is

\[
\omega(X(\alpha, t), t) = \nabla_\alpha X(\alpha, t) \omega_0(\alpha), \text{ where } \omega_0(\alpha) = \omega|_{t=0}.
\]

Consider some interpretation of the formula (1.33). \( \nabla_\alpha X(\alpha, t) \) is \( 3 \times 3 \) real matrix with \( \det(\nabla_\alpha X(\alpha, t)) = 1 \) since the fluid is incompressible. Then the product of three complex eigenvalues is \(-1\). If one of the eigenvalues is \(-1\), then two others are \( \lambda \) and \( \lambda^{-1} \) with \( |\lambda| \geq 1 \). In the case \( |\lambda| > 1 \), the formula (1.33) shows that the vorticity increases when \( \omega_0 \) aligns roughly with the complex eigenvector associated with \( \lambda \) and decreases when directions of \( \omega_0 \) and the eigenvector associated with \( \lambda^{-1} \) are close.

In particular, the formula (1.33) leads to the following significant result for ideal fluid flows in two dimensions.

**Corollary 1.2.** Let \( X(\alpha, t) \) be the smooth particle trajectories corresponding to a divergence-free velocity field. Then the vorticity \( \omega(x, t) \) satisfies

\[
\omega(X(\alpha, t), t) = \omega_0(\alpha), \quad \alpha \in \mathbb{R}^2,
\]

and the vorticity \( \omega_0(\alpha) \) is conserved along particle trajectories for two-dimensional inviscid fluid flows.

**Proof.** Let \( \forall t > 0 \), \( X^3(\alpha, t) = \alpha_3 = \text{const} \), and \( X^1_{\alpha_1} = X^2_{\alpha_2} = 0 \). Also, for rotation in the plane parallel to \( X^1 \alpha X^2 \) the vorticity \( \omega_0(\alpha) = [0, 0, k(\alpha)]^t \), where \( k(\alpha) \) is a real-valued function.

Then by the formula (1.33), we have

\[
\omega(X(\alpha, t), t) = \begin{bmatrix} X^1_{\alpha_1} & X^1_{\alpha_2} & 0 \\ X^2_{\alpha_1} & X^2_{\alpha_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ k(\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k(\alpha) \end{bmatrix} = \omega_0(\alpha), \quad \forall t \geq 0.
\]
Another corollary of the Vorticity-Transport Formula shows one remarkable property of vortex lines in inviscid flows.

At first, give a definition of vortex lines in fluid flows.

**Definition 1.6.** We say that a smooth curve $C = \{y(s) \in \mathbb{R}^N : 0 < s < 1 \}$ is a vortex line at fixed time $t$ if it is tangent to the vorticity $\omega$ at each of its points, i.e., provided that

$$\frac{dy(s)}{ds} = \lambda(s) \omega(y(s), t) \quad \text{for some } \lambda(s) \neq 0. \tag{1.35}$$

**Proposition 1.7.** In inviscid, smooth fluid flows, vortex lines move with the fluid.

**Proof.** Let a smooth curve $C = \{y(s) \in \mathbb{R}^N : 0 < s < 1 \}$ be an initial vortex line. Then $\frac{dy(s)}{ds} = \lambda(s) \omega_0(y(s))$, where $\omega_0(\alpha) = \omega|_{t=0}, \alpha = y(s)$.

$C$ evolves with fluid to the curve $C(t) = \{X(y(s), t) \in \mathbb{R}^N : 0 < s < 1 \}$.

We show that $C(t)$ is also a vortex line. By the definition 1.6, we have

$$\frac{dX(y(s), t)}{ds} = \nabla_\alpha X(y(s), t) \frac{dy(s)}{ds} = \nabla_\alpha X(y(s), t) \lambda(s) \omega_0(y(s))$$

The proposition 1.6 gives that $\nabla_\alpha X(y(s), t) \omega_0(y(s)) = \omega(X(y(s), t), t)$.

So $\frac{dX(y(s), t)}{ds} = \lambda(s) \omega(X(y(s), t), t)$, i.e., $C(t)$ is tangent to vortex $\omega$ at each of its points.

We conclude our review of some important properties of the vorticity in ideal fluid flows discussing changing in time of their flux and velocity circulation.

Let $S$ be a bounded, open, smooth surface with smooth, oriented boundary $C$ on it, and $S$ and $C$ evolve with the fluid to $S(t) = \{X(\alpha, t) : \alpha \in S\}$ and $C(t) = \{X(\alpha, t) : \alpha \in C\}$ respectively, where $X(\alpha, t)$ are the smooth particle trajectories corresponding to a divergence-free velocity field $v(x, t)$.

At first, we will get an analog of the transport formula (1.15) for circulation of velocity.

**Proposition 1.8.** Let $C$ be a smooth, oriented, closed curve and let $X(\alpha, t)$ be the smooth particle trajectories corresponding to a divergence-free velocity field $v(x, t)$. Then

$$\frac{d}{dt} \oint_{C(t)} v \cdot dl = \oint_{C(t)} \frac{Dv}{Dt} \cdot dl \tag{1.36}$$
Proof. Change variables \( \alpha \to X(\alpha, t) \). Then \( C \to C(t) \) and an infinitesimal element of \( C, \ d\alpha = (\alpha + d\alpha) - \alpha \to dl = X(\alpha + d\alpha, t) - X(\alpha, t) \). Using a Taylor series expansion to linear order of \( d\alpha \) for smooth particle trajectories \( X(\alpha + d\alpha, t) \) at the point \((\alpha, t)\), we get that the infinitesimal element of the curve \( C(t), \ dl = \nabla_\alpha X(\alpha, t) d\alpha \).

So,

\[
\frac{d}{dt} \oint_{C(t)} v(X(\alpha, t) t) \cdot dl = \frac{d}{dt} \oint_C v(\alpha, t) \cdot \nabla_\alpha X(\alpha, t) d\alpha = \\
\oint_C \left[ \frac{d}{dt} v(\alpha, t) \cdot \nabla_\alpha X(\alpha, t) + v(\alpha, t) \cdot \nabla_\alpha \frac{dX(\alpha, t)}{dt} \right] d\alpha = \\
\oint_C \frac{d}{dt} v(\alpha, t) \cdot \nabla_\alpha X(\alpha, t) d\alpha + \oint_C v(\alpha, t) \cdot \nabla_\alpha v(\alpha, t) d\alpha.
\]

The second integral \( \oint_C v(\alpha, t) \cdot \nabla_\alpha v(\alpha, t) d\alpha = \oint_C v(\alpha, t) dv(\alpha, t) = \oint_C \frac{1}{2} dv(\alpha, t) = \frac{1}{2} \oint_C d(|v(\alpha, t)|^2) = 0 \) as an integral of a perfect differential along a closed contour.

Hence reversing the change of variables in the first integral, we obtain

\[
\frac{d}{dt} \oint_{C(t)} v \cdot dl = \oint_{C(t)} \frac{Dv}{Dt} \cdot dl.
\]

As an immediate application of the formula (1.36), we get the Kelvin’s Conservation Law of Circulation.

**Proposition 1.9** (Kelvin’s Conservation of Circulation). *For a smooth solution \( v \) to the Euler equation, the circulation \( \Gamma_{C(t)} \) around a curve \( C(t) \) moving with the fluid,

\[
\Gamma_{C(t)} = \oint_{C(t)} v \cdot dl,
\]

is constant in time.

Proof. Differentiate \( \Gamma_{C(t)} \) by \( t \). The formula (1.36) and the Euler equation \( \frac{Dv}{Dt} = -\nabla p \) give

\[
\frac{d}{dt} \Gamma_{C(t)} = \oint_{C(t)} \frac{Dv}{Dt} \cdot dl = -\oint_{C(t)} \nabla p \cdot dl.
\]

The contour \( C(t) \) moves with the fluid along particle trajectories \( X(\alpha, t) \). Because flow is inviscid, there are no tangent forces acting on the fluid. Then \( \nabla p \cdot dl = 0 \), and so \( \frac{d}{dt} \Gamma_{C(t)} = 0 \). Thus, \( \Gamma_{C(t)} \) =const. in time.
The conservation Law of Vorticity Flux is a consequence of this result.

**Corollary 1.3** (Helmholtz’s Conservation of Vorticity Flux). *For a smooth solution* \( v \) *to the Euler equation, the vorticity flux* \( F_{S(t)} \) *through a surface* \( S(t) \) *moving with the fluid,

\[
F_{S(t)} = \int_{S(t)} (\omega \cdot n) \, dS,
\]

where \( n \) is unit vector normal to \( S(t) \), is constant in time.

**Proof.** Let \( C(t) \) be a smooth boundary of the surface \( S(t) \). Apply Stoke’s formula to the previous result

\[
0 = \frac{d}{dt} \Gamma_{C(t)} = \frac{d}{dt} \oint_{C(t)} v \cdot dl = \frac{d}{dt} \int_{C(t)} (\text{curl} v) \cdot dl = \frac{d}{dt} \int_{S(t)} (\omega \cdot n) \, dS = \frac{d}{dt} F_{S(t)}.
\]

So, \( F_{S(t)} \) = const. in time. \( \square \)

Notice that if area of \( S(t) \) decreases, then by the Conservation Law of Vorticity Flux, \( \omega|_{S(t)} \) must roughly increase as we have seen in the Example 1.4 for rotating jets.

**Remark.** *In general,* \( \Gamma_{C(t)} \) *is not conserved in viscous flows with* \( \nu > 0 \).

**Proof.** Applying the Navier-Stokes equation instead of the Euler equation, we have

\[
\frac{d}{dt} \Gamma_{C(t)} = \oint_{C(t)} \frac{Dv}{Dt} \cdot dl = -\oint_{C(t)} \nabla p \cdot dl + \nu \oint_{C(t)} \Delta v \cdot dl.
\]

The first integral is 0 because it is the integral of a conservative vector field around a closed contour.

Also note that \( \text{curl} \omega = \nabla \times \omega = \nabla \times (\nabla \times v) = (\nabla \cdot v)\nabla - (\nabla \cdot \nabla)v = (\text{div} v)\nabla - \Delta v = -\Delta v \) since \( \text{div} v = 0 \).

So, \( \frac{d}{dt} \Gamma_{C(t)} = -\nu \oint_{C(t)} \text{curl} \omega \cdot dl \neq 0 \) in general. \( \square \)

### 1.6 Some Conserved Quantities in Ideal and Viscous Fluid Flows.

We start this section with an important lemma.

**Lemma 1.5.** Let \( w \) be a smooth, divergence-free vector field in \( \mathbb{R}^N \) and let \( q \) be a smooth scalar such that

\[
|w(x)| \leq o \left[ (|x|^{1-N}) \right] \quad \text{as} \quad |x| \to \infty.
\]

\( \quad \) (1.39)
Then $w$ and $\nabla q$ are orthogonal:

$$\int_{\mathbb{R}^N} w \cdot \nabla q \, dx = 0. \quad (1.40)$$

**Proof.** The proof is based on the Green’s formula:

$$\int_U \nabla \phi \cdot \nabla q \, dx = - \int_U q \Delta \phi \, dx + \int_{\partial U} \frac{\partial \phi}{\partial n} q \, dS,$$

where $\phi, q \in C^1(\bar{U}), \ U \subset \mathbb{R}^N$, $n$ is an outward unit vector, normal to $\partial U$, the boundary of $U$.

Let $U = \{ x \in \mathbb{R}^N : |x| \leq R \}$ for some $R > 0$.

Let $\nabla \phi = w$. Then $\Delta \phi = \nabla^2 \phi = \nabla \cdot (\nabla \phi) = \nabla \cdot w = \text{div} w = 0$ and $\frac{\partial \phi}{\partial n} = \nabla \phi \cdot n = w \cdot n$.

So, $\int_{|x|\leq R} w \cdot \nabla q \, dx = \int_{|x|=R} w \cdot q n \, dS$.

Estimate the flux on the right-hand side of the last equation. Let $S_R$ be the surface area of the sphere of radius $R$; $S_R = C_N R^{N-1}$ with $C_N$, a coefficient depending on dimension $N$.

$$\left| \int_{|x|=R} w \cdot q n \, dS \right| \leq \max_{|x|=R} |w \cdot q n| S_R \leq \max_{|x|=R} |w(x)| \cdot |q(x)| C_N R^{N-1} = o(R^{1-N})C_N R^{N-1} \rightarrow 0 \text{ if } R \rightarrow \infty.$$

Thus, $\int_{\mathbb{R}^N} w \cdot \nabla q \, dx = 0$, i.e., $w$ and $\nabla q$ are orthogonal. \[ \Box \]

Recall that the Navier-Stokes and the Euler equations are obtained from the conservation law of momentum, and that the incompressibility condition $\text{div} \, v = 0$ is gotten from the conservation law of mass.

The next proposition lists some other quantities for solution of the Euler equation that are conserved globally in time.

**Proposition 1.10.** Let $v$ (and $\omega = \text{curl} \, v$) and $p$ be a smooth solution to the Euler equation $\frac{Dv}{Dt} = -\Delta p$ and $\text{div} \, v = 0$ in $\mathbb{R}^3$, vanishing sufficiently rapidly as $|x| \rightarrow \infty$. Then the following quantities are conserved for all time:

(i) the total flux $V_3$ of velocity,

$$V_3 = \int_{\mathbb{R}^3} v \, dx \quad (1.41)$$

(ii) the total flux $\Omega_3$ of vorticity,

$$\Omega_3 = \int_{\mathbb{R}^3} \omega \, dx \quad (1.42)$$
(iii) the kinetic energy $E_3$, 
\[ E_3 = \frac{1}{2} \int_{\mathbb{R}^3} |v|^2 \, dx \]  
(1.43)

(iv) the helicity $H_3$, 
\[ H_3 = \int_{\mathbb{R}^3} v \cdot \omega \, dx \]  
(1.44)

(v) the fluid impulse $I_3$, 
\[ I_3 = \frac{1}{2} \int_{\mathbb{R}^3} x \times \omega \, dx \]  
(1.45)

Proof.

(i) Integrate over $\mathbb{R}^3$ the Euler equation \( \frac{\partial v}{\partial t} + (\nabla v)v = -\nabla p \). We have componentwise:

\[ \frac{d}{dt} \int_{\mathbb{R}^3} v^i(x, t) \, dx = -\int_{\mathbb{R}^3} \nabla v^i(x, t) \cdot v(x, t) \, dx - \int_{\mathbb{R}^3} p x^i \, dx, \quad i = 1, 2, 3. \]

Since $v$ vanishes at infinity sufficiently rapidly such that \( |\nabla v^i(x, t)||v(x, t)| = o(|x|^{1-N}) \) as \( |x| \to \infty \), then by the Lemma 1.5, the first integral at the right-hand side of the equation above is zero. Applying to the next integral in this equation the Gauss-Green theorem, we get:

\[ \int_{\mathbb{R}^3} p x^i \, dx = \lim_{R \to \infty} \int_{|x|=R} pn^i \, dS, \]

where \( n^i = \frac{x^i}{R} \) is the \( i^{th} \) component of the outward normal unit vector of the sphere \( |x| = R \).

Since \( p \) vanishes sufficiently rapidly as \( |x| \to \infty \), then for large \( R \), \( p \) must be a constant

So, \( \frac{d}{dt} V_3 = \lim_{R \to \infty} \frac{p}{R} \int_{|x|=R} x_i \, dS = 0. \)

Hence $V_3$ is constant in time.

(ii) Now integrate over $\mathbb{R}^3$ the vorticity equation.

\[ \frac{d\omega}{dt} + \nabla \omega \cdot v = \nabla v \cdot \omega. \]

We have componentwise:

\[ \frac{d}{dt} \int_{\mathbb{R}^3} \omega^i \, dx = \int_{\mathbb{R}^3} \omega \cdot \nabla v^i \, dx - \int_{\mathbb{R}^3} v \cdot \nabla \omega^i \, dx, \quad i = 1, 2, 3. \]

Apply the Green’s formula for both integrals at the right-hand side of this equation.
We get
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \omega^i \, dx = - \int_{\mathbb{R}^3} v^i (\nabla \cdot \omega) \, dx + \lim_{R \to \infty} \int_{|x|=R} (\omega \cdot n) v^i \, dS + \int_{\mathbb{R}^3} \omega^i (\nabla \cdot v) \, dx - \lim_{R \to \infty} \int_{|x|=R} (v \cdot n) \omega^i \, dS.
\]

Because \( \omega = \nabla \times v \), \( \nabla \cdot \omega = \text{div} \omega = \nabla \cdot (\nabla \times v) = 0 \). \( \nabla \cdot v = \text{div} v = 0 \) by the incompressibility. Finally, take into account that \( v \) vanishes sufficiently rapidly as \( |x| \to \infty \).

So, \( \frac{d}{dt} \int_{\mathbb{R}^3} \omega^i \, dx = 0 \), \( i = 1, 2, 3 \), i.e., \( \Omega_3 \) is conserved for all time.

(iii) Apply the transport formula (1.15),
\[
\frac{d}{dt} \int_{X(\Omega, t)} f \, dx = \int_{X(\Omega, t)} [f_t + \text{div}(fv)] \, dx,
\]
for \( \Omega = \mathbb{R}^3 \) and \( f = \frac{1}{2} v \cdot v = \frac{1}{2} |v|^2 \). Then we have
\[
\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^3} |v|^2 \, dx = \int_{\mathbb{R}^3} \left[ v_t \cdot v + \nabla \left( \frac{1}{2} v \cdot v \right) \cdot v + \left( \frac{1}{2} v \cdot v \right) \text{div} v \right] \, dx.
\]

Now \( \text{div} v = 0 \) by the incompressibility. \( \nabla (\frac{1}{2} v \cdot v) = (v \cdot \nabla)v \). Hence
\[
\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^3} |v|^2 \, dx = \int_{\mathbb{R}^3} \left[ v_t + (v \cdot \nabla)v \right] \cdot v \, dx.
\]

Using the Euler equation \( v_t + (v \cdot \nabla)v = -\nabla p \), we get
\[
\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^3} |v|^2 \, dx = -\int_{\mathbb{R}^3} \nabla p \cdot v \, dx.
\]

Finally, since \( v \) and \( p \) vanish sufficiently rapidly as \( |x| \to \infty \), then by the Lemma 1.5, the integral at the right-hand side of the last equation is zero.

So, \( \frac{d}{dt} E_3 = 0 \), and the kinetic energy \( E_3 \) is constant in time.

(iv) Multiply the Euler equation by \( \omega \).
\[
v_t \cdot \omega + (v \cdot \nabla)v \cdot \omega = -\nabla p \cdot \omega.
\]

Differentiating the products \((v \cdot \omega)v\) and \( p \omega \), we have
\[
\text{div} \left( (v \cdot \omega)v \right) = \nabla (v \cdot \omega) \cdot v + (v \cdot \omega) \text{div} v = (v \cdot \nabla)v \cdot \omega + (v \cdot \nabla)\omega \cdot v + (v \cdot \omega) \text{div} v
\]
and \( \text{div}(p \omega) = \nabla p \cdot \omega + p \text{div} \omega \). Recall that \( \text{div} v = 0 \) by the incompressibility.
Hence we get

\[ v_t \cdot \omega + \text{div} ((v \cdot \omega)v) - (v \cdot \nabla)\omega \cdot v = -\text{div}(p\omega) + p \text{ div} \omega. \tag{*} \]

Now multiply the vorticity equation \( \frac{D\omega}{Dt} = (\omega \cdot \nabla)v \) by \( v \) to find

\[ \omega_t \cdot v + (v \cdot \nabla)\omega \cdot v = (\omega \cdot \nabla)v \cdot v = \frac{1}{2} \omega \cdot \nabla \left( v^2 \right). \]

Using that \( \text{div} (v^2\omega) = \nabla \left( v^2 \right) \cdot \omega + v^2 \text{div} \omega \), we obtain

\[ \omega_t \cdot v + (v \cdot \nabla)\omega \cdot v = \frac{1}{2} \text{div} \left( v^2 \omega \right) - \frac{1}{2} v^2 \text{div} \omega. \tag{**} \]

However, \( \text{div} \omega = 0 \) since \( \text{div} \omega = \nabla \cdot \omega = \nabla \cdot (\nabla \times v) = 0 \).

Adding the equations (*) and (**) together, we have

\[ (v \cdot \omega)_t + \text{div}((v \cdot \omega)v) = \frac{1}{2} \text{div} \left( v^2 \omega \right) - \text{div}(p\omega) \]

or

\[ (v \cdot \omega)_t + \text{div} \left[ (v \cdot \omega)v + \left( p - \frac{v^2}{2} \right) \omega \right] = 0. \]

Integrate this equation over \( \mathbb{R}^3 \).

\[ \frac{d}{dt} \int_{\mathbb{R}^3} v \cdot \omega \, dx + \int_{\mathbb{R}^3} \text{div} \left( (v \cdot \omega)v + \left( p - \frac{v^2}{2} \right) \omega \right) \, dx = 0. \]

Apply the Gauss-Green’s formula to the second integral of this equation,

\[ \frac{d}{dt} H_3 + \lim_{R \to \infty} \int_{|x| = R} \left[ (v \cdot \omega)v + \left( p - \frac{v^2}{2} \right) \omega \right] \cdot n \, dS = 0. \]

By the assumption, \( v \) and \( p \) vanish sufficiently rapidly when \( |x| \to \infty \), then \( \omega = \text{curl} \ v \) must vanish sufficiently rapidly at infinity as well. Then the last limit is zero. So, \( \frac{d}{dt} H_3 = 0 \), i.e., the helicity \( H_3 \) is conserved for all time.

(v) To prove the conservation of the fluid impulse \( I_3 \), apply the transport formula (1.15) to \( \Omega = \mathbb{R}^3 \) and \( f = x \times \omega \).

\[ \frac{d}{dt} \int_{\mathbb{R}^3} x \times \omega \, dx = \int_{\mathbb{R}^3} \left[ \frac{\partial}{\partial t} (x \times \omega) + v \cdot \nabla (x \times \omega) + (x \times \omega) \text{div} v \right] \, dx, \]

and apply \( \text{div} v = 0 \). Then using a convective derivative, we have

\[ \frac{d}{dt} I_3 = \int_{\mathbb{R}^3} \left( \frac{D}{Dt} x \times \omega \right) \, dx = \int_{\mathbb{R}^3} \left( \frac{D}{Dt} x \times \omega + x \times \frac{D}{Dt} \omega \right) \, dx. \]
and

\[ \frac{Dx}{Dt} = \frac{\partial x}{\partial t} + v \cdot \nabla x = 0 + (\nabla x)v = Iv = v. \]

By the vorticity equation, \( \frac{D\omega}{Dt} = (\omega \cdot \nabla)v \), it follows that

\[ \frac{d}{dt}I_3 = \int_{\mathbb{R}^3} [v \times \omega + x \times (\omega \cdot \nabla)v] dx, \]

and

\[ v \times \omega + x \times (\omega \cdot \nabla)v = \sum_{j=1}^{3} v \times (\omega^j e_j) + \sum_{j=1}^{3} x \times (\omega^j v_x^j) = \]

\[ \sum_{j=1}^{3} \omega^j (v \times e_j + x \times v_x^j) = \omega \cdot (v \times \nabla x + x \times \nabla v) = \omega \cdot \nabla(x \times v). \]

So,

\[ \frac{dI_3}{dt} = \int_{\mathbb{R}^3} \omega \cdot \nabla(x \times v) dx = \int_{\mathbb{R}^3} \sum_{j=1}^{3} \omega^j (x \times v)_x^j dx. \]

Adding \( \int_{\mathbb{R}^3} \sum_{j=1}^{3} \omega^j (x \times v) dx = \int_{\mathbb{R}^3} \text{div} \omega(x \times v) dx = 0 \) (since \( \text{div} \omega = 0 \)) to the right-hand side of the last equation, we get

\[ \frac{dI_3}{dt} = \int_{\mathbb{R}^3} \sum_{j=1}^{3} [\omega^j (x \times v)_x^j + \omega^j (x \times v)] dx = \]

\[ \int_{\mathbb{R}^3} \sum_{j=1}^{3} [\omega^j (x \times v)]_{x^j} dx = \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{\mathbb{R}^3} \omega^j (x \times v)^i_{x^j} dx \ e_i. \]

Now apply the Gauss-Green’s formula. We obtain

\[ \frac{dI_3}{dt} = \sum_{i,j=1}^{3} \lim_{R \to \infty} \left( \int_{|x|=R} \omega^j (x \times v)^i n^j dS \right) e_i. \]

Exploiting again the fact that both \( v \) and \( \omega \) vanish sufficiently rapidly as \( |x| \to \infty \), we finally have \( \frac{dI_3}{dt} = 0 \), i.e., the fluid impulse \( I_3 \) is conserved for all time.

Notice that for 2D flows, the quantities \( V_2, \Omega_2, E_2, \) and \( I_2 \) are also conserved for all time, while the helicity \( H_2 \equiv 0 \) because \( v = (v^1, v^2, 0)^t, \omega = (0, 0, v^2_{x1} - v^1_{x2})^t \), and so \( v \cdot \omega \equiv 0 \).
At the end of this section, we will consider the effect that viscosity has on the conservation of energy. It is important for the study of the properties of general solution to the Navier-Stokes equation in chapter 3.

**Proposition 1.11.** Let \( v \) be a smooth solution to the Navier-Stokes equation, vanishing sufficiently rapidly as \(|x| \to \infty|\). Then the kinetic energy \( E(t) \) satisfies the ODE

\[
\frac{d}{dt} E(t) = -\nu \int_{\mathbb{R}^N} |\nabla v|^2 \, dx. \tag{1.46}
\]

**Proof.** This proof repeats the steps of the conservation energy proof for inviscid fluid (Prop. 1.10(iii)) with using the Navier-Stokes equation instead of the Euler equation. So, applying the transport formula (1.15) to \( \Omega = \mathbb{R}^N \) and \( f = \frac{1}{2} v \cdot v \), we get

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \frac{1}{2} |v|^2 \, dx = \int_{\mathbb{R}^N} \frac{Dv}{Dt} \cdot v \, dx,
\]

or using the Navier-Stokes equation \( \frac{Dv}{Dt} = \nabla p + \nu \Delta v \), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \frac{1}{2} |v|^2 \, dx = -\int_{\mathbb{R}^N} v \cdot \nabla p \, dx + \nu \int_{\mathbb{R}^N} v \cdot \Delta v \, dx.
\]

Again, the first integral at the right-hand side of this equation is 0 by the Lemma 1.5 since \( v \) and \( p \) vanish sufficiently rapidly as \(|x| \to \infty|\).

For the second integral at the right-hand side of this equation, we apply the Green’s formula

\[
\int_U \phi \cdot \Delta \psi \, dx = -\int_U \nabla \phi \cdot \nabla \psi \, dx + \int_{\partial U} \frac{\partial \psi}{\partial n} \cdot \phi \, dS,
\]

where \( n \) is outward normal unit vector of \( \partial U \), to \( \phi = \psi = v \).

Then we obtain

\[
\frac{d}{dt} E(t) = \nu \left( -\int_{\mathbb{R}^N} \nabla v \cdot \nabla v \, dx + \lim_{R \to \infty} \int_{|x|=R} v(n \cdot \nabla v) \, dS \right).
\]

This limit is zero since \( v \) vanishes at infinity sufficiently rapidly. Hence we get the following rate of change of kinetic energy:

\[
\frac{d}{dt} E(t) = -\nu \int_{\mathbb{R}^N} |\nabla v|^2 \, dx. \tag{1.46}
\]

Thus, for \( \nu > 0 \), the kinetic energy is dissipated in viscous flows.
1.7 Leray’s Formulation of Incompressible Flows.

The goal of this section is to exclude the pressure $p$ from the Navier-Stokes equation by use of the incompressibility condition $\operatorname{div}v = 0$. The fact that the Navier-Stokes equation does not contain the time derivative of $p$ makes this attempt possible.

The following proposition shows the resulting equation with $v$ as the only unknown.

**Proposition 1.12** (Leray’s Formulation of the Navier-Stokes Equation). *Solving the Navier-Stokes equations*

\[
\frac{Dv}{Dt} = -\nabla p + \nu \Delta v \quad (1.47)
\]

and

\[
\operatorname{div} v = 0 \quad (1.48)
\]

with smooth initial velocity $v_0$, $\operatorname{div} v_0 = 0$, is equivalent to solving the evolution equation:

\[
\frac{Dv}{Dt} = C_N \int_{\mathbb{R}^N} \frac{x-y}{|x-y|^N} \operatorname{tr} \left( \nabla v(y,t) \right)^2 dy + \nu \Delta v, \quad v|_{t=0} = v_0. \quad (1.49)
\]

The pressure $p(x, t)$ can be recovered from the velocity $v(x, t)$ by the solution of the Poisson equation

\[
-\Delta p = \operatorname{tr}(\nabla v)^2 = \sum_{i,j=1}^{N} v^i_{x_i} v^j_{x_j}. \quad (1.50)
\]

**Proof.**

1) Exclude the pressure $p$ from the equations (1.47), (1.48).

When we proved Proposition 1.5, we differentiated componentwise the Navier-Stokes equation by $x_j, \ j = 1, 2, \ldots, N$ and got the matrix equation (1.25)

\[
\frac{DV}{Dt} + V^2 = -P + \nu \Delta V,
\]

where $V \equiv \nabla v = [v^i_{x_j}]$ with $\operatorname{tr} V = \operatorname{div} v = 0$ and $P = [p_{x_i x_j}]$ is the Hessian matrix of pressure.

Now take trace of this matrix equation.

\[
\operatorname{tr} \left( \frac{DV}{Dt} \right) = \operatorname{tr} \left( \frac{\partial V}{\partial t} \right) + \operatorname{tr}(v \cdot \nabla V) = \frac{\partial}{\partial t}(\operatorname{tr} V) + v \cdot \nabla(\operatorname{tr} V) = \frac{D}{Dt}(\operatorname{tr} V) = \frac{D}{Dt}(\operatorname{div} v),
\]

\[
\operatorname{tr} (P) = \Delta p \text{ and } \operatorname{tr}(\Delta V) = \Delta (\operatorname{tr} V) = \Delta (\operatorname{div} v).
\]

So, we have

\[
\frac{D}{Dt}(\operatorname{div} v) + \operatorname{tr} V^2 = -\Delta p + \nu \Delta (\operatorname{div} v). \quad (1.51)
\]
Alternatively, we could just take a divergence of the Navier-Stokes equation. Then we get

\[
\nabla \cdot \left[ \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right] = -\nabla \cdot \nabla p + \nabla \cdot \nu \Delta v;
\]

\[
\frac{\partial}{\partial t} (\nabla \cdot v) + \nabla (v \cdot \nabla)v + (v \cdot \nabla)(\nabla \cdot v) = -\Delta p + \nu \Delta (\nabla \cdot v);
\]

\[
\frac{\partial}{\partial t} (\text{div } v) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{N} v_j \frac{\partial}{\partial x_j} \right) v^i + (v \cdot \nabla) \text{div } v = -\Delta p + \nu \Delta (\text{div } v);
\]

\[
\frac{D}{Dt} (\text{div } v) + \sum_{j=1}^{N} \sum_{i=1}^{N} v_j^i v^i_{x_j} = -\Delta p + \nu \Delta (\text{div } v).
\]

Since \( V^2 = V \cdot V = [v_{x_i}] \cdot [v_{x_k}] = \left[ \sum_{i=1}^{N} v_{x_i} v_{x_k} \right] \), then \( \text{tr } V^2 = \sum_{j=1}^{N} \sum_{i=1}^{N} v_{x_i} v_{x_j} \) \((j = k)\).

So, we get equation (1.51).

For incompressible flows \( \text{div } v = 0 \). Thus, the pressure \( p \) could be determined from velocity \( v(x, t) \) as a solution of the following Poisson equation:

\[
-\Delta p = \text{tr} (\nabla v)^2 = \sum_{i,j=1}^{N} v^i_{x_j} v^j_{x_i},
\]

(1.50)

The right-hand side of this equation is smooth in \( \mathbb{R}^N \) and vanishes sufficiently rapidly as \( |x| \to \infty \). Using elementary properties of the Poisson equation (see Theorem 2.3), we have

\[
p(x, t) = \Phi(x) \ast \text{tr} (\nabla v(x, t))^2,
\]

(1.52)

where the fundamental solution of the Laplace’s equation in \( \mathbb{R}^N \setminus \{0\} \) is

\[
\Phi(x) = \begin{cases} 
-\frac{1}{2\pi} \log |x|, & \text{if } N = 2, \\
\frac{\Gamma(N/2)}{2\pi^{N/2}(N-2)} \frac{1}{|x|^{N-2}}, & \text{if } N \geq 3.
\end{cases}
\]

(1.53)

Now to compute \( \nabla p \), first see that

\[
\nabla (|x|) = \nabla \left( \sum_{j=1}^{N} x_j^2 \right)^{1/2} = \sum_{i=1}^{N} \frac{1}{2|x|} \left( \sum_{j=1}^{N} 2x_j \delta_{ij} \right) e_i = \sum_{i=1}^{N} \frac{x_i}{|x|} e_i = \frac{x}{|x|},
\]

where Kronecker symbol \( \delta_{ij} = \begin{cases} 
0, & \text{if } i \neq j \\
1, & \text{if } i = j
\end{cases} \) and \( e_i \) is the vector whose i-th component is 1 and the other components are 0.

Then for \( N = 2 \), \( \nabla \Phi(x) = -\frac{1}{2\pi} \frac{1}{|x|} \nabla (|x|) = -\frac{x}{2\pi|x|^2} \).
and for $N > 2$, $\nabla \Phi(x) = -\frac{\Gamma(N/2)}{2\pi^{N/2}} |x|^{1-N} \nabla (|x|) = -\frac{\Gamma(N/2)}{2\pi^{N/2}} \frac{x}{|x|^N}$.

i. e., $\nabla \Phi(x) = -C_N \frac{x}{|x|^N}$ with $C_N = \begin{cases} \frac{1}{2\pi^{N/2}}, & \text{if } N = 2, \\ \frac{\Gamma(N/2)}{2\pi^{N/2}}, & \text{if } N > 2. \end{cases}$

Hence $\nabla p(x,t) = \nabla \int_{R^N} \Phi(x - y) \text{tr} (\nabla v(y,t))^2 \, dy = \int_{R^N} \nabla \Phi(x - y) \text{tr} (\nabla v(y,t))^2 \, dy$.

Finally,

$$-\nabla p(x,t) = C_N \int_{R^N} \frac{x - y}{|x - y|^N} \text{tr} (\nabla v(y,t))^2 \, dy.$$ (1.54)

Substitute $\nabla p$ from this equation into the Navier-Stokes equation.

$$\frac{Dv}{Dt} = C_N \int_{R^N} \frac{x - y}{|x - y|^N} \text{tr} (\nabla v(y,t))^2 \, dy + \nu \Delta v.$$ (1.55)

This closed evolution equation is of unknown $v$ only.

2) Prove that the solution $v$ to this equation is automatically divergence free, so incompressibility equation (1.48) can be dropped.

Actually, the evolution equation (1.55) is the Navier-Stokes equation when $p(x,t)$ satisfies the equation (1.50). Applying the divergence operator to this (or the Navier-Stokes) equation, we got the equation (1.51) as we saw above. Using the equation (1.50), we reduce the equation (1.51) to the following one.

$$\frac{D}{Dt}(\text{div} v) = \nu \Delta (\text{div} v).$$ (1.56)

From the initial condition (1.3), to supply the incompressibility stipulation (1.2), we have:

$$\text{div} v|_{t=0} = \text{div} v_0 = 0.$$ (1.57)

Then smooth solution $\text{div} v$ must be bounded on a short time interval.

Denote $\text{div} v = u$. Now we will apply the energy method to prove the uniqueness of solution of the initial value problem

$$\left\{ \begin{array}{l} \frac{Du}{Dt} = \nu \Delta u, \quad (x, t) \in R^N \times [0, \infty), \\ u| = 0, \quad x \in R^N \end{array} \right.$$ (1.58)

in the class of bounded functions.

Let $u_1$ and $u_2$ are two solutions of this IVP.

Then $\frac{\partial(u_1 - u_2)}{\partial t} = -(v \cdot \nabla)(u_1 - u_2) + \nu \Delta (u_1 - u_2)$.

Multiply this equation by $(u_1 - u_2)$.

$$(u_1 - u_2) \frac{\partial}{\partial t} (u_1 - u_2) = -(u_1 - u_2)(v \cdot \nabla)(u_1 - u_2) + (u_1 - u_2)\nu \Delta (u_1 - u_2)$$ or

$$\frac{1}{2} \frac{\partial}{\partial t} (u_1 - u_2)^2 = -\frac{1}{2}(v \cdot \nabla)(u_1 - u_2)^2 + \nu (u_1 - u_2) \Delta (u_1 - u_2).$$
Integrate the last equation over $\mathbb{R}^N$.

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} (u_1 - u_2)^2 \, dx = -\frac{1}{2} \int_{\mathbb{R}^N} (v \cdot \nabla)(u_1 - u_2)^2 \, dx + \nu \int_{\mathbb{R}^N} (u_1 - u_2) \Delta (u_1 - u_2) \, dx.\]$$

Apply an integration by parts to the first integral on the right-hand side of this equation,

$$-\frac{1}{2} \int_{\mathbb{R}^N} (v \cdot \nabla)(u_1 - u_2)^2 \, dx = -\frac{1}{2} \sum_{j=1}^{N} \int_{\mathbb{R}^N} v_j \frac{\partial}{\partial x_j} (u_1 - u_2)^2 \, dx =$$

$$\frac{1}{2} \sum_{j=1}^{N} \int_{\mathbb{R}^N} (u_1 - u_2)^2 v_j^2 \, dx - \frac{1}{2} \sum_{j=1}^{N} \lim_{R \to \infty} \int_{|x|=R} (u_1 - u_2)^2 v_j n_j^0 \, dS =$$

$$\frac{1}{2} \int_{\mathbb{R}^N} (u_1 - u_2)^2 \text{div} v \, dx.$$

For the second integral, apply the Green’s formula

$$\nu \int_{\mathbb{R}^N} (u_1 - u_2) \cdot \Delta (u_1 - u_2) \, dx = -\nu \int_{\mathbb{R}^N} \nabla (u_1 - u_2) \cdot \nabla (u_1 - u_2) \, dx$$

$$+ \lim_{R \to \infty} \int_{|x|=R} (u_1 - u_2) \cdot \nabla (u_1 - u_2) n_0 \, dS = -\nu \int_{\mathbb{R}^N} |\nabla (u_1 - u_2)|^2 \, dx.$$

Here $n_0^j$ is the outward normal unit vector, and both limits are 0 since $v$ (and so $u = \text{div} v$) vanishes sufficiently rapidly at infinity.

Then we have:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} (u_1 - u_2)^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^N} \text{div} v (u_1 - u_2)^2 \, dx - \nu \int_{\mathbb{R}^N} |\nabla (u_1 - u_2)|^2 \, dx$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^N} \text{div} v (u_1 - u_2)^2 \, dx.$$

Now we use the boundness of $\text{div} v$. Hence there exists a constant $C > 0$ such that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} (u_1 - u_2)^2 \, dx \leq C \int_{\mathbb{R}^N} (u_1 - u_2)^2 \, dx.$$

After that we will use Grönwall’s lemma (see section 2.1) for $\phi(t) = 2C$,

$$\eta(t) = \int_{\mathbb{R}^N} (u_1 - u_2)^2 \, dx, \text{ and } \psi(t) \equiv 0.$$

So, $\eta(t) \leq \phi(t) \eta(0) + \psi(t)$ on $[0, T]$ for some $T > 0$ implies that $\forall t \in [0, T]$,

$$\eta(t) \leq e^{\int_0^T \phi(s) \, ds} \left[ \eta(0) + \int_0^T \psi(s) \, ds \right].$$
Therefore we get:

\[
\int_{\mathbb{R}^N} (u_1 - u_2)^2 dx \leq e^{2CT} \left[ \int_{\mathbb{R}^N} (u_1 - u_2)^2 dx \right]_{t=0}.
\]

Since \( u_1|_{t=0} = u_2|_{t=0} \), then \( u_1 \equiv u_2, \forall t \in [0, T] \), i.e., the solution \( u \) of the IVP (1.58) is unique. So, the trivial solution \( u \equiv 0 \) is the only solution of this problem in class of bounded functions.

Thus, \( \text{div} v \equiv 0 \) for \( v \) satisfying the evolution equation (1.55) with the initial condition

\[
v|_{t=0} = v_0, \quad \text{div} v_0 = 0.
\]

We have proved that the IVP with the Navier-Stokes equation (1.2)-(1.3) is equivalent to the IVP (1.55), (1.59).

By the Lemma 1.5, a divergence-free vector field \( v \) and the gradient of scalar \( p \) are orthogonal in \( L^2(\mathbb{R}^N) \) if both \( v \) and \( p \) vanish sufficiently rapidly as \( |x| \to \infty \), i.e.,

\[
(v, \nabla p)_0 = \int_{\mathbb{R}^N} v \cdot \nabla p dx = 0.
\]

Denote the projection operator on the space of divergence-free vector fields by \( \mathcal{P} \).

So

\[
\mathcal{P} v = w, \quad \text{div} w = 0;
\]

\[
\mathcal{P} v = v \text{ if and only if } \text{div} v = 0,
\]

\[
\mathcal{P} (\nabla p) = 0.
\]

A general concept of decomposing a vector field into the divergence free part and the gradient part, orthogonal to it, is very convenient for studying incompressible flows and will be considered in detail in the next chapter in the Hodge’s Decomposition section.

From this point of view, the equation (1.55) is a projection of the Navier-Stokes equation on the space of divergence-free vector fields. Using the projection operator \( \mathcal{P} \), the equation (1.55) can be written as following:

\[
v_t = \mathcal{P} (-v \cdot \nabla v) + \nu \Delta v.
\]

This Leray’s formulation of the Navier-Stokes equation and the energy method will be used in Chapter 3.
In this chapter, we review some properties of the Fourier transform, the Poisson equation, Sobolev spaces, mollifiers, and Leray’s projection operator. Also, we will prove here Hodge’s decomposition of vector fields and Grönwall’s inequality we have used in the previous chapter.

2.1 The Grönwall’s Inequality.

Grönwall’s inequality is useful to derive various estimates in energy methods. This inequality often is used in differential or integral forms.

**Lemma 2.1** (Grönwall’s inequality in differential form.) If:

1) \( \eta(t) \geq 0 \) and is differentiable on \([0, T]\),
2) \( \varphi(t), \psi(t) \geq 0 \) and are integrable on \([0, T]\),
3) \( \eta'(t) \leq \varphi(t) \eta(t) + \psi(t) \),

then \( \forall t \in [0, T], \)

\[
\eta(t) \leq e^{\int_0^t \varphi(s) \, ds} \left[ \eta(0) + \int_0^t \psi(s) \, ds \right].
\] (2.1)

**Proof.** Perform a substitution \( \eta(t) = \mu(t) e^{\int_0^t \varphi(s) \, ds} \). Note that \( \eta(0) = \mu(0) \).

Then from the inequality 3), we have

\[
\mu'(t) e^{\int_0^t \varphi(s) \, ds} + \mu(t) e^{\int_0^t \varphi(s) \, ds} \varphi(t) \leq \varphi(t) \mu(t) e^{\int_0^t \varphi(s) \, ds} + \psi(t)
\]

or

\[
\mu'(t) \leq \psi(t) e^{-\int_0^t \varphi(s) \, ds} \leq \psi(t) \text{ since } e^{-\int_0^t \varphi(s) \, ds} \leq 1.
\]

Integrating the last inequality on \([0, t], 0 \leq t \leq T\), we get

\[
\int_0^t \mu'(s) \, ds \leq \int_0^t \psi(s) \, ds.
\]

So, \( \mu(t) \leq \int_0^t \psi(s) \, ds + \mu(0) \).

Multiplying this inequality by \( e^{\int_0^t \varphi(s) \, ds} \), we obtain (2.1).

\[
\eta(t) \leq e^{\int_0^t \varphi(s) \, ds} \left[ \eta(0) + \int_0^t \psi(s) \, ds \right].
\]
Lemma 2.2 (Grönwall’s inequality in integral form.). If:

1) $c(t) \geq 0$ and $c(t) \in C^1([0, T])$,
2) $u(t), q(t) \in C([0, T])$,
3) $q(t) \leq c(t) + \int_0^t u(s)q(s)\,ds$,

then

$$q(t) \leq c(t) + \int_0^t c(s)u(s)e^{\int_s^t u(\tau)\,d\tau}\,ds$$

(2.2)

or equivalently,

$$q(t) \leq c(0)e^{\int_0^t u(s)\,ds} + \int_0^t c'(s)e^{\int_s^t u(\tau)\,d\tau}\,ds.$$  (2.2a)

Proof.

1) Define $v(s) := e^{-\int_0^s u(\tau)\,d\tau}\int_0^s u(\tau)q(\tau)\,d\tau$. Note that $v \in C^1([0, T])$ and $v(0) = 0$.

Then $v'(s) = u(s)e^{-\int_0^s u(\tau)\,d\tau}[q(s) - \int_0^s u(\tau)q(\tau)\,d\tau]$. Since by the hypothesis 3)

$q(t) - \int_0^t u(s)q(s)\,ds \leq c(t)$, then $v'(s) \leq u(s)c(s)e^{-\int_0^s u(\tau)\,d\tau}$.

Integrating this inequality on $[0, t], \; t \in [0, T]$ and taking in count that $v(0) = 0$, we have $v(t) \leq \int_0^t u(s)c(s)e^{-\int_0^s u(\tau)\,d\tau}\,ds$.

2) By the definition of $v$, $\int_0^t u(s)q(s)\,ds = v(t)e^{\int_0^t u(s)\,ds}$. Applying the estimate for $v$ obtained above, we get

$$\int_0^t u(s)q(s)\,ds \leq e^{\int_0^t u(s)\,ds} \int_0^t u(s)c(s)e^{-\int_0^s u(\tau)\,d\tau}\,ds = \int_0^t u(s)c(s)e^{\int_s^t u(\tau)\,d\tau}\,ds.$$  

Use this bound along with the hypothesis 3)

$$q(t) \leq c(t) + \int_0^t u(s)c(s)e^{\int_s^t u(\tau)\,d\tau}\,ds.$$  (2.2)

Integrating by parts at the right-hand side of this inequality gives us the following result,

$$q(t) \leq c(t) + \int_0^t c(s)\frac{d}{ds}\left(-e^{\int_s^t u(\tau)\,d\tau}\right)\,ds$$

$$= c(t) - c(s)e^{\int_s^t u(\tau)\,d\tau}\bigg|_{s=0}^{s=t} + \int_0^t c'(s)e^{\int_s^t u(\tau)\,d\tau}\,ds$$

$$= c(t) - c(t) + c(0)e^{\int_0^t u(\tau)\,d\tau} + \int_0^t c'(s)e^{\int_s^t u(\tau)\,d\tau}\,ds.$$  (2.2a)
2.2 Elementary Properties of the Fourier Transform.

In this section, we review some basic properties of the Fourier transform.

**Definition 2.1.** The Fourier transform \( \hat{f} \) for a function \( f \in L^1(\mathbb{R}^N) \) is

\[
\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x \cdot \xi} f(x) \, dx, \quad \xi \in \mathbb{R}^N.
\]  

(2.3)

The inverse transform for \( f \) is

\[
\check{f}(\xi) = \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} f(x) \, dx, \quad \xi \in \mathbb{R}^N.
\]  

(2.4)

Note that \( \check{f}(x) = \hat{f}(-x) \). Also, \( |e^{\pm 2\pi i x \cdot \xi}| = 1 \), and \( |\int_{\mathbb{R}^N} e^{\pm 2\pi i x \cdot \xi} f(x) \, dx| \leq \int_{\mathbb{R}^N} |e^{\pm 2\pi i x \cdot \xi} f(x)| \, dx = \|f\|_{L^1} < \infty \) since \( f \in L^1(\mathbb{R}^N) \).

So, \( \hat{f}, \check{f} \in L^\infty(\mathbb{R}^N) \), and \( \|\hat{f}\|_{L^\infty}, \|\check{f}\|_{L^\infty} \leq \|f\|_{L^1} \).

The Fourier transform is very convenient to apply to derivatives and convolutions. The last one is described by the following lemma.

**Lemma 2.3.** If \( f, g \in L^1(\mathbb{R}^N) \), then the Fourier transform of their convolution is the product of their Fourier transforms:

\[
\widehat{f \ast g} = \hat{f} \hat{g}.
\]  

(2.5)

**Proof.** We will use Fubini’s theorem to prove (2.5).

\[
\widehat{f \ast g}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x \cdot \xi} \left[ \int_{\mathbb{R}^N} f(x-y) g(y) \, dy \right] \, dx
\]

\[
= \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} e^{-2\pi i (x-y) \cdot \xi} f(x-y) g(y) \, dy \right) \, dx.
\]

Make substitution \( z(x) = x - y \), so \( dz = dx \). Then

\[
\widehat{f \ast g}(\xi) = \int_{\mathbb{R}^N \times \mathbb{R}^N} e^{-2\pi i z \cdot \xi} e^{-2\pi i y \cdot \xi} f(z) g(y) \, dy \, dz
\]

\[
= \int_{\mathbb{R}^N} e^{-2\pi i z \cdot \xi} f(z) \, dz \int_{\mathbb{R}^N} e^{-2\pi i y \cdot \xi} g(y) \, dy = \hat{f}(\xi) \hat{g}(\xi).
\]

Next three lemmas will help us to prove some properties of the Fourier transform.
Lemma 2.4. Let \( f, g \in L^1(\mathbb{R}^N) \). Then

\[
\int_{\mathbb{R}^N} f(x)\hat{g}(x) \, dx = \int_{\mathbb{R}^N} \hat{f}(y)g(y) \, dy. \tag{2.6}
\]

Proof. First, show that both integrals in the equation (2.6) are finite.

\[
\left| \int_{\mathbb{R}^N} f(x)\hat{g}(x) \, dx \right| \leq \sup_{x \in \mathbb{R}^N} |\hat{g}(x)| \int_{\mathbb{R}^N} |f(x)| \, dx = \|\hat{g}\|_{L^\infty} \|f\|_{L^1} \leq \|g\|_{L^1} \|f\|_{L^1} < \infty.
\]

Similarly

\[
\left| \int_{\mathbb{R}^N} \hat{f}(y)g(y) \, dy \right| \leq \|f\|_{L^1} \|g\|_{L^1} < \infty.
\]

Now use Fubini’s theorem.

\[
\int_{\mathbb{R}^N} f(x)\hat{g}(x) \, dx = \int_{\mathbb{R}^N} f(x) \left[ \int_{\mathbb{R}^N} e^{-2\pi i y \cdot x} g(y) \, dy \right] \, dx = \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} e^{-2\pi i y \cdot x} f(x) g(y) \, dy \right] \, dx
\]

\[
\int_{\mathbb{R}^N} g(y) \left[ \int_{\mathbb{R}^N} e^{-2\pi i x \cdot y} f(x) \, dx \right] \, dy = \int_{\mathbb{R}^N} g(y) \hat{f}(y) \, dy.
\]

Similarly, it could be shown that

\[
\int_{\mathbb{R}^N} f(x)\hat{g}(x) \, dx = \int_{\mathbb{R}^N} \hat{f}(y)g(y) \, dy. \tag{2.7}
\]

Here we consider one example of directly finding the Fourier transform. We will use this example later in proving Plancherel’s theorem.

Example 2.1. Let \( f_\varepsilon := e^{-\varepsilon|x|^2}, \varepsilon > 0 \). Then \( \hat{f_\varepsilon}(\xi) = \left( \frac{\pi}{\varepsilon} \right)^{N/2} e^{-(\pi|\xi|^2)/\varepsilon} \).

Proof. First, show that \( f_\varepsilon \in L^1(\mathbb{R}^N) \). Using the Fubini’s theorem, we have

\[
\int_{\mathbb{R}^N} |f_\varepsilon(x)| \, dx = \int_{\mathbb{R}^N} e^{-\varepsilon|x|^2} \, dx = \int_{\mathbb{R}^N} e^{-\varepsilon \sum_{n=1}^{N} x_n^2} \, dx_1 dx_2 \ldots dx_n = \prod_{n=1}^{N} \int_{-\infty}^{\infty} e^{-\varepsilon x_n^2} \, dx_n = \prod_{n=1}^{N} \frac{1}{\sqrt{\pi} \varepsilon} \int_{-\infty}^{\infty} e^{-(\sqrt{\pi} x_n)^2} \, d(\sqrt{\pi} x_n) = \prod_{n=1}^{N} \sqrt{\frac{\pi}{\varepsilon}} = \left( \frac{\pi}{\varepsilon} \right)^{N/2} < \infty.
\]
Then the Fourier transform

\[ \hat{f}_\varepsilon(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x \cdot \xi} e^{-\varepsilon |x|^2} \, dx = \int_{\mathbb{R}^N} e^{-\sum_{n=1}^{N} \left( \varepsilon x_n^2 + 2 \sqrt{\varepsilon} x_n \pi i \xi_n / \sqrt{\varepsilon} \right)} \, dx_1 dx_2 \ldots dx_N = \]

\[ \int_{\mathbb{R}^N} e^{-\sum_{n=1}^{N} \left[ \left( \sqrt{\varepsilon} x_n + \pi i \xi_n / \sqrt{\varepsilon} \right)^2 - (\pi i \xi_n)^2 / \varepsilon \right]} \, dx_1 dx_2 \ldots dx_N = \]

\[ e^{-\left( \pi |\xi|^2 / \varepsilon \right)} \prod_{n=1}^{N} \frac{1}{\sqrt{\varepsilon_n}} \int_{-\infty}^{\infty} e^{-\left( \sqrt{\varepsilon} x_n + \pi i \xi_n / \sqrt{\varepsilon} \right)^2} \, dx_n = \]

\[ e^{-\left( \pi |\xi|^2 / \varepsilon \right)} \prod_{n=1}^{N} \sqrt{\frac{\varepsilon_n}{\varepsilon}} = \left( \frac{\varepsilon}{\varepsilon_n} \right)^{N/2} e^{-\left( \pi |\xi|^2 / \varepsilon \right)} , \]

where we have omitted the details of the contour shift arising from the substitution \( x_n \to \left( \sqrt{\varepsilon} x_n + \frac{\pi i \xi_n}{\sqrt{\varepsilon}} \right) . \)

\[ \Box \]

Note that applying the Lemma 2.3 to \( f_\varepsilon(x) \) and to some function \( g \in L^1(\mathbb{R}^N) \), we get

\[ \int_{\mathbb{R}^N} \hat{g}(x) e^{-\varepsilon |x|^2} \, dx = \left( \frac{\pi}{\varepsilon} \right)^{N/2} \int_{\mathbb{R}^N} g(y) e^{-\left( \pi |y|^2 / \varepsilon \right)} \, dy . \] (2.8)

**Lemma 2.5.** Let \( f \in L^p(\mathbb{R}^N) , \quad 1 \leq p < \infty \). Then \( \lim_{h \to 0} \| f_h - f \|_{L^p} = 0 \), where \( f_h(x) = f(x + h) \).

**Proof.**

1) Since \( C_0^\infty(\mathbb{R}^N) \) is dense in \( L^p(\mathbb{R}^N) \), then for each \( p \in [1, \infty) \), \( \forall \varepsilon > 0 \) and \( \forall f \in L^p(\mathbb{R}^N) \), \( \exists g \in C_0^\infty \) such that \( \| f - g \|_{L^p} < \frac{\varepsilon}{3} \).

2) \( g(x) \) is uniformly continuous because it is continuous on its compact support.

So \( g_h \to g \) uniformly as \( h \to 0 \) where \( g_h(x) = g(x + h) \).

Let \( g(x) \) has support in some compact \( V \subset \mathbb{R}^N \). For sufficiently small \( |h| > 0 \), \( g_h \) has support in the compact \( V \). Then \( \| g_h - g \|_{L^p(\mathbb{R}^N)} = \| g_h - g \|_{L^p(V)} \leq \sup_{x \in V} |g_h(x) - g(x)| \cdot [\text{vol}(V)]^{1/p} \to 0 \) as \( h \to 0 \) by the uniform convergence of \( g \) on \( V \).

So, \( \forall \varepsilon > 0 \), \( \exists r > 0 \) s.t. \( \forall h \in B_r(0) \), \( \| g_h - g \|_{L^p} < \frac{\varepsilon}{3} \).

3) Note that by the substitution \( y = x + h \), we have

\[ \| f_h - g_h \|_{L^p(\mathbb{R}^N)} = \| f - g \|_{L^p(\mathbb{R}^N)} < \frac{\varepsilon}{3} , \quad \forall \varepsilon > 0 . \]
4) Thus $\forall \varepsilon > 0$ and $\forall f \in L^p(\mathbb{R}^N)$, $\exists r > 0$ and $\exists g \in C_0^\infty (\mathbb{R}^N)$ s.t. $\forall h \in B_r(0)$, 
\[ \| f_h - f \|_{L^p} = \| f_h - g_h + g_h - g + g - f \|_{L^p} \leq \| f_h - g_h \|_{L^p} + \| g_h - g \|_{L^p} + \| f - g \|_{L^p} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \] i.e., $\lim_{h \to 0} \| f_h - f \|_{L^p} = 0$.

\[ \square \]

**Lemma 2.6.** Let $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Define $g(x) = f(x) \ast \bar{f}(-x)$ where $\bar{f}$ is complex conjugate of $f$. Then $g \in L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$.

**Proof.** Note that since $f \in L^1(\mathbb{R}^N)$, then $f(-x)$, $\bar{f}(x)$, and so $\bar{f}(-x) \in L^1(\mathbb{R}^N)$. Moreover, their $L^1$ norms in $\mathbb{R}^N$ are the same:

\[ \| f(x) \|_{L^1} = \| f(-x) \|_{L^1} = \| \bar{f}(x) \|_{L^1} = \| \bar{f}(-x) \|_{L^1}. \]

1) Show that $g \in L^1(\mathbb{R}^N)$. Use again Fubini’s theorem.

\[ \int_{\mathbb{R}^N} |g(x)| \; dx \leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |f(x-y)| \; |\bar{f}(-y)| \; dy \right) \; dx 
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(-y)| \; |f(x-y)| \; dy \; dx = \int_{\mathbb{R}^N} |f(-y)| \left( \int_{\mathbb{R}^N} |f(x-y)| \; dx \right) \; dy. \]

Use substitution $z = -y$ and $w(x) = x - y = x + z$.

\[ \int_{\mathbb{R}^N} |g(x)| \; dx \leq (-1)^N \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} |f(z)| \left( \int_{\mathbb{R}^N} |f(w)| \; dw \right) (-1)^N \; dz = \int_{\mathbb{R}^N} |f(z)| \; dz \| f \|_{L^1} = \left( \| f \|_{L^1} \right)^2. \]

So, $g \in L^1(\mathbb{R}^N)$ and $\| g \|_{L^1} \leq (\| f \|_{L^1})^2$.

2) Show that $g \in C(\mathbb{R}^N)$.

Remind that by the lemma 2.5 for $p = 1$, $\lim_{h \to 0} \| f_h - f \|_{L^1} = 0$, i.e., $\forall \varepsilon > 0$, $\exists r_1 > 0$ s.t. $\forall h \in B_{r_1}(0)$, $\| f_h - f \|_{L^1} < \frac{\varepsilon}{2\| f \|_{L^\infty}}$.

Similarly $\lim_{h \to 0} \| \bar{f}[-(x + h)] - \bar{f}(-x) \|_{L^1} = 0$, i.e., $\forall \varepsilon > 0$, $\exists r_2 > 0$ s.t. $\forall h \in B_{r_2}(0)$, $\| \bar{f}[-(x + h)] - \bar{f}(-x) \|_{L^1} < \frac{\varepsilon}{2\| f \|_{L^\infty}}$. 

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Now \( \forall x \in \mathbb{R}^N \) and \( \forall \varepsilon > 0, \exists r = \min \{r_1, r_2\} > 0 \) s.t. \( \forall h \in B_r(0) \),

\[
|g(x + h) - g(x)| = |f(x + h) * \tilde{f}(-x - h) - f(x) * \tilde{f}(-x)| \leq \\
|f(x + h) * \tilde{f}(-x - h) - f(x) * \tilde{f}(-x - h)| \\
+ |f(x) * \tilde{f}(-x - h) - f(x) * \tilde{f}(-x)| = \\
|\tilde{f}(-x - h)| \left| \int_{\mathbb{R}^N} \left[ f(x + h - y) - f(x - y) \right] \tilde{f}(-y - h) \, dy \right| \\
+ \left| \int_{\mathbb{R}^N} f(x \leq y) \left[ \tilde{f}(-y - h) - \tilde{f}(-y) \right] \, dy \right|
\]

where \( z(y) = x - y \).

So, \( |g(x + h) - g(x)| \leq \|\tilde{f}\|_{L^\infty} \|f_h - f\|_{L^1} + \|f\|_{L^\infty} \|\tilde{f}(-y - h) - \tilde{f}(-y)\|_{L^1} < \\
\|f\|_{L^\infty} \left( \frac{\varepsilon}{2\|\tilde{f}\|_{L^\infty}} + \frac{\varepsilon}{2\|f\|_{L^\infty}} \right) = \varepsilon, \forall \varepsilon > 0, \forall x \in \mathbb{R}^N. \)

Thus, \( g \in C(\mathbb{R}^N) \).

\[\square\]

An important property of the Fourier transform is that it keeps the \( L^2 \) norm unchanged as the following theorem shows.

**Theorem 2.1** (Plancherel’s Theorem). Let \( f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \). Then \( \hat{f}, \tilde{f} \in L^2(\mathbb{R}^N) \) and

\[
\|\hat{u}\|_{L^2} = \|\tilde{u}\|_{L^2} = \|u\|_{L^2}.
\]  \hspace{1cm} (2.9)

**Proof.**

1) Define again \( g(x) := f(x) * \varphi(x) \) with \( \varphi(x) := \tilde{f}(-x) \).

By the Lemma 2.6, \( g \in L^1(\mathbb{R}^N) \) with \( L^1 \) norm bounded by \( \|f\|_{L^1}^2 \). The Lemma 2.3 gives that \( \hat{g}(\xi) = \hat{f}(\xi) \hat{\varphi}(\xi) \).

Show that \( \hat{f}(\xi) \) and \( \hat{\varphi}(\xi) \) are complex conjugate.

\[
\hat{\varphi}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x \xi} \tilde{f}(-x) \, dx.
\]

Perform the substitution \( y = -x \). So, the components \( y_n = -x_n, 1 \leq n \leq N \) and
\[ dx = dx_1 dx_2 \ldots dx_n = (-1)^N dy_1 dy_2 \ldots dy_n = (-1)^N dy. \]

Then
\[ \hat{\phi}(\xi) = (-1)^N \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} e^{2\pi i y \cdot \xi} \hat{f}(y)(-1)^N dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} e^{-2\pi i y \cdot \xi} \hat{f}(y) dy = \int_{\mathbb{R}^N} e^{-2\pi i y \cdot \xi} \hat{f}(y) dy = \hat{\bar{\phi}}(\xi). \]

Hence \( \hat{g}(\xi) = \hat{\bar{\phi}}(\xi) \hat{\bar{f}}(\xi) = |\hat{\bar{f}}(\xi)|^2 \), i.e., \( \hat{g}(\xi) \) is real-valued non-negative function.

2) Show that \( \hat{g} \in L^1(\mathbb{R}^N) \) and then \( \hat{f} \in L^2(\mathbb{R}^N) \) since
\[ \|\hat{g}\|_{L^1} = \int_{\mathbb{R}^N} |\hat{g}(\xi)| d\xi = \int_{\mathbb{R}^N} |\hat{\bar{f}}(\xi)|^2 d\xi = \|\hat{\bar{f}}\|_{L^2}^2. \]

We will use the function \( f_\varepsilon(x) = e^{-\varepsilon|x|^2} \), \( \varepsilon > 0 \) from example 2.1.
\[ \int_{\mathbb{R}^N} |\hat{g}(\xi)| d\xi = \int_{\mathbb{R}^N} g(\xi) d\xi = \int_{\mathbb{R}^N} g(\xi) \lim_{\varepsilon \to 0^+} e^{-\varepsilon|x|^2} d\xi \text{ since } \lim_{\varepsilon \to 0^+} e^{-\varepsilon|x|^2} = 1. \]

Note that \( f_\varepsilon(\xi) \) is increasing as \( \varepsilon \to 0^+ \).

Apply the Monotone Convergence theorem and then equation (2.8).
\[ \int_{\mathbb{R}^N} |\hat{g}(\xi)| d\xi = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \hat{g}(\xi) e^{-\varepsilon|x|^2} d\xi = \lim_{\varepsilon \to 0^+} \left( \frac{\pi}{\varepsilon} \right)^{N/2} \int_{\mathbb{R}^N} g(x) e^{-\pi^2|x|^2/\varepsilon} dx. \]

Perform substitution \( y = \frac{x}{\sqrt{\varepsilon}} \). So, \( x_j = \frac{\sqrt{\varepsilon}}{\pi} y_j, j = 1, 2, \ldots, N, \) and
\[ dx = dx_1 dx_2 \ldots dx_N = \frac{\varepsilon^{N/2}}{\pi^N} dy. \]

Then
\[ \int_{\mathbb{R}^N} |\hat{g}(\xi)| d\xi = \lim_{\varepsilon \to 0^+} \left( \frac{\pi}{\varepsilon} \right)^{N/2} \varepsilon^{N/2} \int_{\mathbb{R}^N} g \left( \frac{\sqrt{\varepsilon} y}{\pi} \right) e^{-|y|^2} dy. \]

Since \( e^{-|y|^2} \leq 1 \), and so \( g \left( \frac{\sqrt{\varepsilon} y}{\pi} \right) e^{-|y|^2} \leq g \left( \frac{\sqrt{\varepsilon} y}{\pi} \right) \in L^1(\mathbb{R}^N) \), then we can apply now the Dominated Convergence theorem and the Fubini’s theorem after that.
\[ \int_{\mathbb{R}^N} |\hat{g}(\xi)| d\xi = \pi^{-N/2} \int_{\mathbb{R}^N} \lim_{\varepsilon \to 0^+} g \left( \frac{\sqrt{\varepsilon} y}{\pi} \right) e^{-|y|^2} dy = \]
\[ \pi^{-N/2} g(0) \int_{\mathbb{R}^N} e^{-\sum_{j=1}^{N} y_j^2} dy = \pi^{-N/2} g(0) \prod_{j=1}^{N} \int_{-\infty}^{\infty} e^{-y_j^2} dy_j = \]
\[ \pi^{-N/2} g(0) \pi^{N/2} = g(0). \]

Thus, \( \hat{g} \in L^1(\mathbb{R}^N), \hat{f} \in L^2(\mathbb{R}^N), \) and \( \|\hat{g}\|_{L^1} = \|\hat{f}\|_{L^2}^2 = g(0). \)

Following the same arguments, we can also show that \( \hat{f} \in L^2(\mathbb{R}^N). \)
3) The relation (2.9) follows from the last result.

\[ \|\hat{f}\|_{L^2}^2 = g(0) = (f \ast \varphi)(0) = \int_{\mathbb{R}^n} f(0-x)\varphi(x)\,dx = \int_{\mathbb{R}^n} f(-x)\hat{f}(-x)\,dx = \int_{\mathbb{R}^n} |f(-x)|^2 \,dx. \]

After substitution \( y = -x \), we have

\[ \|\hat{f}\|_{L^2}^2 = \int_{\mathbb{R}^n} |f(y)|^2 \,dy = \|f\|_{L^2}^2. \]

Then

\[ \|\hat{f}\|_{L^2} = \|f\|_{L^2} \text{ in } \mathbb{R}^N. \] (2.9a)

By the same way, we can get

\[ \|\hat{g}\|_{L^2} = \|g\|_{L^2} \text{ in } \mathbb{R}^N. \] (2.9b)

Based on Plancherel’s theorem, we can define the Fourier transform in \( L^2(\mathbb{R}^N) \). Since \( L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) is dense in \( L^2(\mathbb{R}^N) \), given \( f \in L^2(\mathbb{R}^N) \), there exists a sequence \( \{f_k\} \subset L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) such that \( \|f_k - f\|_{L^2} \to 0 \) as \( k \to \infty \). Then \( \{f_k\} \) is a Cauchy sequence in \( L^2(\mathbb{R}^N) \). Hence, the sequence \( \{\hat{f}_k\} \subset L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) is a Cauchy sequence in \( L^2(\mathbb{R}^N) \) as well, because by Plancherel’s theorem, \( \|\hat{f}_{k_1} - \hat{f}_{k_2}\|_{L^2} = \|\hat{f}_{k_1} - \hat{f}_{k_2}\|_{L^2} = \|f_{k_1} - f_{k_2}\|_{L^2}, \quad k_1, k_2 \in \mathbb{N} \). However, \( L^2(\mathbb{R}^N) \) is complete, so, \( \exists \hat{f} \in L^2(\mathbb{R}^N) \) such that \( \hat{f} := \lim_{k \to \infty} \hat{f}_k \). Also, \( \|\hat{f}\|_{L^2} = \lim_{k \to \infty} \|\hat{f}_k\|_{L^2} = \lim_{k \to \infty} \|f_k\|_{L^2} = \|f\|_{L^2}. \)

In the same way, we can define the inverse Fourier transform in \( L^2(\mathbb{R}^N) \) as \( \hat{f}(\xi) := \lim_{k \to \infty} \hat{f}_k(\xi) \).

Lemma 2.4 could be extended to \( L^2(\mathbb{R}^N) \).

Lemma 2.7. Let \( f, g \in L^2(\mathbb{R}^N) \). Then

\[ \int_{\mathbb{R}^N} \hat{f}(x)g(x) \,dx = \int_{\mathbb{R}^N} f(y)\hat{g}(y) \,dy. \] (2.7)

Proof. Recall again that \( L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) is dense in \( L^2(\mathbb{R}^N) \). So, there exist sequences \( \{f_k\}, \{g_k\} \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) such that \( f_k \to f \) and \( g_k \to g \) as \( k \to \infty \), i.e., \( \forall \varepsilon > 0, \exists k_0(\varepsilon) \in \mathbb{N} \) s.t. \( \forall k \geq k_0 \quad \|f_k - f\|_{L^2}, \|g_k - g\|_{L^2} < \varepsilon \).

Note that by the Plancherel’s theorem, \( \|f_k - f\|_{L^2} = \|\hat{f}_k - \hat{f}\|_{L^2} \) and \( \|g_k - g\|_{L^2} = \|\hat{g}_k - \hat{g}\|_{L^2}. \)
1) Show that \( \| \hat{f}_k \|_{L^2} \) is finite.
\[
\| \hat{f}_k \|_{L^2} = \| \hat{f}_k - \hat{f} \|_{L^2} \leq \| \hat{f}_k - \hat{f} \|_{L^2} + \| \hat{f} \|_{L^2} < \epsilon + \| f \|_{L^2}, \quad \forall k > k_0(\epsilon), \quad \epsilon > 0.
\]

2) Show that for the \( L^2 \) inner product, \((\hat{f}_k, g_k) \to (\hat{f}, g)\) as \( k \to \infty \).

Consider
\[
\left| \int_{\mathbb{R}^N} \hat{f}_k g_k \, dx - \int_{\mathbb{R}^N} \hat{f} g \, dx \right| = \left| \int_{\mathbb{R}^N} \hat{f}_k g_k \, dx - \int_{\mathbb{R}^N} \hat{f}_k g \, dx + \int_{\mathbb{R}^N} \hat{f}_k g \, dx - \int_{\mathbb{R}^N} \hat{f} g \, dx \right| \\
\leq \left| \int_{\mathbb{R}^N} \hat{f}_k (g_k - g) \, dx \right| + \left| \int_{\mathbb{R}^N} (\hat{f}_k - \hat{f}) \, g \, dx \right|.
\]

Applying the Cauchy-Schwartz inequality, we get
\[
\left| (\hat{f}_k, g_k) - (\hat{f}, g) \right| \leq \| \hat{f}_k \|_{L^2} \| g_k - g \|_{L^2} + \| \hat{f}_k - \hat{f} \|_{L^2} \| g \|_{L^2} < \\
(\| \hat{f}_k \|_{L^2} + \| g \|_{L^2}) \epsilon < (\epsilon + \| f \|_{L^2} + \| g \|_{L^2}) \epsilon, \quad \forall k \geq k_0, \forall \epsilon > 0.
\]

So, \((\hat{f}_k, g_k) \to (\hat{f}, g)\) as \( k \to \infty \).

Similarly, it could be shown that \((f_k, \hat{g}_k) \to (f, \hat{g})\) as \( k \to \infty \).

3) Using the Lemma 2.4 for \( f_k, g_k \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \), we obtain the equation (2.7) for \( f, g \in L^2(\mathbb{R}^N) \).

\[
\int_{\mathbb{R}^N} \hat{f}(x) \hat{g}(x) \, dx = \lim_{k \to \infty} \int_{\mathbb{R}^N} \hat{f}_k(x) \hat{g}_k(x) \, dx = \\
\lim_{k \to \infty} \int_{\mathbb{R}^N} f_k(x) \hat{g}_k(x) \, dx = \int_{\mathbb{R}^N} f(x) \hat{g}(x) \, dx.
\]

We conclude our review of the Fourier transform by the following theorem.

**Theorem 2.2** (Properties of the Fourier transform). Let \( f, g \in L^2(\mathbb{R}^N) \). Then:

(i) \[
\int_{\mathbb{R}^N} f(x) \hat{g}(x) \, dx = \int_{\mathbb{R}^N} \hat{f}(\xi) \hat{g}(\xi) \, d\xi,
\]

(ii) \forall multiindices \( \alpha \) such that \( D^\alpha f \in L^2(\mathbb{R}^N) \),

\[
\hat{D^\alpha f} = (2\pi i)^\alpha \hat{f}(\xi),
\]
(iii) if \( f, g \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \), then
\[
\hat{f} \ast \hat{g} = \hat{f} \hat{g}
\] (2.5)

and
\[
(f \ast g) = f \hat{g}
\] (2.12)

(iv)
\[
(\hat{f}) = f
\] (2.13)

and
\[
(\hat{f}) = f.
\] (2.14)

So, the Fourier transform is a bijection \( L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N) \).

**Proof.**

(i) Apply the Plancherel’s theorem for \((f + \alpha g), \alpha \in \mathbb{C}, \text{ in } \mathbb{R}^N\).

\[
\|f + \alpha g\|_{L^2}^2 = \|\hat{f} + \alpha \hat{g}\|_{L^2}^2.
\]

Then
\[
\int_{\mathbb{R}^N} (f + \alpha g)(f + \alpha g) \, dx = \int_{\mathbb{R}^N} (\hat{f} + \alpha \hat{g})(\hat{f} + \alpha \hat{g}) \, d\xi
\]
or
\[
\int_{\mathbb{R}^N} (|f|^2 + |\alpha|^2 |g|^2 + \alpha \bar{g} f + \bar{\alpha} f g) \, dx = \int_{\mathbb{R}^N} (|\hat{f}|^2 + |\alpha|^2 |\hat{g}|^2 + \alpha \bar{\hat{g}} \hat{f} + \bar{\alpha} \hat{f} \bar{\hat{g}}) \, d\xi,
\]
\[
\|f\|_{L^2} + |\alpha|^2 \|g\|_{L^2} + \int_{\mathbb{R}^N} (\alpha \bar{g} f + \bar{\alpha} f g) \, dx = \|\hat{f}\|_{L^2} + |\alpha|^2 \|\hat{g}\|_{L^2} + \int_{\mathbb{R}^N} (\alpha \bar{\hat{g}} \hat{f} + \bar{\alpha} \hat{f} \bar{\hat{g}}) \, d\xi.
\]

Hence
\[
\int_{\mathbb{R}^N} (\alpha \bar{g} f + \bar{\alpha} f g) \, dx = \int_{\mathbb{R}^N} (\alpha \bar{\hat{g}} \hat{f} + \bar{\alpha} \hat{f} \bar{\hat{g}}) \, d\xi.
\]

For \( \alpha = 1 \), we have:
\[
\int_{\mathbb{R}^N} (g \bar{f} + \bar{g} f) \, dx = \int_{\mathbb{R}^N} (\bar{g} \hat{f} + \hat{g} \bar{f}) \, d\xi.
\]

For \( \alpha = i \), we have:
\[
\int_{\mathbb{R}^N} (g \bar{f} - f \bar{g}) \, dx = \int_{\mathbb{R}^N} (\bar{g} \hat{f} - \hat{g} \bar{f}) \, d\xi.
\]
Subtracting the last equation from the previous one, we obtain
\[ 2 \int_{\mathbb{R}^N} f \, g \, dx = 2 \int_{\mathbb{R}^N} \hat{f} \, \overline{g} \, d\xi. \quad (2.10) \]

(ii) At first, we will prove the property (ii) for \( C_0^\infty(\mathbb{R}^N) \) by the mathematical induction. Let \( f \in C_0^\infty(\mathbb{R}^N) \), and \( R > 0 \) is large enough such that \( \text{spt}(f) \subset B(0, R) \). Use the definition of the Fourier transform and then integrate by parts, to find the transform of \( \frac{\partial f}{\partial x_j} \), \( 1 \leq j \leq N \).

\[
\frac{\partial \hat{f}}{\partial x_j}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i \xi \cdot x} \frac{\partial f(x)}{\partial x_j} \, dx = \int_{B(0,R)} e^{-2\pi i \xi \cdot x} \frac{\partial f(x)}{\partial x_j} \, dx = \int_{\partial B(0,R)} e^{-2\pi i \xi \cdot x} f(x) \nu^j(x) \, dS(x) - \int_{B(0,R)} \frac{\partial}{\partial x_j} (e^{-2\pi i \xi \cdot x}) f(x) \, dx = 0 - (-2\pi i \xi_j) \int_{B(0,R)} e^{-2\pi i \xi \cdot x} f(x) \, dx = 2\pi i \xi_j \int_{\mathbb{R}^N} e^{-2\pi i \xi \cdot x} f(x) \, dx = 2\pi i \xi_j \hat{f}(\xi).
\]

Here \( \nu^j \) is the \( j \)-th component of the unit normal outward vector of \( \partial B(0,R) \).

For another step of the mathematical induction proof, assume that the property (ii) holds for \( D^\beta f \), that is, for all multiindices \( \beta \) such that \( |\beta| = |\alpha| - 1 \), i.e., \( D^\beta \hat{f}(\xi) = (2\pi i \xi)^\beta \hat{f}(\xi) \).

It follows that \( D^\alpha f(x) = (D^\beta f)_{x_j}(x) \) for some \( 1 \leq j \leq N \). Then
\[
\hat{D^\alpha f}(\xi) = (2\pi i \xi_j) \hat{D^\beta f}(\xi) = (2\pi i \xi_j)(2\pi i \xi)^\beta \hat{f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi).
\]

So, the property (ii) has been proved on \( C_0^\infty(\mathbb{R}^N) \).

Now we prove that this property extends to any function \( f \in L^2(\mathbb{R}^N) \) such that \( D^\alpha f \in L^2(\mathbb{R}^N) \) for multiindex \( \alpha \).

\( C_0^\infty(\mathbb{R}^N) \) is dense in \( L^2(\mathbb{R}^N) \), i.e., \( \forall f \in L^2(\mathbb{R}^N), \exists \{ f_k \} \subset C_0^\infty(\mathbb{R}^N) \) such that \( f_k \to f \) as \( k \to \infty \).

\( \{ f_k \} \) converges uniformly on compact subsets of \( \mathbb{R}^N \).

So, \( D^\alpha f_k \to D^\alpha f \) as \( k \to \infty \).

Then using the definition of the Fourier transform in \( L^2(\mathbb{R}^N) \) and the property (ii) in \( C_0^\infty(\mathbb{R}^N) \), we get
\[
\hat{D^\alpha f}(\xi) = \lim_{k \to \infty} \hat{D^\alpha f_k}(\xi) = \lim_{k \to \infty} (2\pi i \xi)^\alpha \hat{f}_k(\xi) = (2\pi i \xi)^\alpha \lim_{k \to \infty} \hat{f}_k(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi).
\]

(iii) By the Lemma 2.3, the equation (2.5) holds in \( L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) that is dense in \( L^2(\mathbb{R}^N) \). Applying the density argument by the same way, as we did above proving
(ii), we get the equation (2.5) in $L^2(\mathbb{R}^N)$.
Similarly, we can prove that
\[(f \ast g)\hat{=} \hat{f} \hat{g}.	ag{2.12}\]

(iv) Show that $\forall g \in L^2(\mathbb{R}^N)$, $\hat{\bar{\hat{g}}} = \hat{\bar{\hat{g}}} = \hat{\bar{g}}$.

Next, using the Lemma 2.7, the last relation, and the property (i), we have
\[
\int_{\mathbb{R}^N} (\hat{\hat{f}})\hat{g} \, dx = \int_{\mathbb{R}^N} \hat{f} \hat{\bar{g}} \, dx = \int_{\mathbb{R}^N} \hat{f} \hat{\bar{g}} \, dx = \int_{\mathbb{R}^N} f \bar{g} \, dx.
\]

Hence $\int_{\mathbb{R}^N} (\hat{\hat{f}} - f) \hat{g} \, dx = 0$, $\forall g \in L^2(\mathbb{R}^N)$.

Therefore $\hat{\hat{f}} = f$.

The same proof works to show that
\[
\hat{\hat{f}} = f.	ag{2.14}\]

Thus, the Fourier transform is a bijection $L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$.

\[
\square
\]

The Fourier transform is a powerful tool in solution of PDE. In particular, it could be applied to solve the Poisson equation.

2.3 Elementary Properties of the Poisson Equation.

In this section, we review solution of the Poisson equation in $\mathbb{R}^N$ using the Fourier transform.

Lemma 2.8. Let $f \in C^2_0(\mathbb{R}^N)$. Then for $N \geq 3$, $(\Phi \ast f)(x)$ is the solution of the Poisson equation $-\Delta u = f$ where the fundamental solution of Laplace’s equation is
\[
\Phi(x) = \frac{\Gamma(N/2)}{2\pi^{N/2}(N - 2)} \frac{1}{|x|^{N-2}}, \quad x \in \mathbb{R}^N \setminus \{0\}.	ag{2.15}\]

Proof.

1) Find the Fourier transform of $\Delta u$ using the property (2.11) of the Fourier transform,
\[
\hat{\Delta u}(\xi) = (\sum_{j=1}^{N} D^2_j u)(\xi) = \sum_{j=1}^{N} (2\pi i \xi_j)^2 \hat{u}(\xi) = -4\pi^2 |\xi|^2 \hat{u}(\xi).
\]
Then applying the Fourier transform to both sides of the Poisson equation, we have

\[ 4\pi^2|\xi|^2 \hat{u}(\xi) = \hat{f}(\xi) \quad \text{or} \quad \hat{u}(\xi) = \frac{1}{4\pi^2|\xi|^2} \hat{f}(\xi). \]

Take the inverse Fourier transform from both sides of the last equation and apply the properties of the inverse Fourier transform (2.13) and (2.12).

We obtain

\[ u(x) = \frac{1}{4\pi^2} \left( \frac{1}{|\xi|^2} \right) \hat{f}(x) = \Phi(x) * f(x), \]

where the fundamental solution of Laplace’s equation is

\[ \Phi(x) = \frac{1}{4\pi^2} \left( \frac{1}{|\xi|^2} \right) \hat{f}(x). \]

Note that \( \frac{1}{|\xi|^2} \) is not integrable for \( N = 2 \) at \( \xi = 0 \). So, we consider here the case \( N > 2 \) only.

2)

(a) To find this inverse Fourier transform, we will use identity:

\[ \frac{1}{|\xi|^2} = \int_0^\infty e^{-t|\xi|^2} dt. \]

Then

\[ \left( \frac{1}{|\xi|^2} \right) \hat{f}(x) = \int_0^\infty \left( \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} e^{-t|\xi|^2} d\xi \right) dt = \int_0^\infty \left( \prod_{j=1}^N \int_{-\infty}^{\infty} e^{2\pi i x_j \xi_j} e^{-t \sum_{j=1}^N \xi_j^2} d\xi_j \right) dt = \int_0^\infty \left( \prod_{j=1}^N I_j(t) \right) dt \]

where \( I_j(t) = \int_{-\infty}^{\infty} e^{-(\sqrt{t}\xi_j)^2+2\pi i x_j \xi_j} d\xi_j \).

(b)

\[ I_j(t) = \int_{-\infty}^{\infty} e^{-\left(\sqrt{t}\xi_j\right)^2+\frac{2\pi i x_j}{\sqrt{t}} \cdot \xi_j + \frac{\pi^2 x_j^2}{t} - \frac{\pi^2 x_j^2}{t}} d\xi_j = e^{-\frac{\pi^2 x_j^2}{t}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{t}\xi_j - \frac{\pi x_j}{\sqrt{t}}\right)^2} d\xi_j. \]
Make the substitution $z(\xi_j) = \sqrt{t} \xi_j - \frac{\pi i x_j}{\sqrt{t}}$. Then we have

$$I_j(t) = \frac{1}{\sqrt{t}} e^{-\frac{\pi^2 x_j^2}{t}} \int_{\Lambda_j} e^{-z^2} \, dz$$

where $\Lambda_j := \{ z \in \mathbb{C} | \Im z = -\frac{\pi x_j}{\sqrt{t}} \}$.

Without lost of generality, assume that $x_j \leq 0$.

Now define the counterclockwise oriented contour $\lambda = \bigcup_{k=1}^4 \lambda_k$ with

$\lambda_1 = \{(x, y) | -R \leq x \leq R, \ y = 0 \}$, $\lambda_2 = \{(x, y) | x = R, \ 0 \leq y \leq -\frac{\pi x_j}{\sqrt{t}} \}$,

$\lambda_3 = \{(-x, y) | -R \leq x \leq R, \ y = -\frac{\pi x_j}{\sqrt{t}} \}$,

$\lambda_4 = \{(x, -\frac{\pi x_j}{\sqrt{t}} - y) | x = -R, \ 0 \leq y \leq -\frac{\pi x_j}{\sqrt{t}} \}$ for some $R > 0$.

$e^{-z^2}$ is analytic on $\mathbb{C}$. So, by the Cauchy integral theorem, $\oint_{\lambda} e^{-z^2} \, dz = 0$.

Then $\sum_{k=1}^4 \int_{\lambda_k} e^{-z^2} \, dz = 0$, and $\int_{\lambda_1} e^{-z^2} \, dz = \int_{\lambda_2} e^{-z^2} \, dz - \int_{\lambda_3} e^{-z^2} \, dz - \int_{\lambda_4} e^{-z^2} \, dz$.

Now take the limit when $R \to \infty$.

$$\lim_{R \to \infty} \int_{\lambda_1} e^{-z^2} \, dz = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi},$$

$$\lim_{R \to \infty} -\int_{\lambda_3} e^{-z^2} \, dz = \lim_{R \to \infty} \int_{\lambda_3} e^{-z^2} \, dz = \int_{\Lambda_j} e^{-z^2} \, dz.$$  

Also, $\left| \int_{\lambda_{2,4}} e^{-z^2} \, dz \right| = \left| \int_{0}^{\frac{\pi x_j}{\sqrt{t}}} e^{-(\pm R + iy)^2} \, dy \right| \leq \int_{0}^{\frac{\pi x_j}{\sqrt{t}}} \left| e^{-R^2} e^{\mp 2 R i y} e^{yi^2} \right| \, dy = e^{-R^2} \int_{0}^{\frac{\pi x_j}{\sqrt{t}}} e^{y^2} \, dy \to 0$ as $R \to \infty$.

Therefore $\int_{\Lambda_j} e^{-z^2} \, dz = \sqrt{\pi}$ and $I_j(t) = \sqrt{\frac{\pi}{t}} e^{-\frac{\pi^2 x_j^2}{t}}$.

(c) Thus,

$$\left( \frac{1}{|\xi|^2} \right)^\vee(x) = \int_{0}^{\infty} \prod_{j=1}^{N} \sqrt{\frac{\pi}{t}} e^{-\frac{\pi^2 x_j^2}{t}} \, dt = \pi^{N/2} \int_{0}^{\infty} \frac{1}{t^{N/2}} e^{-\sum_{j=1}^{N} \frac{\pi^2 x_j^2}{t}} \, dt$$

$$= \pi^{N/2} \int_{0}^{\infty} \frac{1}{t^{N/2}} e^{-\frac{\pi^2 |x|^2}{t}} \, dt.$$
Note that applying L’Hopital rule \([((N + 1)/2)]\) times, we have

\[
\lim_{t \to 0^+} e^{-\frac{\pi^2|x|^2}{t}} = \lim_{(\frac{1}{t}) \to \infty} \left(\frac{1}{t}\right)^{N/2} e^{\pi^2|x|^2(\frac{1}{t})} = 0 \quad \text{for } N \in \mathbb{N}.
\]

However, when \(t \to \infty\), \(e^{-\frac{\pi^2|x|^2}{t}} \to 1\). So, our integrand is integrable at infinity only if \(N/2 > 1\), i.e., for \(N \geq 3\).

Make the change of variable \(s = \frac{\pi^2|x|^2}{t}\).

\[
\left(\frac{1}{|\xi|^2}\right)^{(N/2)}(x) = \pi^{N/2} \int_{\mathbb{R}^N} \frac{s^{N/2}}{\pi^N |x|^N e^{-s}} \left(-\frac{\pi^2|x|^2}{s^2}\right) ds = \pi^{2-N/2} \int_0^\infty s^{(N/2)-2} e^{-s} ds.
\]

So, \(\Phi(x) = \frac{1}{4\pi^2} \left(\frac{1}{|\xi|^2}\right)^{(N/2)}(x) = \frac{\Gamma(N/2) - 1}{4\pi^{N/2} |x|^{N-2}}\)

where the gamma function \(\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \text{Re } z > 0\).

Since \(\Gamma(z + 1) = z\Gamma(z)\), then \(\Gamma(N/2) - 1 = \frac{\Gamma(N/2)}{(N/2) - 1}\). Then

\[
\Phi(x) = \frac{\Gamma(N/2)}{2\pi^{N/2}(N-2)} \frac{1}{|x|^{N-2}}, \quad x \in \mathbb{R}^N \setminus \{0\}.
\]

\(\square\)

Next, we consider the case \(N = 2\) separately.

**Lemma 2.9.** Let \(f \in C^2_0(\mathbb{R}^2)\). The \((\Phi * f)(x)\) is the solution of the Poisson equation

\[-\Delta u = f\]

where the fundamental solution of the Laplace’s equation

\[
\Phi(x) = -\frac{1}{2\pi} \log |x|, \quad x \in \mathbb{R}^2 \setminus \{0\}
\]

\(\phi(x) = -\frac{1}{2\pi} \log |x|, \quad x \in \mathbb{R}^2 \setminus \{0\}\)

\(\phi(x) = -\frac{1}{2\pi} \log |x|, \quad x \in \mathbb{R}^2 \setminus \{0\}\)

**Proof.**

1) Assume that for \(N = 2\), like in the case \(N \geq 3\), \(\Phi(x)\) is a radial function satisfying the Laplace’s equation, i.e.,

\[
\Phi(x) = v(r) \quad \text{where } r = |x| \quad \text{and } \Delta \Phi(x) = 0.
\]

Then

\[
\Phi_{x_j} = v'(r) r_{x_j} = v'(r) \frac{x_j}{r}, \quad j = 1, 2, \quad r > 0.
\]
Hence

$$\Phi_{x_jx_j} = v''(r) x_j^2 + v'(r) r x_j x_j = v''(r) \left( \frac{x_j}{r} \right)^2 + v'(r) \frac{r-x_j x_j}{r^2}.$$ 

Therefore

$$\Delta \Phi = \Phi_{x_1x_1} + \Phi_{x_2x_2} = v''(r) \left( \frac{x_1^2 + x_2^2}{r^2} \right) + v'(r) \left( \frac{2}{r} - \frac{x_1^2 + x_2^2}{r^2} \right) = v''(r) + \frac{1}{r} v'(r) = 0.$$ 

For $w = v'(r)$, we get a separable ODE of the first order

$$w' + \frac{1}{r} w = 0.$$ 

After separation variables and integration, we have

$$\log |w| = - \log r + \log |C_1|, \quad C_1 \in \mathbb{R} \setminus \{0\}, \quad \text{or} \quad v' = \frac{C_1}{r}.$$ 

After a second integration, we obtain

$$v(r) = C_1 \log r + C_2, \quad C_2 \in \mathbb{R}.$$ 

So, choosing $C_1 = -\frac{1}{2\pi}$ and $C_2 = 0$, we get

$$\Phi(x) = -\frac{1}{2\pi} \log |x|, \quad x \in \mathbb{R}^2 \setminus \{0\}. \quad (2.16)$$ 

$\Phi(x)$ from (2.16) is defined as the fundamental solution of the Laplace’s equation in $\mathbb{R}^2 \setminus \{0\}$. 

2) Show that

$$u(x) = (\Phi * f)(x) = \int_{\mathbb{R}^2} \Phi(x-y) f(y) \, dy \in C^2(\mathbb{R}^2).$$ 

(a) Note that $\Phi$ has a singularity when $x = y$. To avoid this problem, use the fact that a convolution is commutative. Indeed, changing the variable of integration by $z(y) = x - y$, we get

$$u(x) = \int_{\mathbb{R}^2} \Phi(z) f(x-z) \, dz = (f * \Phi)(x).$$ 

Take some $h \in \mathbb{R}$, $e_1 = (1, 0)$, $e_2 = (0, 1)$, $j = 1, 2$ and consider the ratio

$$\frac{u(x+he_j) - u(x)}{h} = \int_{\mathbb{R}^2} \Phi(z) \frac{f(x+he_j-z) - f(x-z)}{h} \, dz.$$ 

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Recall that $f \in C^2_0(\mathbb{R}^2)$. Then by Taylor’s theorem, we have

$$f(x + he_j - z) = f(x - z) + \frac{\partial f(x - z)}{\partial x_j} h + \frac{\partial^2 f(x + e_1 e_j - z)}{\partial x_j^2} h^2$$

with $c$ between 0 and $h$.

So,

$$\frac{f(x + he_j - z) - f(x - z)}{h} \rightarrow \frac{\partial f(x - z)}{\partial x_j} \quad \text{as } h \rightarrow 0.$$

(b) Show that pointwise convergence above is uniform convergence.

Take $h = \frac{1}{n}$ and define

$$g_n(x) := \frac{f(x + \frac{e_j}{n} - z) - f(x - z)}{1/n}, \quad M(x) := \sup_{x \in B(0,n)} |g_n(x) - \frac{\partial f(x - z)}{\partial x_j}|.$$

By the previous result, $g_n \rightarrow \frac{\partial f(x - z)}{\partial x_j}$ as $n \rightarrow \infty$.

$M(x)$ is continuous in $\mathbb{R}^2$ since $f \in C^2(\mathbb{R}^2)$. Take $R > 0$ so big that

$$B(0,R) \supset \text{spt}(g_n), \text{spt} \left(\frac{\partial f(x - z)}{\partial x_j}\right).$$

Then $\forall n \in \mathbb{N}, \exists x_n \in B(0,R)$-compact, such that $M(x_n) = \sup_{x \in B(0,R)} |g_n(x) - \frac{\partial f(x - z)}{\partial x_j}| \rightarrow 0$ as $n \rightarrow \infty$.

We get that $\sup_{x \in B(0,R)} \left| g_n(x) - \frac{\partial f(x - z)}{\partial x_j} \right| \rightarrow 0$ as $n \rightarrow \infty$.

So, $g_n(x) \rightarrow \frac{\partial f(x - z)}{\partial x_j}$ uniformly as $n \rightarrow \infty$.

Therefore,

$$\frac{f(x + he_j - z) - f(x - z)}{h} \rightarrow \frac{\partial f(x - z)}{\partial x_j}.$$ (c) Now, employing the uniform convergence of the integrand, we obtain

$$\frac{\partial u(x)}{\partial x_j} = \lim_{h \rightarrow 0} \frac{u(x + he_j) - u(x)}{h} = \lim_{h \rightarrow \infty} \int_{\mathbb{R}^2} \Phi(z) \frac{f(x + he_j - z) - f(x - z)}{h} dz$$

$$= \int_{\mathbb{R}^2} \Phi(z) \frac{\partial f(x - z)}{\partial x_j} dz = \left(\frac{\partial f}{\partial x_j}\right)(x).$$

Repeating the same argument, we can get

$$\frac{\partial^2 u(x)}{\partial x_j \partial x_k} = \int_{\mathbb{R}^2} \Phi(z) \frac{\partial^2 f(x - z)}{\partial x_j \partial x_k} dz = \left(\frac{\partial f}{\partial x_j \partial x_k} * \Phi\right)(x), \quad j, k = 1, 2.$$ (b) It remains to show that $u = \Phi * f$ satisfies the Poisson equation $-\Delta u = f$ in $\mathbb{R}^2$.

According to the previous part of our proof,

$$\Delta u(x) = (\Phi * \Delta f)(x) = \int_{\mathbb{R}^2} \Phi(z) \Delta_x f(x - z) dz.$$
Choose some $\varepsilon > 0$. Note that the integrand is $C^\infty$ function in $\mathbb{R}^2 \setminus B(0, \varepsilon)$.

Represent

$$\Delta u(x) = I_\varepsilon + J_\varepsilon,$$

where $I_\varepsilon := \int_{B(0,\varepsilon)} \Phi(z) \Delta_x f(x - z) \, dz$ and $J_\varepsilon := \int_{\mathbb{R}^2 \setminus B(0,\varepsilon)} \Phi(z) \Delta_x f(x - z) \, dz$

and we will take the limit as $\varepsilon \to 0^+$ afterwards.

(a) Show that $I_\varepsilon \to 0$ as $\varepsilon \to 0^+$.

Estimate $I_\varepsilon$.

$$|I_\varepsilon| \leq \sup_{x \in \mathbb{R}^2} |\Delta f(x)| \int_{B(0,\varepsilon)} |\Phi(z)| \, dz \leq C \int_{B(0,\varepsilon)} |\Phi(z)| \, dz, \quad C \in \mathbb{R},$$

since $f \in C^2_0(\mathbb{R}^2)$, and $\Delta f$ is continuous on compact set and so is bounded.

Switch to polar coordinates because $\Phi$ is a radial function.

$$|I_\varepsilon| \leq C \int_{B(0,\varepsilon)} \frac{1}{2\pi} |\log r| \, dr \, d\theta = \frac{C}{2\pi} \int_0^{2\pi} d\theta \int_{\varepsilon}^0 |\log r| \, dr = C \int_{\varepsilon}^0 |\log r| \, dr.$$

Assume that $0 < \varepsilon < 1$. So, $|\log r| = -\log r$. Integrate by parts taking into account that the integrand is singular at the origin.

$$|I_\varepsilon| \leq -C \lim_{\delta \to 0^+} \left( \frac{r^2}{2} \log r \bigg|_\varepsilon^\delta - \frac{1}{2} \int_\delta^\varepsilon r \, dr \right) =$$

$$\frac{C}{2} \left[ -\varepsilon^2 \log \varepsilon + \frac{\varepsilon^2}{2} + \lim_{\delta \to 0^+} \left( \delta^2 \log \delta - \frac{\delta^2}{2} \right) \right] \leq \frac{C}{2} \varepsilon^2 (-\log \varepsilon) = \frac{C}{2} \varepsilon^2 |\log \varepsilon|.$$

Here, to find the limit, we applied L’Hopital’s rule.

$$\lim_{\delta \to 0^+} \delta^2 \log \delta = \lim_{\delta \to 0^+} \frac{\log \delta}{\delta^{-2}} = \left( \frac{\infty}{\infty} \right) = \lim_{\delta \to 0^+} \frac{\delta^{-1}}{2\delta^{-3}} = -\frac{1}{2} \lim_{\delta \to 0^+} \delta^2 = 0.$$

So, $|I_\varepsilon| \to 0$ as $\varepsilon \to 0^+$.

(b) Show that $J_\varepsilon \to -f(x)$ as $\varepsilon \to 0^+$.

i. Since $\Delta_x f(x - z) = \Delta_z f(x - z)(-1)^2$, then

$$J_\varepsilon = \int_{\mathbb{R}^2 \setminus B(0,\varepsilon)} \Phi(z) \Delta_x f(x - z) \, dz.$$

Let again $R > \varepsilon > 0$ be so big that $B(0, R) \supset \text{sup}(f)$.

So, $J_\varepsilon = \int_{B(0,R) \setminus B(0,\varepsilon)} \Phi(z) \Delta_z f(x - z) \, dz$. 

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Now integrate by parts. We have

\[
J_\varepsilon = \lim_{R \to \infty} \int_{B(0,R) \setminus B(0,\varepsilon)} \Phi(z) \nabla_z [\nabla_z f(x - z)] \, dz = \\
\lim_{R \to \infty} \left[ -\int_{B(0,R) \setminus B(0,\varepsilon)} \nabla \Phi(z) \nabla_z f(x - z) \, dz + \\
\int_{\partial B(0,\varepsilon)} \Phi(z) \frac{\partial f(x - z)}{\partial \nu} \, dS(z) + \int_{\partial B(0,R)} \Phi(z) \frac{\partial f(x - z)}{\partial \nu} \, dS(z) \right].
\]

Note that \( \nu \) is an exterior normal unit vector for the annulus \( B(0, R) \setminus B(0, \varepsilon) \), i.e., \( \nu \) is the outward unit vector normal to the outer circle \( \partial B(0, R) \), and it is the inward unit vector normal to the inner circle \( \partial B(0, \varepsilon) \).

If \( R \to \infty \), then the last integral is 0 since \( f \) vanishes at infinity.

Hence

\[
J_\varepsilon = K_\varepsilon + L_\varepsilon
\]

where \( K_\varepsilon := -\int_{B(0,R) \setminus B(0,\varepsilon)} \nabla \Phi(z) \nabla_z f(x - z) \, dz \)

and \( L_\varepsilon := \int_{\partial B(0,\varepsilon)} \Phi(z) \frac{\partial f(x - z)}{\partial \nu} \, dS(z) \).

ii. Show that \( L_\varepsilon \to 0 \) as \( \varepsilon \to 0^+ \).

At first, estimate \( \left| \frac{\partial f(x - z)}{\partial \nu} \right| \) on \( \partial B(0, \varepsilon) \) using the Cauchy-Schwartz inequality.

\[
\left| \frac{\partial f(x - z)}{\partial \nu} \right| = |\nu \cdot \nabla f(x - z)| \leq |\nu| |\nabla f(x - z)| \\
\leq 1 \sup \left| \nabla f(x - z) \right| \leq \| \nabla f \|_{L^\infty(\mathbb{R}^2)} < \infty \text{ since } f \in C^2_0(\mathbb{R}^2).
\]

Then \( |L_\varepsilon| \leq \| \nabla f \|_{L^\infty} \int_{\partial B(0,\varepsilon)} |\Phi(z)| \, dS(z) \).

Since \( |\Phi(z)| = \frac{1}{2\pi} |\log \varepsilon| \text{ const on } \partial B(0, \varepsilon) \) as a radial function, then \( |L_\varepsilon| \leq \| \nabla f \|_{L^\infty} \frac{1}{2\pi} |\log \varepsilon| 2\pi \varepsilon \text{ because } \int_{\partial B(0,\varepsilon)} dS(z) = 2\pi \varepsilon \), the length of the circumference \( \partial B(0, \varepsilon) \).

Applying L'Hopital's rule, we have

\[
\lim_{\varepsilon \to 0^+} \varepsilon |\log \varepsilon| = - \lim_{\varepsilon \to 0^+} \frac{\log \varepsilon}{1/\varepsilon} = \left( \frac{\infty}{\infty} \right) = \lim_{\varepsilon \to 0^+} \frac{\varepsilon - 1}{-\varepsilon^2} = \lim_{\varepsilon \to 0^+} \varepsilon = 0.
\]

So, \( |L_\varepsilon| \to 0 \) as \( \varepsilon \to 0^+ \).

iii. Finally, show that \( K_\varepsilon \to -f(x) \) as \( \varepsilon \to 0^+ \). Perform integration by parts
for $K_\varepsilon$ and use again the fact that $f$ vanishes at infinity.

\[ K_\varepsilon = -\int_{\mathbb{R}^2 \setminus B(0,\varepsilon)} \nabla \Phi(z) \cdot \nabla f(x - z) \, dz = \int_{\mathbb{R}^2 \setminus B(0,\varepsilon)} \Delta \Phi(z) f(x - z) \, dz - \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi(z)}{\partial \nu} f(x - z) \, dS(z). \]

Since $\Phi(z)$ is a solution of the Laplace’s equation in $\mathbb{R}^2 \setminus \{0\}$, then $\Delta \Phi(z) \equiv 0$ in $\mathbb{R}^2 \setminus \{0\}$, and so

\[ K_\varepsilon = -\int_{\partial B(0,\varepsilon)} \frac{\partial \Phi(z)}{\partial \nu} f(x - z) \, dS(z). \]

Now calculate $\frac{\partial \Phi(z)}{\partial \nu}$ on the circumference $\partial B(0,\varepsilon)$.

\[ \frac{\partial \Phi(z)}{\partial \nu} = \nu \cdot \nabla \Phi(z). \]

\[ \frac{\partial \Phi(z)}{\partial z_j} = -\frac{1}{2\pi} \frac{\partial}{\partial z_j} \log |z| = -\frac{1}{2\pi} \frac{1}{|z|} \frac{\partial}{\partial z_j} \left((z_1^2 + z_2^2)^{1/2}\right) = -\frac{1}{2\pi|z|} \frac{1}{2}(z_1^2 + z_2^2)^{-1/2} 2z_j = -\frac{z_j}{2\pi|z|^2}, \quad j = 1, 2. \]

Then $\nabla \Phi(z) = -\frac{1}{2\pi|z|^2} z$.

Since the vector $z$ is an outward pointing vector normal to the circumference $\partial B(0,\varepsilon)$, then $\nu = -\frac{z}{|z|}$ is an inward pointing vector normal to the $\partial B(0,\varepsilon)$.

So, $\frac{\partial \Phi(z)}{\partial \nu} = \left(-\frac{z}{|z|}\right)\left(-\frac{z}{2\pi|z|^2}\right) = \frac{|z|^2}{2\pi|z|^2} = \frac{1}{2\pi|z|^2} = \frac{1}{2\pi\varepsilon}$ on $B(0,\varepsilon)$.

Therefore $K_\varepsilon = -\frac{1}{2\pi\varepsilon} \int_{\partial B(0,\varepsilon)} f(x - z) \, dS(z)$.

The integrand $f$ is continuous on $\partial B(0,\varepsilon)$. Apply the Mean Value theorem to get

\[ K_\varepsilon = -\frac{1}{(2\pi\varepsilon)} f(x - z^*) (2\pi\varepsilon) = f(x - z^*), \]

where $z^*$ is some point on the $\partial B(0,\varepsilon)$.

Note that a choice of the constant for $\Phi(x)$ in the equation (2.16) is explained by the necessity of cancellation by $2\pi$ to obtain $-f(x)$ as $\varepsilon \rightarrow 0^+$.

Exploiting the continuity of $f$ once more, we attain that $K_\varepsilon \rightarrow -f(x)$ as $\varepsilon \rightarrow 0^+$ because $z^* \rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

Thus, $\Delta u(x) = -f(x)$.

We summarize both lemmas in the following theorem.
Theorem 2.3. The fundamental solution of the Laplace’s equation in \( \mathbb{R}^2 \setminus \{0\} \) is

\[
\Phi(x) = \begin{cases} 
\frac{-1}{2\pi} \log |x|, & \text{if } N = 2, \\
\frac{1}{2\pi^{N/2}(N-2)} \frac{1}{|x|^{N-2}}, & \text{if } N \geq 3.
\end{cases}
\] (2.15-2.16)

If \( f \in C_0^2(\mathbb{R}^N) \), then there exists a solution \( u \in C^2(\mathbb{R}^N) \) to the Poisson equation \(-\Delta u = f\) in \( \mathbb{R}^N \) given by the convolution \( u(x) = (\Phi \ast f)(x) \).

In the next section, we introduce Sobolev spaces we will need in the chapter 3.

2.4 Calculus Inequalities in the Sobolev Spaces.

Here we review some useful properties of Sobolev spaces.

Definition 2.2. Let \( p \in [1, \infty) \) and \( k \in \mathbb{Z}^+ \cup \{0\} \).

The Sobolev space \( W^{k,p}(\mathbb{R}^N) := \{ f \in L^p(\mathbb{R}^N) \mid D^\alpha f \in L^p(\mathbb{R}^N) \text{ for } |\alpha| \leq k \} \)

where \( D^\alpha \) is the weak derivative. \( D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_N^{\alpha_N}, D^0 f = f, |\alpha| = \sum_{j=1}^N \alpha_j. \)

The norm \( \|f\|_{W^{k,p}(\mathbb{R}^N)} := \left( \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|^p_{L^p(\mathbb{R}^N)} \right)^{1/p}. \)

In particular, for the case \( p = 2 \), we use notation \( H^k(\mathbb{R}^N) := W^{k,2}(\mathbb{R}^N) \), and so the norm

\[
\|f\|_{H^k(\mathbb{R}^N)} := \left( \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|^2_{L^2(\mathbb{R}^N)} \right)^{1/2}. \] (2.17)

We will use the following notation for the derivatives of \( k \)-th order:

\[ D^k f = \{ D^\alpha f \mid |\alpha| = k \}; \quad \|D^k f\|_{L^p(\mathbb{R}^N)} = \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p(\mathbb{R}^N)}. \] (2.18)

Together with the norm defined by the equation (2.17), we will use else two other norms defined by

\[
\|f\|_{H^k(\mathbb{R}^N)} = \left[ \int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^k \, d\xi \right]^{1/2}.
\] (2.19)

where \( \hat{f}(\xi) \) denotes the Fourier transform of \( f(x) \), and

\[
\|f\|_{H^k(\mathbb{R}^N)} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_{L^2(\mathbb{R}^N)}. \] (2.20)

The Sobolev space \( H^k \) generalizes to the case \( k = s \in \mathbb{R} \) with norm defined by the equation
The following lemma justifies an equivalence of all these norms in $H^k(\mathbb{R}^N)$.

**Lemma 2.10.** The norms defined by the equations (2.17), (2.19), and (2.20) are equivalent in $H^k(\mathbb{R}^N), \ k \in \mathbb{Z}^+ \cup \{0\}$.

**Proof.**

1) Prove the equivalency in $H^k(\mathbb{R}^N)$ of the norms defined by the equations (2.17) and (2.19).

(a) Starting with the norm defined by (2.17) and applying Plancherel’s theorem and the property of the Fourier transform (2.11), we have

$$
\|f\|_{H^k(\mathbb{R}^N)}^2 = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_{L^2(\mathbb{R}^N)}^2 = \sum_{0 \leq |\alpha| \leq k} \|\hat{D}\hat{f}\|_{L^2(\mathbb{R}^N)}^2 = \sum_{0 \leq |\alpha| \leq k} \prod_{j=1}^N (2\pi i \xi_j)^{\alpha_j} \|\hat{f}\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} \left( \sum_{0 \leq |\alpha| \leq k} (4\pi^2)^{|\alpha|} |\xi^\alpha|^2 \right) |\hat{f}(\xi)|^2 d\xi
$$

where $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_N^{\alpha_N} = \prod_{j=1}^N \xi_j^{\alpha_j}$.

Hence we have the following estimate:

$$
\int_{\mathbb{R}^N} \left( \sum_{0 \leq |\alpha| \leq k} |\xi^\alpha|^2 \right) |\hat{f}(\xi)|^2 d\xi \leq \|f\|_{H^k(\mathbb{R}^N)}^2 \leq (4\pi^2)^k \int_{\mathbb{R}^N} \left( \sum_{0 \leq |\alpha| \leq k} |\xi^\alpha|^2 \right) |\hat{f}(\xi)|^2 d\xi
$$

Comparing the bounds above with the $H^k$-norm in (2.19), we can see that it is sufficient to show that $\sum_{0 \leq |\alpha| \leq k} |\xi^\alpha|^2$ and $(1 + |\xi|^2)^k$ are comparable.

(b) Show that

$$
\sum_{0 \leq |\alpha| \leq k} |\xi^\alpha|^2 = \sum_{m=0}^k (|\xi|^2)^m.
$$
Indeed,
\[
\sum_{0 \leq |\alpha| \leq k} |\xi^\alpha|^2 = 1 + \sum_{j=1}^{N} \xi_j^2 + \sum_{|\alpha|=2} |\xi^\alpha|^2 + \cdots + \sum_{|\alpha|=k} |\xi^\alpha|^2.
\]
where
\[
\sum_{|\alpha|=2} |\xi^\alpha|^2 = \sum_{j=1}^{N} (\xi_j^2)^2 + 2 \sum_{l \neq j} (\xi_l \xi_j)^2 = \left( \sum_{j=1}^{N} \xi_j^2 \right)^2 = (|\xi|^2)^2.
\]

Here products \(\xi_l \xi_j\) and \(\xi_j \xi_l\) correspond to derivatives \(D_{lj}\) and \(D_{jl}\) respectively. Assume that
\[
\sum_{|\alpha|=k-1} |\xi^\alpha|^2 = (|\xi|^2)^{k-1}
\]
Then
\[
\sum_{|\alpha|=k} |\xi^\alpha|^2 = \sum_{j=1}^{N} \xi_j^2 \left( \sum_{|\alpha|=k-1} |\xi^\alpha|^2 \right) = |\xi|^2 (|\xi|^2)^{k-1} = (|\xi|^2)^k.
\]
So, by mathematical induction,
\[
\sum_{0 \leq |\alpha| \leq k} |\xi^\alpha|^2 = \sum_{m=0}^{k} (|\xi|^2)^m.
\]
(c) Now consider
\[
(1 + |\xi|^2)^k = \sum_{m=0}^{k} \left( \begin{array}{c} k \\ m \end{array} \right) (|\xi|^2)^m
\]
The coefficients \(\left( \begin{array}{c} k \\ m \end{array} \right) \geq 1.\) Then
\[
\sum_{0 \leq |\alpha| \leq k} |\xi^\alpha|^2 = \sum_{m=0}^{k} (|\xi|^2)^m \leq (1 + |\xi|^2)^k.
\]
On the other hand,
\[
(1 + |\xi|^2)^k \leq \left( \begin{array}{c} k \\ \lfloor k/2 \rfloor \end{array} \right) \sum_{0 \leq |\alpha| \leq k} |\xi^\alpha|^2,
\]
where \(\left( \begin{array}{c} k \\ \lfloor k/2 \rfloor \end{array} \right)\) is the biggest of binomial coefficients \(\left( \begin{array}{c} k \\ m \end{array} \right), \ 0 \leq m \leq k.\)
Thus, the \(H^k\)-norm defined by the equations (2.17) and (2.19) are equivalent.
2) Show that the equations (2.17) and (2.20) define equivalent norms in \(H^k(\mathbb{R}^N)\).
(a) Enumerate and denote all \(\|D^\alpha f\|_{L^2(\mathbb{R}^N)}, 0 \leq |\alpha| \leq k\) as \(a_m, 0 \leq m \leq M\) with \(M = \sum_{l=0}^{k} N^l = \frac{N^{k+1}-1}{N-1}\). Then \(\|D^\alpha f\|_{L^2(\mathbb{R}^N)} = \sum_{m=0}^{M} a_m\), and \(\left(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|^2_{L^2(\mathbb{R}^N)}\right)^{1/2} = \left(\sum_{m=0}^{M} a_m^2\right)^{1/2}\) where \(a_m \geq 0, m \in [0, M]\).

It follows that

\[
\sum_{m=0}^{M} a_m^2 \leq \left(\sum_{m=0}^{M} a_m\right)^2 = \sum_{m=0}^{M} a_m^2 + 2 \sum_{m \neq l} a_m a_l.
\]

(b) On the other hand, since \(2a_m a_l \leq a_m^2 + a_l^2\),

\[
\left(\sum_{m=0}^{M} a_m\right)^2 \leq \sum_{m=0}^{M} a_m^2 + \sum_{m \neq l} (a + m^2 + a_l^2) = \sum_{m=0}^{M} a_m^2 + M \sum_{m=0}^{M} a_m^2 = (M + 1) \sum_{m=0}^{M} a_m^2.
\]

So, \(\sum_{m=0}^{M} a_m \leq \sqrt{M + 1} \left(\sum_{m=0}^{M} a_m^2\right)^{1/2}\).

Therefore, the two \(H^k\)-norms defined by the equations (2.17) and (2.20) are equivalent.

\[\square\]

The next lemma asserts that an \(L^2\) bound on a higher derivative implies a pointwise bound on lower derivative.

**Lemma 2.11 (Sobolev Inequality).** If \(s > N/2\), \(k \in \mathbb{Z}^+ \cup \{0\}\), then \(H^{s+k} \subset C^k(\mathbb{R}^N)\), and there exists \(C > 0\) such that

\[
|v|_{C^k(\mathbb{R}^N)} \leq C \|v\|_{H^{s+k}(\mathbb{R}^N)}, \quad \forall v \in H^{s+k}(\mathbb{R}^N).
\]  

(2.21)
Proof.

1) Using the property (2.13) of the Fourier transform and the definition (2.4) of the inverse Fourier transform, we have, \( \forall x \in \mathbb{R}^N, \forall |\alpha| \leq s + k, \)

\[
|D^\alpha v(x)| = |(\widehat{D^\alpha v})(x)| = \left| \int_{\mathbb{R}^N} e^{2\pi ix \cdot \xi} (\widehat{D^\alpha v})(\xi) \, d\xi \right| 
\leq \int_{\mathbb{R}^N} |e^{2\pi ix \cdot \xi}| |(\widehat{D^\alpha v})(\xi)| \, d\xi = \int_{\mathbb{R}^N} |(\widehat{D^\alpha v})(\xi)| \, d\xi = \left\| \widehat{D^\alpha v} \right\|_{L^1(\mathbb{R}^N)}.
\]

So, \( \sup_{x \in \mathbb{R}^N} |D^\alpha v(x)| \leq \left\| \widehat{D^\alpha v} \right\|_{L^1(\mathbb{R}^N)}. \)

2) In proving the previous lemma about an equivalence of the norms in Sobolev space, we got that there exists \( C_1 > 0 \) such that

\[
\sum_{0 \leq |\alpha| \leq k} |\xi^\alpha| \leq C_1 \left( \sum_{0 \leq |\alpha| \leq k} |\xi^\alpha|^2 \right)^{1/2}, \text{ and } \sum_{0 \leq |\alpha| \leq k} |\xi^\alpha|^2 \leq (1 + |\xi|^2)^k
\]

so, \( \sum_{0 \leq |\alpha| \leq k} |\xi^\alpha| \leq C_1 (1 + |\xi|^2)^{k/2}. \)

Then applying the property of the Fourier transform (2.11), the last inequality, and then the Cauchy-Schwartz inequality after that, we have

\[
\sum_{0 \leq |\alpha| \leq k} \left\| \widehat{D^\alpha v} \right\|_{L^1(\mathbb{R}^N)} = \sum_{0 \leq |\alpha| \leq k} \int_{\mathbb{R}^N} |2\pi i|^{\alpha}| \xi^\alpha| \left| \hat{v}(\xi) \right| \, d\xi = \\
\int_{\mathbb{R}^N} \left( \sum_{0 \leq |\alpha| \leq k} (2\pi)^{|\alpha|} |\xi^\alpha| \right) \left| \hat{v}(\xi) \right| \, d\xi \leq (2\pi)^k \int_{\mathbb{R}^N} \left( \sum_{0 \leq |\alpha| \leq k} |\xi^\alpha| \right) \left| \hat{v}(\xi) \right| \, d\xi \leq \\
C_1 (2\pi)^k \int_{\mathbb{R}^N} (1 + |\xi|^2)^{k/2} \left| \hat{v}(\xi) \right| \, d\xi = C_1 (2\pi)^k \int_{\mathbb{R}^N} \frac{(1 + |\xi|^2)^{(s+k)/2}}{(1 + |\xi|^2)^{s/2}} \left| \hat{v}(\xi) \right| \, d\xi \leq \\
C_1 (2\pi)^k \left\| (1 + |\xi|^2)^{(s+k)/2} \hat{v}(\xi) \right\|_{L^2(\mathbb{R}^N)} \left\| (1 + |\xi|^2)^{-s/2} \right\|_{L^2(\mathbb{R}^N)} = \\
C_1 (2\pi)^k \left[ \int_{\mathbb{R}^N} (1 + |\xi|^2)^{s+k} \left| \hat{v}(\xi) \right|^2 \, d\xi \right]^{1/2} \left[ \int_{\mathbb{R}^N} (1 + |\xi|^2)^{-s} \, d\xi \right]^{1/2}. 
\]

So, using the norm in Sobolev space defined by the equation (2.19), we get

\[
\sum_{0 \leq |\alpha| \leq k} \left\| \widehat{D^\alpha v} \right\|_{L^1(\mathbb{R}^N)} \leq C_1 (2\pi)^k \left\| v \right\|_{H^{s+k}(\mathbb{R}^N)} \left[ \int_{\mathbb{R}^N} (1 + |\xi|^2)^{-s} \, d\xi \right]^{1/2}. 
\]
3) Need to show that \( \int_{\mathbb{R}^N} (1 + |\xi|^2)^{-s} \, d\xi < \infty. \)

Turn to spherical coordinates in \( \mathbb{R}^N. \)

\[
\int_{\mathbb{R}^N} (1 + |\xi|^2)^{-s} \, d\xi = \omega_N \int_0^\infty (1 + r^2)^{-s} r^{N-1} \, dr
\]

where \( \omega_N \) is the surface area of a unit ball in \( \mathbb{R}^N. \)

Denote \( C_2 = \int_0^1 (1 + r^2)^{-s} r^{N-1} \, dr < \infty. \)

Then \( \int_{\mathbb{R}^N} (1 + |\xi|^2)^{-s} \, d\xi = \omega_N (C_2 + \int_1^\infty (1 + r^2)^{-s} r^{N-1} \, dr). \)

Now using that \( r^2 \geq 1 \) in the last integral, we have

\[
\int_{\mathbb{R}^N} (1 + |\xi|^2)^{-s} \, d\xi \leq \omega_N (C_2 + \int_1^\infty 2^{-s} r^{-2s+N-1} \, dr).
\]

Since by the hypothesis \( s > \frac{N}{2}, \) then \( -2s < -N, \) and \( -2s + N - 1 < -1. \)

Hence the integral at the right-hand side of the last inequality is convergent and is equal to \( (2s - N)^{-1}. \)

So, \( \int_{\mathbb{R}^N} (1 + |\xi|^2)^{-s} \, d\xi \leq C_3 < \infty \) where \( C_3 = \omega_N (C_2 + \frac{2^{-s}}{2s-N}). \)

4) We obtained

\[
\sum_{0 \leq |\alpha| \leq k} \| \hat{D}^\alpha v \|_{L^1(\mathbb{R}^N)} \leq C \| v \|_{H^{s+k}(\mathbb{R}^N)} \quad \text{where} \quad C = C_1 (2\pi)^k C_3.
\]

Then

\[
\sum_{0 \leq |\alpha| \leq k} \sup_{x \in \mathbb{R}^N} |D^\alpha v(x)| \leq C \| v \|_{H^{s+k}(\mathbb{R}^N)}.
\]

Hence

\[
\sup_{0 \leq |\alpha| \leq k} \sup_{x \in \mathbb{R}^N} |D^\alpha v(x)| = |v|_{C^k(\mathbb{R}^N)} \leq C \| v \|_{H^{s+k}(\mathbb{R}^N)}.
\] (2.21)

Before we consider important calculus inequalities in Sobolev spaces, we will prove several useful propositions known as Gagliardo-Nirenberg estimates, following Chapter 13, Section 3 of [10].
Up to the end of this section, we will assume that all functional spaces are considered over \( \mathbb{R}^N \) without mentioning that explicitly.

**Proposition 2.1.** For real \( k \geq 1, \ 1 \leq p \leq k, \ 1 \leq j \leq N, \) we have

\[
\|D_j u\|_{L^{2k/p}}^2 \leq C \|u\|_{L^2} \|D_j^2 u\|_{L^q} \quad \forall u \in C_0^\infty, \tag{2.22}
\]

hence \( \forall u \in L^q \cap H^q, \) where \( q_1 = \frac{2k}{p+1}, \ q_2 = \frac{2k}{p-1}. \)

**Proof.**

1) Let \( v \in C_0^\infty, \) real \( q \geq 2. \)

Show that \( v|v|^{q-2} \in C_0^1. \)

(a) For \( v \neq 0, \) \( D_j(|v|) = D_j(\sqrt{v^2}) = \frac{2v}{2|v|} D_j v = \text{sign}(v) D_j v. \)

Then \( D_j(v|v|^{q-2}) = D_j v|v|^{q-2} + (q-2)v|v|^{q-3} \frac{v}{|v|} D_j v = |v|^{q-2} D_j v \) for \( v \neq 0. \)

(b) Let \( v(x_0) = 0 \) at some \( x_0 \in \mathbb{R}^N. \) Show that \( D_j(v|v|^{q-2})|_{x=x_0} = 0. \)

\[
\lim_{h \to 0} \frac{(v|v|^{q-2})(x_0 + e_j h) - (v|v|^{q-2})(x_0)}{h} = \frac{v(x_0 + e_j h) - v(x_0)}{h}.
\]

So, \( \forall v \in C_0^\infty, \) \( (v|v|^{q-2}) \in C_0^1. \)

2) Let \( v = D_j u. \) Consider

\[
D_j(u v|v|^{q-2}) = D_j u v|v|^{q-2} + u(q-1)|v|^{q-2} D_j v = |v|^q + u(q-1)|v|^{q-2} D_j v.
\]

Hence \( |v|^q = D_j(u v|v|^{q-2}) - u(q-1)|v|^{q-2} D_j v \)

or \( |D_j u|^q = D_j(u D_j u|D_j u|^{q-2}) - u(q-1)|D_j u|^{q-2} D_j^2 u. \)

3) Integrating the last equality, we have

\[
\|D_j u\|_{L^q}^q = \int_{\mathbb{R}^N} |D_j u|^q \, dx = \int_{\mathbb{R}^N} D_j(u D_j u|D_j u|^{q-2}) \, dx - (q-1) \int_{\mathbb{R}^N} u|D_j u|^{q-2} D_j^2 u \, dx.
\]
The first integral

\[
\int_{\mathbb{R}^N} D_j(u D_j u |D_j u|^{q-2}) \, dx = \int_{\mathbb{R}^{N-1}} (u D_j u |D_j u|^{q-2}) \big|_\infty^\infty \, d^{N-1} x = 0,
\]

since \( u \) vanishes at infinity.

Then applying the Generalized H older’s inequality for 3 factors in the integrand of the second integral, we have

\[
\|D_j u\|_{L^q} \leq |q - 1| \left| \int_{\mathbb{R}^N} u |D_j u|^{q-2} D_{jj}^2 u \, dx \right| \leq |q - 1| \int_{\mathbb{R}^N} |u| |D_j u|^{q-2} D_{jj}^2 u \, dx \leq |q - 1| \|u\|_{L^{q_2}} \|D_{jj}^2 u\|_{L^{q_1}} \|D_j u|^{q-2}\|_{L^{q/(q-2)}}
\]

where \( \frac{1}{q_1} + \frac{1}{q_2} + \frac{q-2}{q} = 1 \) or \( \frac{1}{q_1} + \frac{1}{q_2} = \frac{2}{q} = \frac{p}{k} \) or \( q = \frac{2k}{p} \).

Note that

\[
\|D_j u|^{q-2}\|_{L^{q/(q-2)}} = \left( \int_{\mathbb{R}^N} |D_j u|^{(q-2)\frac{q}{q-2}} \, dx \right)^{1/(q-2)} = \|D_j u\|_{L^q}^{q-2}.
\]

So, for \( q = \frac{2k}{p} \), we obtained

\[
\|D_j u\|_{L^q} \leq |q - 1| \|u\|_{L^{q_2}} \|D_{jj}^2 u\|_{L^{q_1}} \|D_j u|^{q-2}\|_{L^q}.
\]

Hence,

\[
\|D_j u\|_{L^q}^2 \leq C \|u\|_{L^{q_2}} \|D_{jj}^2 u\|_{L^{q_1}} \text{ with } C = \frac{2k}{p} - 1. \quad (2.22)
\]

The next proposition is an application of the estimate (2.22).

**Proposition 2.2.** Let \( l, m, n \in \mathbb{Z}^+ \) and \( k, p \in \mathbb{R} \). If \( n \leq p \leq k + 1 - m \) and \( l \geq n \), then for sufficiently small \( \varepsilon > 0 \)

\[
\|D^l u\|_{L^{2k/p}} \leq C \varepsilon \|D^{l-n} u\|_{L^{2k/(p-n)}} + C(\varepsilon) \|D^{l+m} u\|_{L^{2k/(p+m)}}. \quad (2.23)
\]

**Proof.** Let \( k, p \in \mathbb{R} \), \( k \geq 1 \), \( 1 \leq p \leq k \).

1) Substitute \( D^{l-1} u \) instead of \( u \) in the inequality (2.22) and take the square root of both sides of the inequality

\[
\|D_j D^{l-1} u\|_{L^{2k/p}} \leq \sqrt{C} \|D^{l-1} u\|_{L^{2k/(p-1)}}^{1/2} \|D_{jj}^2 D^{l-1} u\|_{L^{2k(p+1)}}^{1/2}.
\]
Summing on $j$ and recalling (2.18), we have

$$
\sum_{j=1}^{N} \sum_{|\alpha| = l-1} \| D_j D^\alpha u \|_{L^{2k/p}} \leq
\sqrt{C} \sum_{j=1}^{N} \left( \sum_{|\alpha| = l-1} \| D^\alpha u \|_{L^{2k/(p-1)}} \right)^{1/2} \left( \sum_{|\alpha| = l-1} \| D_{jj}^2 D^\alpha u \|_{L^{2k/(p+1)}} \right)^{1/2}.
$$

Note that the left-hand side of the last inequality is

$$
\sum_{|\beta| = l} \| D^\beta u \|_{L^{2k/p}} = \| D^l u \|_{L^{2k/p}}.
$$

Denoting

$$
a_j = \left( \sum_{|\alpha| = l-1} \| D^\alpha u \|_{L^{2k/(p-1)}} \right)^{1/2}, \quad b_j = \left( \sum_{|\alpha| = l-1} \| D_{jj}^2 D^\alpha u \|_{L^{2k/(p+1)}} \right)^{1/2},
$$

and applying the Cauchy-Schwartz inequality to the right-hand side of the last inequality, we get

$$
\sum_{j=1}^{N} a_j b_j \leq \left( \sum_{j=1}^{N} a_j^2 \right)^{1/2} \left( \sum_{j=1}^{N} b_j^2 \right)^{1/2}.
$$

Then we obtain

$$
\| D^l u \|_{L^{2k/p}} \leq \sqrt{C} \sqrt{N} \left( \sum_{|\alpha| = l-1} \| D^\alpha u \|_{L^{2k/(p-1)}} \right)^{1/2} \left( \sum_{|\alpha| = l+1} \| D^\alpha u \|_{L^{2k/(p+1)}} \right)^{1/2}.
$$

Rewrite the last factor on the right-hand side by adding the norms of all mixed derivatives of the second order of $D^\alpha u$, $|\alpha| = l - 1$, to obtain

$$
\| D^l u \|_{L^{2k/p}} \leq \sqrt{C} \sqrt{N} \left( \sum_{|\alpha| = l} \| D^\alpha u \|_{L^{2k/(p-1)}} \right)^{1/2} \left( \sum_{|\alpha| = l+1} \| D^\alpha u \|_{L^{2k/(p+1)}} \right)^{1/2}
$$
or

$$
\| D^l u \|_{L^{2k/p}} \leq \sqrt{C} \sqrt{N} \| D^{l-1} u \|_{L^{2k/(p-1)}} \| D^{l+1} u \|_{L^{2k/(p+1)}}^{1/2}.
$$
After squaring, we have
\[
\|D^l u\|^2_{L^{2k/p}} \leq CN \|D^{l-1} u\|_{L^{2k/(p-1)}} \|D^{l+1} u\|_{L^{2k/(p+1)}}. \tag{2.24}
\]

Return to the previous inequality and apply the Cauchy inequality with \(\varepsilon\):
\[
abla \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}, \quad a, b \in \mathbb{R}, \quad \varepsilon > 0.
\]

We get
\[
\|D^l u\|_{L^{2k/p}} \leq C_1 \varepsilon \|D^{l-1} u\|_{L^{2k/(p-1)}} + C(\varepsilon) \|D^{l+1} u\|_{L^{2k/(p+1)}}, \quad \forall \varepsilon > 0 \tag{2.25}
\]
where \(C_1 = \sqrt{CN}\) and \(C(\varepsilon) = \frac{C_1}{\varepsilon}\).

2) Take \(l \geq 2\) and \(2 \leq p \leq k\). Using in the inequality (2.25), \((p-1)\) and \((l-1)\) instead of \(p\) and \(l\) correspondingly, we get
\[
\|D^{l-1} u\|_{L^{2k/(p-1)}} \leq C_1 \varepsilon_1 \|D^{l-2} u\|_{L^{2k/(p-2)}} + C(\varepsilon_1) \|D^l u\|_{L^{2k/p}}, \quad \varepsilon_1 > 0. \tag{2.26}
\]

Now replace \(\|D^{l-1} u\|_{L^{2k/(p-1)}}\) in the inequality (2.25) by the right-hand side of the inequality (2.26).
\[
\|D^l u\|_{L^{2k/p}} \leq C_1^2 \varepsilon_1 \|D^{l-2} u\|_{L^{2k/(p-2)}} + C_1(\varepsilon_1) \|D^l u\|_{L^{2k/p}} + C(\varepsilon) \|D^{l+1} u\|_{L^{2k/(p+1)}}
\]
or
\[
[1 - C_1 \varepsilon C(\varepsilon_1)] \|D^l u\|_{L^{2k/p}} \leq C_1^2 \varepsilon_1 \|D^{l-2} u\|_{L^{2k/(p-2)}} + C(\varepsilon) \|D^{l+1} u\|_{L^{2k/(p+1)}}.
\]

Fix \(\varepsilon_1\) and pick \(\varepsilon\) so small that
\[
C_1 \varepsilon C(\varepsilon_1) \leq \frac{1}{2}.
\]
Then
\[
\|D^l u\|_{L^{2k/p}} \leq C_2 \varepsilon \|D^{l-2} u\|_{L^{2k/(p-2)}} + C_1(\varepsilon) \|D^{l+1} u\|_{L^{2k/(p+1)}} \tag{2.27}
\]
where \(C_2 = 2C_1^2 \varepsilon_1\) and \(C_1(\varepsilon) = 2C(\varepsilon)\).

3) Repeat this procedure. Take \(l \geq 3, \ 3 \leq p \leq k\) and use \((p-1)\) and \((l-1)\) instead of \(p\) and \(l\) correspondingly in the inequality (2.27).
\[
\|D^{l-1} u\|_{L^{2k/(p-1)}} \leq C_2 \varepsilon_2 \|D^{l-3} u\|_{L^{2k/(p-3)}} + C_1(\varepsilon_2) \|D^l u\|_{L^{2k/p}}, \quad \varepsilon_2 > 0.
\]

Replace \(\|D^{l-1} u\|_{L^{2k/(p-1)}}\) in the inequality (2.25) by the right-hand side of the last inequality to obtain
\[
\|D^l u\|_{L^{2k/p}} \leq C_1 C_2 \varepsilon_2 \|D^{l-3} u\|_{L^{2k/(p-3)}} + C_1 \varepsilon C(\varepsilon_2) \|D^l u\|_{L^{2k/p}} + C(\varepsilon) \|D^{l+1} u\|_{L^{2k/(p+1)}}.
\]
Fix $\varepsilon_2$ and pick $\varepsilon$ so small that $C_1\varepsilon C_1(\varepsilon_2) \leq \frac{1}{2}$. Then

$$\|D^l u\|_{L^{2k/p}} \leq C_3 \varepsilon \|D^{l-3} u\|_{L^{2k/(p-3)}} + C_2(\varepsilon) \|D^{l+1} u\|_{L^{2k/(p+1)}}$$

where $C_3 = 2C_1C_2\varepsilon_2$ and $C_2(\varepsilon) = 2C_1(\varepsilon)$.

4) Assume that $l \geq n$ and $n \leq p \leq k$.
Continue to repeat the procedure described above. After repeating $(n-1)$ times, we will have

$$\|D^l u\|_{L^{2k/p}} \leq C \varepsilon \|D^{l-n} u\|_{L^{2k/(p-n)}} + C(\varepsilon) \|D^{l+1} u\|_{L^{2k/(p+1)}}, \quad \forall \varepsilon > 0. \quad (2.28)$$

5) Now we will work on the second term on the right-hand side of the inequality (2.28). Let $l \geq n$, $n \leq p \leq k-1$. Apply the inequality (2.25) using now $(p+1)$ and $(l+1)$ instead of $p$ and $l$ correspondingly. We get

$$\|D^{l+1} u\|_{L^{2k/(p+1)}} \leq C_1\varepsilon' \|D^l u\|_{L^{2k/p}} + C(\varepsilon') \|D^{l+2} u\|_{L^{2k/(p+2)}}, \quad \forall \varepsilon' > 0 \quad (*)$$

Substituting the right-hand side of this inequality into the inequality (2.28) for $\|D^{l+1} u\|_{L^{2k/(p+1)}}$, we have

$$\|D^l u\|_{L^{2k/p}} \leq C \varepsilon \|D^{l-n} u\|_{L^{2k/(p-n)}} + C(\varepsilon)C_1\varepsilon' \|D^l u\|_{L^{2k/p}} + C(\varepsilon)C(\varepsilon') \|D^{l+2} u\|_{L^{2k/(p+2)}}.$$  

Fix $\varepsilon'$ and pick $\varepsilon$ so small that $C(\varepsilon)C_1\varepsilon' \leq \frac{1}{2}$. Then

$$\|D^l u\|_{L^{2k/p}} \leq C' \varepsilon \|D^{l-n} u\|_{L^{2k/(p-n)}} + C'(\varepsilon) \|D^{l+2} u\|_{L^{2k/(p+2)}}, \quad (**)$$

where $C' = 2C$ and $C'(\varepsilon) = 2C(\varepsilon)C_1\varepsilon'$.

6) Take $l \geq n$, $n \leq p \leq k-2$ and use now $(p+2)$ and $(l+2)$ instead of $p$ and $l$ correspondingly in the inequality (2.25).

$$\|D^{l+2} u\|_{L^{2k/(p+2)}} \leq C \varepsilon'' \|D^{l+1} u\|_{L^{2k/(p+1)}} + C(\varepsilon'') \|D^{l+3} u\|_{L^{2k/(p+3)}}, \quad \forall \varepsilon'' > 0.$$  

In the last inequality replace $\|D^{l+1} u\|_{L^{2k/(p+1)}}$ by the right-hand side of the inequality (*).

$$\|D^{l+2} u\|_{L^{2k/(p+2)}} \leq C C_1 \varepsilon'' \varepsilon''' \|D^l u\|_{L^{2k/p}} + C \varepsilon'' C(\varepsilon') \|D^{l+2} u\|_{L^{2k/(p+2)}} + C(\varepsilon'') \|D^{l+3} u\|_{L^{2k/(p+3)}}.$$  

Fix $\varepsilon'$ and pick $\varepsilon''$ so small that $C\varepsilon'' C(\varepsilon') \leq \frac{1}{2}$. Then

$$\|D^{l+2} u\|_{L^{2k/(p+2)}} \leq C'' \varepsilon'' \varepsilon''' \|D^l u\|_{L^{2k/p}} + C''(\varepsilon'') \|D^{l+3} u\|_{L^{2k/(p+3)}}.$$
where $C'' = 2CC_1\varepsilon'$ and $C''(\varepsilon'') = 2C(\varepsilon'')$.

Use this inequality to replace $\|D^{l+2}u\|_{L^{2k/(p+2)}}$ in the inequality (**),

$$\|D^l u\|_{L^{2k/p}} \leq C'\varepsilon \|D^{l-n}u\|_{L^{2k/(p-n)}} + C'(\varepsilon)C''(\varepsilon') \|D^{l+3}u\|_{L^{2k/(p+3)}}.$$  

After that, fix $\varepsilon''$ and pick $\varepsilon$ so small that $C'(\varepsilon)C''(\varepsilon') \leq \frac{1}{2}$. Then

$$\|D^l u\|_{L^{2k/p}} \leq C''\varepsilon \|D^{l-n}u\|_{L^{2k/(p-n)}} + C''(\varepsilon) \|D^{l+3}u\|_{L^{2k/(p+3)}}.$$  

where $C'' = 4C$ and $C''(\varepsilon) = 8C(\varepsilon)C_1\varepsilon'C(\varepsilon'')$.

7) Assume $n \leq p \leq k - (m - 1)$, $l \geq n$. Repeat $(m - 1)$ times the procedure described in part 6). For sufficiently small $\varepsilon$, we will obtain the following inequality:

$$\|D^l u\|_{L^{2k/p}} \leq C\varepsilon \|D^{l-n}u\|_{L^{2k/(p-n)}} + C(\varepsilon) \|D^{l+m}u\|_{L^{2k/(p+m)}}. \quad (2.23)$$

We underline a special case of Proposition 2.2 in the following corollary.

**Corollary 2.1.** If positive integers are satisfying $l \leq p \leq k - 1$, then

$$\|D^l u\|_{L^{2k/p}} \leq C\varepsilon \|u\|_{L^{2k/(p-l)}} + C(\varepsilon) \|D^{l-k-p}u\|_{L^2} \quad (2.29)$$

In particular, if $p = l < k$, then

$$\|D^l u\|_{L^{2k/p}} \leq C\varepsilon \|u\|_{L^\infty} + C(\varepsilon) \|D^k u\|_{L^2}, \quad (2.30)$$

for all $u \in C_0^\infty$.

**Proof.** If $l = n$, $l \leq p \leq k - (m - 1)$, then the inequality (2.23) gives for sufficiently small $\varepsilon > 0$

$$\|D^l u\|_{L^{2k/p}} \leq C\varepsilon \|u\|_{L^{2k/(p-l)}} + C(\varepsilon) \|D^{l-m}u\|_{L^{2k/(p+m)}} \quad (2.31)$$

Hence for $p = k - m$ or $m = k - p$, we obtain

$$\|D^l u\|_{L^{2k/p}} \leq C\varepsilon \|u\|_{L^{2k/(p-l)}} + C(\varepsilon) \|D^{l-k-p}u\|_{L^2}. \quad (2.29)$$

The inequality (2.30) follows for $p = l$. \qed
Next, we need two propositions to estimate \( \| D^j u \|_{L^{2k/p}} \) by a product that is often more convenient than a sum like in the inequality (2.29).

**Proposition 2.3.** Let \( l, m, \mu \) be nonnegative integers satisfying \( l \leq \max(\mu, m) \), and \( p_1, q, r \in [1, \infty] \). Suppose the estimate

\[
\| D^l u \|_{L^q} \leq C_1 \| D^\mu u \|_{L^r} + C_2 \| D^m u \|_{L^{p_1}}
\]  

(2.32)

is valid \( \forall u \in C_0^\infty \). Then

\[
\| D^l u \|_{L^q} \leq (C_1 + C_2) \| D^\mu u \|_{L^r}^{\beta/(\alpha + \beta)} \| D^m u \|_{L^{p_1}}^{\alpha/(\alpha + \beta)}
\]  

with \( \alpha = \frac{N}{q} - \frac{N}{r} + \mu - l \), \( \beta = -\frac{N}{q} + \frac{N}{p_1} - m + l \).  

(2.33)

provided \( \alpha, \beta \neq 0 \). In this case, \( \alpha \) and \( \beta \) have the same sign.

**Proof.** Use the following notations:

\[
Q = \| D^l u \|_{L^q}, \quad R = \| D^\mu u \|_{L^r}, \quad P = \| D^m u \|_{L^{p_1}}.
\]

So, the inequality (2.32) is

\[
Q \leq C_1 R + C_2 P.
\]  

(2.32a)

Replace \( u(x) \) in the inequality (2.32) by \( u(sx) \) for some \( s > 0 \).

\[
Q = \| D^l u \|_{L^q} = \sum_{|\gamma| = l} \| D^\gamma u \|_{L^q} = \sum_{|\gamma| = l} \left( \int_{\mathbb{R}^N} |D^\gamma u(sx)|^q \, dx \right)^{1/q}.
\]

Change to the variable \( t = sx \). So \( dt = sdx \) and \( dt = S^N \, dx \).

\[
D^\gamma u(sx) = D^\gamma u(t) \left( \frac{dt}{dx} \right)^\gamma = D^\gamma u(t) s^{|\gamma|} = D^\gamma u(t) s^l.
\]

Then

\[
Q = \sum_{|\gamma| = l} \left( \int_{\mathbb{R}^N} |D^\gamma u(t)|^q s^{|\gamma| - N} \, dt \right)^{1/q} = s^{l-(N/q)} \sum_{|\gamma| = l} \left( \int_{\mathbb{R}^N} |D^\gamma u(t)|^q \, dt \right)^{1/q} =
\]

\[
s^{l-(N/q)} \sum_{|\gamma| = l} \| D^\gamma u \|_{L^q} = s^{l-(N/q)} \| D^l u \|_{L^q} = s^{l-(N/q)} Q.
\]

Similarly, we get \( R = s^{\mu-(N/r)} R \) and \( P = s^{m-(N/p_1)} P \).
Thus, we got the estimate (2.33).

\[ s^{1-(N/q)} Q \leq C_1 s^{u-(N/r)} R + C_2 s^{m-(N/p_1)} P, \quad \forall s > 0 \]

or

\[ Q \leq C_1 s^{u-(N/r)-1+(N/q)} R + C_2 s^{m-(N/p_1)-1+(N/q)} P = C_1 s^{\alpha} R + C_2 S^{-\beta} P. \]

If \( \alpha > 0 \) and \( \beta < 0 \), then for \( s \to 0^+ \), the last inequality implies \( Q = 0 \). If \( \alpha < 0 \) and \( \beta > 0 \), then when \( s \to \infty \), we get \( Q = 0 \) as well. However, \( Q \neq 0 \), \( \forall u \in C_{0}^{\infty} \). So, \( \alpha \) and \( \beta \) must have the same sign.

Now choose \( s \) such that \( s^{\alpha} R = s^{-\beta} P \), i.e., \( s^{\alpha+\beta} = \frac{P}{R} \) or \( s = \left( \frac{P}{R} \right)^{1/(\alpha+\beta)} \).

Then \( s^{-\beta} P = s^{\alpha} R = \left( \frac{P}{R} \right)^{\alpha/(\alpha+\beta)} R = P^{\alpha/(\alpha+\beta)} R^{1-[\alpha/(\alpha+\beta)]} = P^{\alpha/(\alpha+\beta)} R^{\beta/(\alpha+\beta)}. \)

So, \( Q \leq (C_1 + C_2) P^{\alpha/(\alpha+\beta)} R^{\beta/(\alpha+\beta)}. \)

Thus, we got the estimate (2.33). \( \square \)

**Proposition 2.4.** If \( l, p, \) and \( k \) are positive integers satisfying \( l \leq p \leq k - 1 \), then

\[ \left\| D^l u \right\|_{L^{2k/p}} \leq C \left\| u \right\|_{L^{(k-p)/(k+l-p)}} \left\| D^{k+l-p} u \right\|_{L^2}^{l/(k+l-p)} . \tag{2.35} \]

In particular, if \( p = l < k \), then

\[ \left\| D^l u \right\|_{L^{2k/l}} \leq C \left\| u \right\|_{L^{(p/k)}} \left\| D^k u \right\|_{L^2}^{l/k}) . \tag{2.36} \]

**Proof.** Apply the Proposition 2.3 to the estimate (2.29). Comparing that estimate with the hypothesis (2.32), we take

\[ \mu = 0, \quad m = k + l - p, \quad q = \frac{2k}{p}, \quad r = \frac{2k}{p-1}, \quad p_1 = 2. \]

Then the relations (2.34) give

\[ \alpha = \frac{Np}{2k} - \frac{N(l-p)}{2k} + 0 - l = l \left( \frac{N}{2k} - 1 \right) = \frac{l(N-2k)}{2k}, \]

and \( \beta = -\frac{Np}{2k} + \frac{N}{2} - k - l + p + l = -\frac{Np + Nk + (p-k)2k}{2k} = \frac{(N-2k)(l-p)}{2k}. \)

Hence \( \alpha + \beta = \frac{N-2k}{2k} (l + k - p) \), \( \frac{\beta}{\alpha+\beta} = \frac{k-p}{l+k-p} \), and \( \frac{\alpha}{\alpha+\beta} = \frac{l}{l+k-p} \).

So, the estimate (2.33) implies

\[ \left\| D^l u \right\|_{L^{2k/p}} \leq C \left\| u \right\|_{L^{2k/(p-1)}} \left\| D^{k+l-p} u \right\|_{L^2}^{l/(k+l-p)} . \tag{2.35} \]

For \( p = l \), this inequality modifies to (2.36). \( \square \)

To prove the next proposition, we need to consider the following small lemma.
Lemma 2.12. If $k, l, m \in \mathbb{Z}^+$ and $k = l + m$, then $\forall a, b \geq 0, \exists C \in (0, 1)$ such that
\[ a^{m/k} b^{l/k} \leq C(a + b). \]

Proof. Let $0 \leq m \leq k$ and $l = k - m$. Keeping only one term in the binomial formula, we have
\[ (a + b)^k \geq \binom{k}{m} a^m b^l. \]
Raising both sides of this inequality to the power $1/k$, we get
\[ a + b \geq \left( \frac{k}{m} \right)^{1/k} a^{m/k} b^{l/k}. \]
So,
\[ a^{m/k} b^{l/k} \leq C(a + b) \quad \text{with} \quad C = \left( \frac{k}{m} \right)^{-1/k} < 1. \]

Proposition 2.5. If $|\beta| + |\gamma| = k$, then
\[ \| (D^\beta f)(D^\gamma g) \|_{L^2} \leq C \| f \|_{L^\infty} \| D^k g \|_{L^2} + C \| D^k f \|_{L^2} \| g \|_{L^\infty}, \quad \forall f, g \in C_0 \cap H^k. \quad (2.37) \]

Proof. Let $|\beta| = l, |\gamma| = m$, and so $l + m = k$. Then $\frac{l}{2k} + \frac{m}{2k} = \frac{1}{2}$, and using the Hölder’s inequality, we have
\[ \| (D^\beta f)(D^\gamma g) \|_{L^2} \leq \| D^\beta f \|_{L^{2k/l}} \| D^\gamma g \|_{L^{2k/m}}. \]
Apply the estimate (2.36) for each factor of the last inequality to find
\[ \| (D^\beta f)(D^\gamma g) \|_{L^2} \leq \left( C_1 \| f \|_{L^\infty} \| D^k f \|_{L^2}^{l/k} \right) \left( C_2 \| g \|_{L^\infty} \| D^k g \|_{L^2}^{m/k} \right) = C_1 C_2 \left( \| f \|_{L^\infty} \| D^k g \|_{L^2} \right)^{m/k} \left( \| D^k f \|_{L^2} \| g \|_{L^\infty} \right)^{l/k}. \]
Now use Lemma 2.12. We obtain
\[ \| (D^\beta f)(D^\gamma g) \|_{L^2} \leq C \| f \|_{L^\infty} \| D^k g \|_{L^2} + C \| D^k f \|_{L^2} \| g \|_{L^\infty} \quad (2.37) \]
with $C = C_1 C_2 C_3$ where $C_1$ and $C_2$ are the constants from the estimate (2.36), and $C_3$ is the constant from the Lemma 2.12. \qed

Beginning from this point and thereafter, we will use an abbreviated notation $\| u \|_k$ instead
of $\|u\|_{H^k(\mathbb{R}^N)}$. So, $\|u\|_0$ means $\|u\|_{L^2(\mathbb{R}^N)}$.

The following theorem is an application of the Gagliardo-Nirenberg estimates proved in the Propositions 2.1-2.5.

**Theorem 2.4 (Calculus Inequalities in the Sobolev Space).**

(i) For all $m \in \mathbb{Z}^+ \cup \{0\}$, there exists $C > 0$ such that, for all $u, v \in L^\infty(\mathbb{R}^N) \cap H^m(\mathbb{R}^N)$,

$$\|uv\|_m \leq C(\|u\|_{L^\infty} \|v\|_m + \|u\|_m \|v\|_{L^\infty}),$$  \hspace{1cm} (2.38)

$$\sum_{0 \leq |\alpha| \leq m} \|D^\alpha(uv) - uD^\alpha v\|_0 \leq C(\|\nabla u\|_{L^\infty} \|v\|_{m-1} + \|u\|_m \|v\|_{L^\infty}),$$  \hspace{1cm} (2.39)

(ii) For all $s > N/2$, $H^s(\mathbb{R}^N)$ is a Banach algebra. That is, there exists $C > 0$ such that, for all $u, v \in H^s(\mathbb{R}^N)$,

$$\|uv\|_s \leq C \|u\|_s \|v\|_s.$$  \hspace{1cm} (2.40)

**Proof.**

(i)

(a) Let $\alpha, \beta, \gamma$ be multi-indices, and $|\beta| + |\gamma| = |\alpha| = m$.

By definition

$$\|D^m(uv)\|_0 := \sum_{|\alpha| = m} \|D^\alpha(uv)\|_0.$$

Use the Leibnitz’s formula in $\mathbb{R}^N$, so that

$$\|D^m(uv)\|_0 = \sum_{|\alpha| = m} \sum_{\beta + \gamma = \alpha} \binom{\alpha}{\beta}(D^\beta u)(D^\gamma v) \|_0 \text{ with } \binom{\alpha}{\beta} = \prod_{j=1}^N \binom{\alpha_j}{\beta_j}.$$

Apply the triangle-inequality and the estimate (2.37) after that to obtain

$$\|D^m(uv)\|_0 \leq \sum_{|\alpha| = m} \sum_{|\beta| \leq |\alpha|} \binom{\alpha}{\beta} \|D^\beta u\|L^\infty \|D^\gamma v\| L^\infty \leq$$

$$\sum_{|\alpha| = m} \sum_{|\beta| \leq |\alpha|} \binom{\alpha}{\beta} C_\beta \left(\|u\|_{L^\infty} \|D^{\alpha} v\|_0 + \|D^{\alpha} u\|_0 \|v\|_{L^\infty}\right) =$$

$$C_m \left(\|u\|_{L^\infty} \|D^m v\|_0 + \|D^m u\|_0 \|v\|_{L^\infty}\right) \text{ with } C_m = \sum_{|\alpha| = m} \sum_{|\beta| \leq |\alpha|} \binom{\alpha}{\beta} C_\beta.$$
(b) Similarly, we can get the following estimate for $\|D^{m-1}(uv)\|_0$.

$$\|D^{m-1}(uv)\|_0 \leq C_{m-1} \left( \|u\|_{L^\infty} \|D^{m-1}v\|_0 + \|D^{m-1}u\|_0 \|v\|_{L^\infty} \right).$$

Continuing in descending order of $m$, we finally have

$$\|D(uv)\|_0 \leq C_1 \left( \|u\|_{L^\infty} \|Dv\|_0 \right).$$

Also,

$$\|uv\|_0 \leq \|u\|_{L^\infty} \|v\|_0 \leq \|u\|_{L^\infty} \|v\|_0 + \|u\|_0 \|v\|_{L^\infty}.$$ 

Adding all these inequalities, we obtain

$$\|uv\|_m := \sum_{|\alpha| \leq m} \|D^\alpha(uv)\|_0 \leq C \left( \|u\|_{L^\infty} \|v\|_m + \|u\|_m \|v\|_{L^\infty} \right).$$  \hspace{1cm} (2.38)

(c) Prove now the inequality (2.39).

Use again the Leibnitz’s formula in $\mathbb{R}^N$.

$$D^\alpha(uv) = \sum_{\beta + \gamma = \alpha \atop (\beta \geq 0)} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (D^\beta u)(D^\gamma v) = u(D^\alpha v) + \sum_{\beta + \gamma = \alpha \atop (\beta > 0)} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (D^\beta u)(D^\gamma v).$$

Let $e_j$ be a vector with 1 as its $j$-th component and with 0s as all other components, and $j_\beta$ be the number of the first nonzero component of multi-index $\beta$, i.e.,

$$j_\beta = \inf_{i \in I} i \text{ where } I = \{i \in \mathbb{Z}^+ \mid i \leq N \text{ and } \beta_i > 0\}.$$

Denote $\beta' = \beta - e_{j_\beta}$ and $D_{j_\beta}u = u_{j_\beta}$.

Then

$$D^\alpha(uv) - u(D^\alpha v) = \sum_{\beta + \gamma = \alpha \atop (\beta \geq 0)} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) (D^{\beta-e_{j_\beta}}(D_{j_\beta}u)) (D^\gamma v) =$$

$$\sum_{|\beta'|+|\gamma| = |\alpha| - 1} \left( \begin{array}{c} \alpha \\ \beta'+e_{j_\beta} \end{array} \right) (D^{\beta'} u_{j_\beta}) (D^\gamma v).$$

Hence

$$\|D^\alpha(uv) - u(D^\alpha v)\|_0 \leq \sum_{|\beta'|+|\gamma| = |\alpha| - 1} \left( \begin{array}{c} \alpha \\ \beta'+e_{j_\beta} \end{array} \right) \|D^{\beta'} u_{j_\beta}\| (D^\gamma v)\|_0.$$ 

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Apply now the estimate (2.37).

\[ \| D^\alpha (uv) - u(D^\alpha v) \|_0 \leq \sum_{|\beta| \leq |\alpha|} C^{(\alpha)}_{(\beta)} \left( \| u_{j_\beta} \|_{L^\infty} \| D^{|\alpha-1|} v \|_0 + \| D^{\alpha-1} u_{j_\beta} \|_0 \| v \|_{L^\infty} \right). \]

Since

\[ \| u_{j_\beta} \|_{L^\infty} \leq \max_{1 \leq j \leq N} \| u_j \|_{L^\infty} = \| \nabla u \|_{L^\infty} \]

and

\[ \| D^{\alpha-1} u_{j_\beta} \|_0 \leq \| D^{\alpha-1} \nabla u \|_0 = \| D^{\alpha} u \|_0, \]

then

\[ \| D^\alpha (uv) - u(D^\alpha v) \|_0 \leq C_1 \left( \| \nabla u \|_{L^\infty} \| D^{\alpha-1} v \|_0 + \| D^{\alpha} u \|_0 \| v \|_{L^\infty} \right) \]

where \( C_1 = \sum_{|\beta| \leq |\alpha|} C^{(\alpha)}_{(\beta)}. \)

Now, taking a sum by all \( \alpha \) with \( 0 \leq |\alpha| \leq m \), we have

\[ \sum_{0 \leq |\alpha| \leq m} \| D^\alpha (uv) - u(D^\alpha v) \|_0 \leq C_1 \left( \| \nabla u \|_{L^\infty} \| v \|_{m-1} + \| u \|_{m} \| v \|_{L^\infty} \right). \]

(ii) The calculus inequality (2.40) is an implication of the inequality (2.38).

Let \( s \in \mathbb{Z}^+, s > N/2 \). By the Sobolev inequality (2.21) for \( k = 0 \), there exists such \( C > 0 \) that

\[ |v|_C = \sup_{x \in \mathbb{R}^N} |v(x)| \leq C \|v\|_s \]

Then there exists \( C_1 > 0 \) and \( C_2 > 0 \) such that

\[ \|u\|_{L^\infty} \leq C_1 \|u\|_s \quad \text{and} \quad \|v\|_{L^\infty} \leq C_2 \|v\|_s. \]

Applying these last inequalities to enforce the calculus inequality (2.38), we obtain

\[ \|uv\|_s \leq C \left( C_1 \|u\|_s \|v\|_s + C_2 \|u\|_s \|v\|_s \right) = C_3 \|u\|_s \|v\|_s \quad (2.40) \]

where \( C_3 = C \left( C_1 + C_2 \right). \)
These calculus inequalities conclude our review of properties of Sobolev space.

2.5 Properties of the Mollifiers.

Energy methods we will apply in the chapter 3, involves regularization of equations. As a regularizing tool, we will use a mollifier.

**Definition 2.3.** Given any radial function

$$\rho(|x|) \in C^\infty_0(\mathbb{R}^N), \quad \rho \geq 0, \quad \int_{\mathbb{R}^N} \rho \, dx = 1. \tag{2.41}$$

The mollification $J_\varepsilon v$ of functions $v \in L^p(\mathbb{R}^N), \ 1 \leq p \leq \infty$ is defined by

$$\left( J_\varepsilon v \right)(x) = \varepsilon^{-N} \int_{\mathbb{R}^N} \rho\left( \frac{x-y}{\varepsilon} \right) v(y) \, dy, \quad \varepsilon > 0. \tag{2.42}$$

Note that since $\rho\left( \frac{x-y}{\varepsilon} \right) \in C^\infty_0(\mathbb{R}^N)$, then by the Leibnitz’s rule

$$D^\alpha \left( J_\varepsilon v \right)(x) = \varepsilon^{-N} \int_{\mathbb{R}^N} D^\alpha \rho\left( \frac{x-y}{\varepsilon} \right) v(y) \, dy$$

for all multi-indices $\alpha$, i.e., $(J_\varepsilon v)(x) \in C^\infty_0(\mathbb{R}^N)$.

**Theorem 2.5 (Properties of Mollifiers).**

(i) For all $v \in C_0(\mathbb{R}^N)$, $J_\varepsilon v \to v$ uniformly on any compact set $\Omega \subset \mathbb{R}^N$ and

$$\|J_\varepsilon v\|_{L^\infty} \leq \|v\|_{L^\infty}. \tag{2.43}$$

(ii) Mollifiers commute with weak derivatives,

$$D^\alpha J_\varepsilon v = J_\varepsilon D^\alpha v, \quad \forall |\alpha| \leq m, \quad v \in H^m(\mathbb{R}^N). \tag{2.44}$$

(iii) For all $u \in L^p(\mathbb{R}^N)$, $v \in L^q(\mathbb{R}^N)$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_{\mathbb{R}^N} (J_\varepsilon u) v \, dx = \int_{\mathbb{R}^N} u(J_\varepsilon v) \, dx, \quad i.e., \quad (J_\varepsilon u, v)_{L^2} = (u, J_\varepsilon v)_{L^2}. \tag{2.45}$$

(iv) For all $v \in H^s(\mathbb{R}^N)$, $J_\varepsilon v$ converges to $v$ in $H^s(\mathbb{R}^N)$, and the rate of convergence in $H^{s-1}$-norm is linear in $\varepsilon$.

$$\lim_{\varepsilon \to 0^+} \| J_\varepsilon v - v \|_s = 0, \tag{2.46}$$

$$\|J_\varepsilon v - v\|_{s-1} \leq C\varepsilon \|v\|_s. \tag{2.47}$$
(v) For all \( v \in H^m(\mathbb{R}^N), \ k \in \mathbb{Z}^+ \cup \{0\}, \) and \( \epsilon > 0, \)

\[
\| J_\epsilon v \|_{m+k} \leq \frac{C_{mk}}{\epsilon^k} \| v \|_m, \tag{2.48}
\]

\[
\| J_\epsilon D^k v \|_{L^\infty} \leq \frac{C_k}{\epsilon^{(N/2)+k}} \| v \|_0. \tag{2.49}
\]

**Proof.**

(i) (a) First, prove estimate (2.43).

\[
|J_\epsilon v| \leq \epsilon^{-N} \int_{\mathbb{R}^N} \rho(\frac{x-y}{\epsilon})|v(y)| \, dy \leq \| v \|_{L^\infty} \epsilon^{-N} \int_{\mathbb{R}^N} \rho(\frac{x}{\epsilon}) \, dy.
\]

Let \( u(y) = \frac{x-y}{\epsilon} \). Then \( du(y) = \frac{(-1)^N}{\epsilon^N} \, dy \).

Hence

\[
\epsilon^{-N} \int_{\mathbb{R}^N} \rho(\frac{x-y}{\epsilon}) \, dy = \epsilon^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \rho(u) \frac{(-1)^N}{\epsilon^N} \, du = \int_{\mathbb{R}^N} \rho(u) \, du = 1.
\]

So, \( |J_\epsilon v| \leq \| v \|_{L^\infty} \).

Then

\[
\| J_\epsilon v \|_{L^\infty} \leq \| v \|_{L^\infty}. \tag{2.43}
\]

(b) Show that \( J_\epsilon v \to v \) uniformly on any compact set \( \Omega \subset \mathbb{R}^N \).

Since \( \rho(|x|) \in C_0^{\infty}(\mathbb{R}^N) \), then \( \exists \epsilon > 0 \) such that \( \rho(\frac{x}{\epsilon}) = 0 \) for \( |x| > \epsilon \).

Given compact \( \Omega \subset \mathbb{R}^N \). Define \( \Omega_\epsilon := \{ x \in \mathbb{R}^N : \text{dist}(x, \Omega) \leq \epsilon \} \). Then \( \Omega_\epsilon \) is compact as well. Hence \( v \) is uniformly continuous on \( \Omega_\epsilon \):

\[
\forall \epsilon' > 0, \ \exists \delta > 0 \text{ such that } \forall x, y \in \Omega_\epsilon, \ |x - y| \leq \delta \Rightarrow |v(x) - v(y)| < \epsilon'.
\]

Now for \( x \in \Omega \) and for \( \epsilon > \delta \), we have

\[
|(J_\epsilon v)(x) - v(x)| = \left| \epsilon^{-N} \int_{\mathbb{R}^N} \rho(\frac{x-y}{\epsilon})v(y) \, dy - v(x) \right| = 
\]

\[
\left| \epsilon^{-N} \int_{\mathbb{R}^N} \rho(\frac{x-y}{\epsilon})v(y) \, dy - v(x)\epsilon^{-N} \int_{\mathbb{R}^N} \rho(\frac{x-y}{\epsilon}) \, dy \right| = 
\]

\[
\epsilon^{-N} \left| \int_{\mathbb{R}^N} \rho(\frac{x-y}{\epsilon})v(y) \, dy - v(x)\right| \leq \epsilon^{-N} \int_{\mathbb{R}^N} \rho(\frac{x-y}{\epsilon})|v(y) - v(x)| \, dy
\]

\[
< \epsilon^{-N} \epsilon' \int_{\mathbb{R}^N} \rho(\frac{x-y}{\epsilon}) \, dy = \epsilon' \text{ since } \epsilon^{-N} \int_{\mathbb{R}^N} \rho(\frac{x-y}{\epsilon}) \, dy = 1 \text{ by the part (i(a)).}
\]
So, $\forall \epsilon' > 0$, $\exists \epsilon > 0$ such that $\forall x \in \Omega$, $|(J_\epsilon v)(x) - v(x)| < \epsilon'$, i.e., $J_\epsilon v \to v$ uniformly on $\Omega$.

(ii) Apply the Leibnitz rule. Then use the fact that
\[
R \to \rho(\frac{x - y}{\epsilon}) \quad \text{to the difference}
\]
We got the property (2.44).

After that, integrate by parts $|\alpha|$ times taking in count that both $\rho$ and $D^\alpha v$, $0 \leq |\alpha| \leq m$ vanish in infinity.
\[
D^\alpha(J_\epsilon v)(x) = D_\epsilon^\alpha \epsilon^{-N} \int_{\mathbb{R}^N} \rho(\frac{x - y}{\epsilon}) v(y) dy = \epsilon^{-N} \int_{\mathbb{R}^N} D_\epsilon^\alpha [\rho(\frac{x - y}{\epsilon})] v(y) dy =
\]
\[
(-1)^{|\alpha|} \epsilon^{-n} \int_{\mathbb{R}^N} D_y^\alpha [\rho(\frac{x - y}{\epsilon})] v(y) dy = (-\mathcal{X})^{|\alpha|} \epsilon^{-N} (-\mathcal{X})^{|\alpha|} \int_{\mathbb{R}^N} \rho(\frac{x - y}{\epsilon}) D_y^\alpha v(y) dy
\]
\[
= (J_\epsilon D^\alpha v)(x).
\]
We got the property (2.45).

(iii) Let $u \in L^p(\mathbb{R}^N), v \in L^q(\mathbb{R}^N), \text{and } \frac{1}{p} + \frac{1}{q} = 1$.

By the Hölder’s inequality, $\|uv\|_{L^1} \leq \|u\|_{L^p} \|v\|_{L^q}$. Then $(uv) \in L^1(\mathbb{R}^N)$.

So, $(\rho uv) \in L^1(\mathbb{R}^N)$ since $\rho \in C^\infty_0(\mathbb{R}^N)$.

Changing an order of integration in integral over $\mathbb{R}^N \times \mathbb{R}^N$, we have
\[
\int_{\mathbb{R}^N} (J_\epsilon u)(x) v(x) dx = \int_{\mathbb{R}^N} \epsilon^N \left[ \int_{\mathbb{R}^N} \rho(\frac{x - y}{\epsilon}) u(y) dy \right] v(x) dx =
\]
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \epsilon^{-N} \rho(\frac{x - y}{\epsilon}) u(y)v(x) dx dy = \int_{\mathbb{R}^N} u(y) \left[ \epsilon^{-N} \int_{\mathbb{R}^N} \rho(\frac{x - y}{\epsilon}) v(x) dx \right] dy
\]
\[
= \int_{\mathbb{R}^N} u(y) (J_\epsilon v)(y) dy.
\]
We obtained the property (2.45).

(iv) (a) Let $v \in C^s_0(\mathbb{R}^N)$ and let compact $\Omega \supset \cup_{|\alpha| \leq s} \text{spt } D^\alpha v$.

By the property (i), $(J_\epsilon v - v) \to 0$ uniformly on compact $\Omega$ as $\epsilon \to 0^+$.

By the property (ii), $D^\alpha J_\epsilon v = J_\epsilon D^\alpha v$. So, applying the property (i), we get that $(J_\epsilon(D^\alpha v) - D^\alpha v) \to 0$ uniformly on $\Omega$ as $\epsilon \to 0^+$, $\forall \alpha$ satisfying $|\alpha| \leq s$.

Apply now the triangle inequality, the commutation property of mollifiers with derivatives (ii), and the estimate (2.43) to the difference $|D^\alpha(J_\epsilon v) - D^\alpha v|$.

\[
|D^\alpha(J_\epsilon v) - D^\alpha v| \leq |J_\epsilon(D^\alpha v)| + |D^\alpha v| \leq 2\|D^\alpha v\|_{L^\infty}.
\]

Define $g(x) := 2\|D^\alpha v\|_{L^\infty}$. $g(x)$ is integrable on $\mathbb{R}^N$ since
\[
\int_{\mathbb{R}^N} g(x) dx = \int_{\Omega} 2\|D^\alpha v\|_{L^\infty} dx = 2\|D^\alpha v\|_{L^\infty} \text{Vol}(\Omega) < \infty.
\]
So, we can apply the Lebesgue Dominated theorem. Then we get
\[ \forall |\alpha| \leq s, \quad \exists \lim_{\epsilon \to 0} \| J_\epsilon (D^\alpha v) - D^\alpha v \|_0 = 0 \]

or after summation by \( \alpha \), we obtain
\[ \lim_{\epsilon \to 0} \| J_\epsilon v - v \|_{H^s} = 0. \] (2.46)

(b) Since spaces of continuous functions are dense in \( L^2 \), then the equation (2.46) holds \( \forall D^\alpha v \in L^2(\mathbb{R}^N), \forall |\alpha| < s \) or \( \forall v \in H^s(\mathbb{R}^N) \).

(c) For the proof of (2.44), recall that the radial function \( \rho(|x|) \in C_0^\infty(\mathbb{R}^n), \rho \geq 0 \), and \( \int_{\mathbb{R}^N} \rho \, dx = 1 \).

Since \( \rho(|x|) = \rho \left( \left( \sum_{j=1}^N x_j^2 \right)^{1/2} \right) \), then \( \rho \) is an even function of \( x_j, \forall 1 \leq j \leq N \).

(1) The Fourier transform \( \hat{\rho}(\epsilon \xi) = \int_{\mathbb{R}^N} e^{-2\pi i x \cdot \epsilon \xi} \rho(|x|) \, dx \).

Hence \( \hat{\rho}(0) = \int_{\mathbb{R}^N} \rho(|x|) \, dx = 1 \).

Next, show that \( \nabla \hat{\rho}(0) = 0 \).

The j-th component
\[ \left( \nabla \hat{\rho}(\epsilon \xi) \right)_j = \frac{\partial}{\partial \xi_j} \hat{\rho}(\epsilon \xi) = \int_{\mathbb{R}^N} \frac{\partial}{\partial \xi_j} e^{-2\pi i x \cdot \epsilon \xi} \rho(|x|) \, dx = (-2\pi i \epsilon) \int_{\mathbb{R}^N} e^{-2\pi i x \cdot \epsilon \xi} x_j \rho(|x|) \, dx. \]

Then
\[ \left( \nabla \hat{\rho}(0) \right)_j = (-2\pi i \epsilon) \int_{\mathbb{R}^{N-1}} \left( \int_{-\infty}^{\infty} x_j \rho(|x|) \, dx \right) \prod_{k \neq j} dx_k = 0, \]

since \( x_j \rho(|x|) \) is an odd function of \( x_j \) that makes the internal integral over \( \mathbb{R} \) equal to 0.

Using the Taylor extension of the first order at \( \xi = 0 \), we get
\[ \hat{\rho}(\epsilon \xi) = \hat{\rho}(0) + \nabla \hat{\rho}(0) \epsilon \xi + O((\epsilon \xi)^2) = 1 + O(|\epsilon \xi|^2) \]

or
\[ |\hat{\rho}(\epsilon \xi) - 1| \leq C \epsilon^2 |\xi|^2 \] for some constant \( C \).
Estimate the following ratio:

\[
\left| \frac{\hat{\rho}(\epsilon \xi) - 1}{1 + |\xi|^2} \right| \leq \frac{|\hat{\rho}(\epsilon \xi) - 1|^2}{|\xi|^2} \leq C^2 \epsilon^4 |\xi|^4 = C^2 \epsilon^4 |\xi|^2. \quad (*)
\]

Also

\[
|\hat{\rho}(\epsilon \xi)| \leq \int_{\mathbb{R}^N} |e^{-2\pi i \epsilon \xi \cdot x}| \rho(|x|) \, dx = \int_{\mathbb{R}^N} \rho(|x|) \, dx = 1.
\]

Then

\[
|\hat{\rho}(\epsilon \xi) - 1| \leq |\hat{\rho}(\epsilon \xi)| + 1 \leq 2.
\]

Hence

\[
\left| \frac{[\hat{\rho}(\epsilon \xi) - 1]^2}{1 + |\xi|^2} \right| \leq \frac{|\hat{\rho}(\epsilon \xi) - 1|^2}{|\xi|^2} \leq \frac{4}{|\xi|^2}. \quad (**)
\]

For \(|\xi| < \frac{1}{\epsilon}\), by the estimate (\(\ast\)), we have

\[
\left| \frac{[\hat{\rho}(\epsilon \xi) - 1]^2}{1 + |\xi|^2} \right| \leq C^2 \epsilon^4 \frac{1}{\epsilon^2} = C^2 \epsilon^2.
\]

For \(|\xi| \geq \frac{1}{\epsilon}\), the estimate (\(\ast\ast\)) gives

\[
\left| \frac{[\hat{\rho}(\epsilon \xi) - 1]^2}{1 + |\xi|^2} \right| \leq 4 \epsilon^2.
\]

So, \(\forall \xi \in \mathbb{R}^N\), we obtain

\[
\left| \frac{[\hat{\rho}(\epsilon \xi) - 1]^2}{1 + |\xi|^2} \right| \leq C_1 \epsilon^2 \quad \text{with } C_1 = \max\{C^2, 4\}. \quad (***)
\]

(2) Note that mollification of \(v(x)\) is a convolution,

\[
J_\epsilon v(x) = \epsilon^{-N} \int_{\mathbb{R}^N} \rho\left(\frac{x - y}{\epsilon}\right) v(y) \, dy = \rho_\epsilon(x) * v(x), \quad \text{where } \rho_\epsilon(x) = \epsilon^{-N} \rho\left(\frac{x}{\epsilon}\right).
\]

Consider the Fourier transform

\[
\hat{\rho}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i \xi \cdot x} \epsilon^{-N} \rho\left(\frac{x}{\epsilon}\right) \, dx.
\]
Perform the substitution of \( x = \epsilon y \). Then
\[
 dx = \prod_{k=1}^{N} dx_k = \prod_{k=1}^{N} d(\epsilon y_k) = \epsilon^N \prod_{k=1}^{N} dy_k = \epsilon^N dy.
\]

Hence
\[
 \hat{\rho}_\epsilon(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i (\epsilon \xi) \cdot y} \rho(y) dy = \hat{\rho}(\epsilon \xi).
\]

Now, applying the Fourier transform property (2.5) about convolutions, we obtain
\[
 \widehat{J_\epsilon v} (\xi) = \hat{\rho}_\epsilon (\xi) \hat{v} (\xi) = \hat{\rho}(\epsilon \xi) \hat{v}(\xi).
\]

(3) Finally, using the definition of the Sobolev norm (2.19), the last equality, and the bound (***), we get the estimate (2.47).

\[
 \| J_\epsilon v - v \|_{s-1}^2 = \int_{\mathbb{R}^N} (1 + |\xi|^2)^{s-1} \left| \left( \widehat{J_\epsilon v} - v \right)(\xi) \right|^2 d\xi
\]
\[
 = \int_{\mathbb{R}^N} (1 + |\xi|^2)^{s-1} \left| \hat{J_\epsilon v}(\xi) - \hat{v}(\xi) \right|^2 d\xi
\]
\[
 = \int_{\mathbb{R}^N} (1 + |\xi|^2)^{s-1} \left| \hat{\rho}(\epsilon \xi) \hat{v}(\xi) - \hat{v}(\xi) \right|^2 d\xi
\]
\[
 = \int_{\mathbb{R}^N} (1 + |\xi|^2)^{s-1} \left| \hat{\rho}(\epsilon \xi) - 1 \right|^2 \left| \hat{v}(\xi) \right|^2 d\xi
\]
\[
 = \int_{\mathbb{R}^N} \frac{\left| \hat{\rho}(\epsilon \xi) - 1 \right|^2}{1 + |\xi|^2} (1 + |\xi|^2)^s \left| \hat{v}(\xi) \right|^2 d\xi
\]
\[
 \leq C_1 \epsilon^2 \int_{\mathbb{R}^N} (1 + |\xi|^2)^s \left| \hat{v}(\xi) \right|^2 d\xi = C_1 \epsilon^2 \| v \|_s^2.
\]

(v) (a) Prove the important estimate (2.48).

First, note that since \( \rho(x) \in C_0^\infty (\mathbb{R}^N) \), then \( \rho_\beta(x) := D^\beta \rho(x) \in C_0^\infty (\mathbb{R}^N) \).

So, \( \int_{\mathbb{R}^N} |\rho_\beta(x)| dx = C_\beta < \infty \).

Let \( m, k \in \mathbb{Z}^+ \cup \{0\} \), \( \epsilon > 0 \), \( |\alpha| \leq m \), and \( |\beta| \leq k \). Compute

\[
 D^\beta D^\alpha J_\epsilon v(x) = D^\beta D^\alpha \epsilon^{-N} \int_{\mathbb{R}^N} \rho \left( \frac{x-u}{\epsilon} \right) v(y) dy = \epsilon^{-N} D^\alpha \int_{\mathbb{R}^N} D_x^\beta \rho \left( \frac{x-u}{\epsilon} \right) v(y) dy.
\]

Let \( u = \frac{x-u}{\epsilon} \). Then

\[
 D_x^\beta \rho(u) = D_u^\beta \rho(u) (D_x u)^{|\beta|} = \rho_\beta(u) \epsilon^{-|\beta|}.
\]
So,
\[ D^\beta D^\alpha J_\epsilon v(x) = \epsilon^{-|\beta|-N} D^\alpha \int_{\mathbb{R}^N} \rho_\beta \left( \frac{x-y}{\epsilon} \right) v(y) \, dy = \epsilon^{-|\beta|-N} \int_{\mathbb{R}^N} D^\alpha_y \rho_\beta \left( \frac{x-y}{\epsilon} \right) v(y) \, dy. \]

Since \( D^\alpha_y \rho_\beta \left( \frac{x-y}{\epsilon} \right) = (-1)^{|\alpha|} D_y \rho_\beta \left( \frac{x-y}{\epsilon} \right), \) then
\[ D^\beta D^\alpha J_\epsilon v(x) = (-1)^{|\alpha|} \epsilon^{-|\beta|-N} \int_{\mathbb{R}^N} D^\alpha_y \rho_\beta \left( \frac{x-y}{\epsilon} \right) v(y) \, dy. \]

Now integrate by parts \(|\alpha|\) times. We get
\[ D^\beta D^\alpha J_\epsilon v(x) = \epsilon^{-|\beta|-N} \int_{\mathbb{R}^N} \rho_\beta \left( \frac{x-y}{\epsilon} \right) D^\alpha_y v(y) \, dy. \]

Hence
\[
|D^\beta D^\alpha J_\epsilon v(x)|^2 = \epsilon^{-2|\beta|-2N} \left| \int_{\mathbb{R}^N} \rho_\beta \left( \frac{x-y}{\epsilon} \right) D^\alpha_y v(y) \, dy \right|^2 \\
\leq \epsilon^{-2|\beta|-2N} \left[ \int_{\mathbb{R}^N} \left| \rho_\beta \left( \frac{x-y}{\epsilon} \right) \right| \left| D^\alpha_y v(y) \right| \, dy \right]^2 \\
= \epsilon^{-2|\beta|-2N} \left[ \int_{\mathbb{R}^N} \sqrt{\left| \rho_\beta \left( \frac{x-y}{\epsilon} \right) \right|} \sqrt{\left| D^\alpha_y v(y) \right|} \, dy \right]^2.
\]

Applying the Schwartz inequality, we have
\[
|D^\beta D^\alpha J_\epsilon v(x)|^2 \leq \\
e^{-2|\beta|} \left[ \epsilon^{-N} \int_{\mathbb{R}^N} \left| \rho_\beta \left( \frac{x-y}{\epsilon} \right) \right| \, dy \right] \left[ \epsilon^{-N} \int_{\mathbb{R}^N} \left| \rho \left( \frac{x-y}{\epsilon} \right) \right| \left| D^\alpha_y v(y) \right|^2 \, dy \right] \\
\leq C_\beta \epsilon^{-2|\beta|-N} \int_{\mathbb{R}^N} \left| \rho_\beta \left( \frac{x-y}{\epsilon} \right) \right| \left| D^\alpha_y v(y) \right|^2 \, dy
\]

since \( \epsilon^{-N} \int_{\mathbb{R}^N} \left| \rho_\beta \left( \frac{x-y}{\epsilon} \right) \right| \, dy = \int_{\mathbb{R}^N} \left| \rho_\beta (u) \right| \, du = C_\beta \) after substitution \( u = \frac{x-y}{\epsilon}. \)

Estimate now the \( L^2 \)-norm of \( D^\beta D^\alpha J_\epsilon v \) by using the Fubini theorem.
\[
\|D^\beta D^\alpha J_\epsilon v\|_2^2 \leq \frac{C_\beta}{\epsilon^{2|\beta|}} \int_{\mathbb{R}^N} \epsilon^{-N} \left( \int_{\mathbb{R}^N} \left| \rho_\beta \left( \frac{x-y}{\epsilon} \right) \right| \left| D^\alpha_y v(y) \right|^2 \, dy \right) \, dx = \\
\frac{C_\beta}{\epsilon^{2|\beta|}} \int_{\mathbb{R}^N} \left| D^\alpha_y v(y) \right|^2 \left( \int_{\mathbb{R}^N} \epsilon^{-N} \left| \rho_\beta \left( \frac{x-y}{\epsilon} \right) \right| \, dx \right) \, dy \leq \frac{C_\beta^2}{\epsilon^{2|\beta|}} \| D^\alpha v(y) \|_2^2.
\]
So,

$$
\|J_\epsilon v\|_{m+k} = \left( \sum_{|\beta| \leq k} \|D^\beta D^\alpha J_\epsilon v\|_0^2 \right)^{1/2} \leq \left( \sum_{|\beta| \leq k} \frac{C^2_2 \epsilon^{2k}}{2^{2m}} \|D^\alpha v\|_0^2 \right)^{1/2} \leq 
$$

$$
\left( \sum_{|\alpha| \leq m} C^2_{m\alpha} \|D^\alpha v\|_0^2 \right)^{1/2} \leq \left( \frac{C_{m\alpha}}{\epsilon^{2k}} \sum_{|\alpha| \leq m} \|D^\alpha v\|_0^2 \right)^{1/2} = \frac{C_{m\alpha}}{\epsilon^{2k}} \|v\|_m,
$$

where $C^2_{m\alpha} = \max_{|\alpha| \leq m} C^2_{k\alpha}$, $C^2_{k\alpha} = \max_{|\beta| \leq k} C^2_{\beta}$.

Also, we used here that $\epsilon^{-2|\beta|} \leq \epsilon^{-2k}$ since $0 < \epsilon < 1$.

(b) It remains to prove the estimate (2.49).

Let $\rho_k(x) = D^k \rho(x)$. Since mollifiers commute with derivatives by the property (2.44), then we have

$$
|J_\epsilon D^k v(x)| = |D^k J_\epsilon v(x)| = \epsilon^{-N} \left| \int_{\mathbb{R}^N} D^k_x \rho\left(\frac{x-y}{\epsilon}\right) v(y) \, dy \right|.
$$

Because

$$
D^k_x \rho\left(\frac{x-y}{\epsilon}\right) = D^k_u \rho(u) \epsilon^{-k} = \epsilon^{-k} \rho_k\left(\frac{x-y}{\epsilon}\right) \quad \text{with} \quad u = \frac{x-y}{\epsilon},
$$

we get using the Schwartz inequality,

$$
|J_\epsilon D^k v(x)| = \epsilon^{-N-k} \left| \int_{\mathbb{R}^N} \rho_k\left(\frac{x-y}{\epsilon}\right) v(y) \, dy \right| \leq \epsilon^{-(N/2)-k} \epsilon^{-N/2} \left[ \int_{\mathbb{R}^N} \left| \rho_k\left(\frac{x-y}{\epsilon}\right) \right|^2 \, dy \right]^{1/2} \left[ \int_{\mathbb{R}^N} \left| v(y) \right|^2 \, dy \right]^{1/2} \leq \frac{C_k}{\epsilon^{(N/2)+k}} \|v\|_0,
$$

where $C_k^2 = \epsilon^{-N} \int_{\mathbb{R}^N} \left| \rho_k\left(\frac{x-y}{\epsilon}\right) \right|^2 \, dy = \int_{\mathbb{R}^N} \left| \rho_k(u) \right|^2 \, du$ since $|\rho_k(u)|^2 \in C_0(\mathbb{R}^N)$.

So,

$$
\|J_\epsilon D^k v(x)\|_{L^\infty} \leq \frac{C_k}{\epsilon^{(N/2)+k}} \|v\|_0. \quad (2.49)
$$

□
2.6 The Hodge’s Decomposition of Vector Fields and Properties of the Leray’s Projection Operator.

At first, prove Hodge’s decomposition in $\mathbb{R}^N$.

**Proposition 2.6** (Hodge’s Decomposition in $\mathbb{R}^N$). Every vector field $v \in L^2(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ has a unique decomposition:

$$v = w + \nabla q, \quad \text{div} w = 0,$$

(2.50)

where $w$ is a divergence free vector field and $q$ is a scalar in $\mathbb{R}^N$ with the following properties:

(i) $w, \nabla q \in L^2(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$,

(ii) $w \perp \nabla q$ in $L^2$, i.e., $(w, \nabla q)_{L^2} = 0$,

(iii) for any multi-index $\beta$ of the derivative $D^\beta, \ |\beta| \geq 0$,

$$\|D^\beta v\|_0^2 = \|D^\beta w\|_0^2 + \|\nabla D^\beta q\|_0^2. \quad (2.51)$$

**Proof.**

(i) (a) Show that $w, \nabla q \in C^\infty(\mathbb{R}^N)$.

From the equations (2.50), we get

$$\text{div} v = \text{div}(w + \nabla q) = \text{div} w + \text{div}(\nabla q) = 0 + \Delta q = \Delta q.$$

So,

$$\Delta q = \text{div} v \quad (2.52)$$

Thus, $q \in C^\infty(\mathbb{R}^N)$ as a solution of the Poisson equation (2.52). Then $w \in C^\infty(\mathbb{R}^N)$ as well since $w = v - \nabla q$. This part also proves the existence of $q$ and $w$ such that (2.50) holds.

(b) Prove that $w, \nabla q \in L^2(\mathbb{R}^N)$.

Assume $v \in C^\infty(\mathbb{R}^N)$ with bounded spt $v \subset \{x \in \mathbb{R}^N | |x| < R\}$ for some $R > 0$.

As a solution of the Poisson equation (2.52),

$$q(x) = -\Phi(x) \ast \text{div} v(x)$$

where $\Phi(x)$ is the fundamental solution of the Laplace’s equation given by (2.15) and (2.16).

Then

$$\nabla q(x) = -\int_{|y| \leq R} \nabla_x \Phi(x - y) \text{div} v(y) \, dy$$

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Now
\[ \nabla \Phi(x) = \begin{cases} 
- \frac{1}{2\pi} \frac{x}{|x|^2}, & \text{if } N = 2 \\
- \frac{\Gamma(N/2)}{2\pi^{N/2}} \frac{x}{|x|^N}, & \text{if } N \geq 3 
\end{cases} \]

where \( \omega_N \) is the surface area of a unit sphere in \( \mathbb{R}^N \).

So,
\[ \nabla q(x) = \frac{1}{\omega_N} \int_{|y| \leq R} \frac{x - y}{|x - y|^N} \text{div}v(y) \, dy. \]

Using the Taylor expansion, we have for large \( |x| \)
\[ |x - y|^{-N} = \left( |x| \left| \frac{x}{|x|} - \frac{y}{|x|} \right| \right)^{-N} = |x|^{-N} \left( \frac{x}{|x|} - \frac{y}{|x|} \right)^{2(-N/2)} = \]
\[ |x|^{-N} \left( \frac{x^2}{|x|^2} - 2 \frac{x \cdot y}{|x|^2} + \frac{y^2}{|x|^2} \right)^{-N/2} = |x|^{-N} \left[ 1 + \left( -2 \frac{x \cdot y}{|x|^2} + \frac{y^2}{|x|^2} \right)^{2(-N/2)} \right] = \]
\[ |x|^{-N} \left[ 1 - \frac{N}{2} \left( -2 \frac{x \cdot y}{|x|^2} + \frac{y^2}{|x|^2} \right) + \frac{N(N+1)}{2} \left( -2 \frac{x \cdot y}{|x|^2} + \frac{y^2}{|x|^2} \right)^2 + \ldots \right]. \]

This binomial series converges if \( -2 \frac{x \cdot y}{|x|^2} + \frac{y^2}{|x|^2} < 1 \).

Assume that
\[ \left| -2 \frac{x \cdot y}{|x|^2} + \frac{y^2}{|x|^2} \right| \leq 2 \frac{|x||y|}{|x|^2} + \frac{|y|^2}{|x|^2} < 1 \]

Then \( \frac{|y|}{|x|} < \sqrt{2} - 1 \) or \( |x| > (\sqrt{2} + 1)|y| \).

So, for \( |y| \leq R \) and \( |x| > (\sqrt{2} + 1)R \), we have
\[ \left| -2 \frac{x \cdot y}{|x|^2} + \frac{y^2}{|x|^2} \right| \leq 2 \frac{|y|}{|x|} + \frac{|y|^2}{|x|^2} \leq \frac{2R}{|x|} + \frac{R^2}{|x|^2} \leq \frac{C}{|x|} = O \left( \frac{1}{|x|} \right), \]
and
\[ |x-y|^{-N} = |x|^{-N} \left[ 1 + \mathcal{O}\left(-2\frac{x \cdot y}{|x|^2} + \frac{y^2}{|x|^2}\right) \right] = |x|^{-N} \left[ 1 + \mathcal{O}\left(\frac{1}{|x|}\right) \right]. \]

Then
\[
\frac{x-y}{|x-y|^N} = \frac{x-y}{|x|^N} \left[ 1 + \mathcal{O}\left(\frac{1}{|x|}\right) \right] = \frac{x-y}{|x|^N} + \frac{x-y}{|x|^N} \mathcal{O}\left(\frac{1}{|x|}\right) =
\]
\[
\frac{x}{|x|^N} - \frac{y}{|x|^N} + (x-y) \mathcal{O}\left(\frac{1}{|x|^{N+1}}\right).
\]

Also, since \( v \in C^\infty(\mathbb{R}^N) \), then \( \exists C_0 > 0 \) such that \( |\text{div} v(y)| \leq C_0 \) on \( |y| \leq R \).

Then
\[
\nabla q(x) = \frac{1}{\omega_N} \left[ \frac{x}{|x|^N} \int_{|y| \leq R} \text{div} v(y) \, dy - \frac{1}{|x|^N} \int_{|y| \leq R} y \text{div} v(y) \, dy + \mathcal{O}\left(\frac{1}{|x|^{N+1}}\right) \int_{|y| \leq R} (x-y) \text{div} v(y) \, dy \right].
\]

Applying the Gauss-Green theorem, we get
\[
\int_{|y| \leq R} \text{div} v(y) \, dy = \int_{|y|=R} v(y) \nu(y) \, dS = 0 \quad \text{since spt} \, v \subset \{ x \in \mathbb{R}^N \mid |x| < R \}.
\]

Estimate the remaining two integrals.
\[
\left| \int_{|y| \leq R} y \text{div} v(y) \, dy \right| \leq \int_{|y| \leq R} |y| \left| \text{div} v(y) \right| \, dy \leq RC_0 \text{Vol} \left( B(0, R) \right),
\]
\[
\left| \int_{|y| \leq R} (x-y) \text{div} v(y) \, dy \right| \leq \int_{|y| \leq R} |x-y| \left| \text{div} v(y) \right| \, dy \leq C_0 \int_{|y| \leq R} (|x| + |y|) \, dy \leq 2C_0|x| \text{Vol} \left( B(0, R) \right).
\]

So,
\[
\nabla q(x) = \frac{1}{\omega_N} \left[ 0 + \mathcal{O}(|x|^{-N}) + |x|\mathcal{O}(|x|^{-N-1}) \right] = \mathcal{O}(|x|^{-N}),
\]
i.e., \( |\nabla q(x)| \leq C_1 |x|^{-N} \) for some \( C_1 > 0 \) and \( |x| > (\sqrt{2} + 1)R \).
Now it is convenient to use polar coordinates to check if $\nabla q \in L^2(\mathbb{R}^N)$.

$$\int_{|x| \geq 3R} |\nabla q(x)|^2 \,dx \leq C_1^2 \int_{|x| \geq 3R} |x|^{-2N} \,dx = C_1^2 \omega_N \int_{3R}^{\infty} r^{-2N} r^{N-1} \,dr =$$

$$C_1^2 \omega_N \lim_{a \to \infty} \left(-\frac{1}{N a^N}\right) = C_1^2 \omega_N \lim_{a \to \infty} \left(-\frac{1}{a^N} + \frac{1}{(3a)^N}\right) = C_1^2 \omega_N \frac{N}{N^3 R^N} < \infty.$$

Thus, $\nabla q \in L^2(\mathbb{R}^N)$. Hence $w = v - \nabla q \in L^2(\mathbb{R}^N)$ as well.

(ii) Show that $w \perp \nabla q$ in $L^2(\mathbb{R}^N)$.

(a) Recall that $v$ vanishes sufficiently rapidly at infinity. Let $v = \mathcal{O}(|x|^{-N})$ as $|x| \to \infty$. From the part (i), we have that $\nabla q(x) = \mathcal{O}(|x|^{-N})$ as $|x| \to \infty$. Then $w = v - \nabla q = \mathcal{O}(|x|^{-N})$.

(b) Estimate $q(x)$ for large $|x|$.

$$q(x) = -\int_{|y| \leq R} \Phi(x - y) \text{div} v(y) \,dy$$

since $\text{spt} \, v \subset B(0, R)$,

and $\Phi(x)$ is given by the equations (2.15) for $N > 2$ and (2.16) for $N = 2$.

In case $N > 2$,

$$q(x) = \frac{1}{\omega_N (2 - N)} \int_{|y| \leq R} |x - y|^{-N+2} \text{div} v(y) \,dy.$$

Similarly as we did in the part (i), we get for large $|x|$

$$|x - y|^{-N+2} = |x|^{-N+2} \left[1 + \mathcal{O}\left(-2 \frac{x \cdot y}{|x|^2} + \frac{y^2}{|x|^2}\right)\right].$$

Then

$$|q(x)| \leq \frac{|x|^{-N+2}}{\omega_N (2 - N)} \left[\int_{|y| \leq R} \text{div} v(y) \,dy + C_2 \int_{|y| \leq R} \left| -2 \frac{x \cdot y}{|x|^2} + \frac{y^2}{|x|^2} \right| |\text{div} v(y)| \,dy \right] \leq$$

$$\frac{|x|^{-N+2}}{\omega_N (N - 2)} \left[0 + 2C_2 \frac{|x|}{|x|^2} \int_{|y| \leq R} |y| |\text{div} v(y)| \,dy + C_2 \frac{|x|^2}{|x|^2} \int_{|y| \leq R} |y|^2 |\text{div} v(y)| \,dy \right] \leq$$

$$\leq \frac{|x|^{-N+2}}{\omega_N (N - 2)} \left(2C_2 RC_0 + C_2 R^2 C_0\right) \text{Vol} B(0, R) \leq \frac{C_3 |x|^{-N+2}}{|x|} = \mathcal{O}(|x|^{-N+1}).$$
In case $N = 2$,
\[ q(x) = \frac{1}{2\pi} \int_{|y| \leq R} \ln |x - y| \text{div} v(y) \, dy. \]

\[ |x - y| \leq |x| + |y| \leq 2|x| \] implies $\ln |x - y| \leq \ln |x| + \ln 2 \leq C_3 \ln |x|$ for large $|x|$. Then

\[ |q(x)| \leq \frac{1}{2\pi} \int_{|y| \leq R} |\ln |x - y|| \text{div} v(y)| \, dy \leq \frac{C_3}{2\pi} C_0 \text{Vol}(B(0, R)) \ln |x| = O(\ln |x|). \]

Thus,
\[ q(x) = \begin{cases} O(\ln |x|), & \text{if } N = 2, \\ O(|x|^{-N+1}), & \text{if } N > 2. \end{cases} \]

(c) So, as $|x| \to \infty$,
\[ |w(x)| |q(x)| = \begin{cases} o(|x|^{-1}), & \text{if } N = 2, \\ O(|x|^{-2N+1}), & \text{if } N > 2. \end{cases} \]

Here for $N = 2$, we used L’Hopital’s rule.

\[ \lim_{|x| \to \infty} \frac{|w(x)||q(x)|}{|x|^{-1}} \leq \lim_{|x| \to \infty} \frac{C|x|^{-2} \ln |x|}{|x|^{-1}} = C \lim_{|x| \to \infty} \frac{\ln |x|}{|x|} = C \lim_{|x| \to \infty} \frac{1}{|x|} = 0. \]

So, $|w(x)||q(x)| = o(|x|^{-N+1})$ as $|x| \to \infty$. Then the conditions of Lemma 1.5 are satisfied, and applying this lemma, we get that $w$ and $\nabla q$ are orthogonal:
\[ \int_{\mathbb{R}^N} w \cdot \nabla q \, dx = 0. \]

Hence
\[ \|v\|_0^2 = \int_{\mathbb{R}^N} v^2 \, dx = \int_{\mathbb{R}^N} (w + \nabla q)^2 \, dx = \int_{\mathbb{R}^N} w^2 \, dx + 2 \int_{\mathbb{R}^N} w \cdot \nabla q \, dx + \int_{\mathbb{R}^N} (\nabla q)^2 \, dx = \|w\|_0^2 + \|\nabla q\|_0^2. \]

Note that the orthogonality of the decomposition (2.50) implies its uniqueness. In detail, if $v = w + \nabla q = w' + \nabla q'$, then the previous estimates apply to $w'$ and $\nabla q'$ so that $(w', \nabla q')_{L^2} = (w', \nabla q)_{L^2} = (w, \nabla q')_{L^2} = 0$, and this will imply that $\|w - w'\|_{L^2} = \|\nabla q - \nabla q'\|_{L^2} = 0$. 

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(iii) Differentiating the equations (2.50) and (2.52) for any multi-index $\beta$, we have

$$D^\beta v = D^\beta w + D^\beta (\nabla q), \quad \text{div } D^\beta w = 0,$$

$$\Delta D^\beta q = \text{div } (D^\beta v).$$

Applying for $D^\beta v$ parts (i) and (ii) of the current proposition, we get that $D^\beta w$ and $D^\beta (\nabla q)$ are orthogonal, and consequently, we have

$$\|D^\beta v\|_0^2 = \|D^\beta w\|_0^2 + \|\nabla D^\beta q\|_0^2 \tag{2.51}$$

For now, we assumed in our proof that $v \in C^\infty_0(\mathbb{R}^N)$.

(iv) Finally, show that all statements hold for $v \in L^2(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$.

(a) For this purpose, consider a function $\rho(x) \in C^\infty_0(\mathbb{R}^N)$ such that $\rho(x) \equiv 1$ for $|x| \leq 1$ and $\rho(x) \equiv 0$ for $|x| \geq 2$.

Define $v_n(x) := v(x)\rho(|x|_n), \quad n \in \mathbb{N}.$

$\{v_n(x)\} \subset C^\infty_0(\mathbb{R}^N), \forall v \in C^\infty(\mathbb{R}^N)$ that implies this proposition holds for $v_n(x) \forall n$.

Also, $\{v_n(x)\} \subset L^2(\mathbb{R}^N)$.

Clearly, $v_n \to v$ as $n \to \infty$.

(b) Since $v_n \in C^\infty_0(\mathbb{R}^N)$, then by the previous proof, $\exists w_n, \nabla q_n \in L^2(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ such that Proposition 2.6 holds for $v_n = w_n + \nabla q_n$.

Need to show that $\exists w, q$ such that $w, \nabla q \in L^2(\mathbb{R}^N), \quad w_n \to w$, and

$\nabla q_n \to \nabla q$ in $L^2(\mathbb{R}^N)$.

Let $n, m \in \mathbb{Z}^+$. $v_n = w_n + \nabla q_n$ and $v_m = w_m + \nabla q_m$.

Subtracting these equations, we get the following decomposition:

$$v_n - v_m = (w_n - w_m) + (\nabla q_n - \nabla q_m), \quad \text{where div } (w_n - w_m) = 0.$$

By Proposition 2.6, $(w_n - w_m) \perp (\nabla q_n - \nabla q_m)$ and

$$\|v_n - v_m\|_0^2 = \|w_n - w_m\|_0^2 + \|\nabla q_n - \nabla q_m\|_0^2.$$

The last equation leads to the following two inequalities:

$$\|v_n - v_m\|_0 \geq \|w_n - w_m\|_0 \quad \text{and} \quad \|v_n - v_m\|_0 \geq \|\nabla q_n - \nabla q_m\|_0.$$

Therefore, $\{v_n\}$ is Cauchy in $L^2(\mathbb{R}^N)$ implies that both $\{w_n\}$ and $\{\nabla q_n\}$ are Cauchy in $L^2(\mathbb{R}^N)$.

So, $\exists w, \alpha \in L^2(\mathbb{R}^N)$ such that $w_n \to w$ and $\nabla q_n \to \alpha$ as $n \to \infty$.

(c) Need to check that $w$ is divergence-free, i.e., $\text{div } w = 0$ in sense of weak deriva-
From Proposition 2.6, we have $\text{div} \ w_n = 0$, $\forall n \in \mathbb{Z}^+$ in sense of weak derivatives.

Let $\beta \in C_0^\infty (\mathbb{R}^N)$. Consider

$$(w_n - w, \nabla \beta)_{L^2(\mathbb{R}^N)} \leq \|w_n - w\|_0 \|\nabla \beta\|_0.$$ 

As $n \to \infty$, $\|w_n - w\|_0 \to 0$ implies $(w_n - w, \nabla \beta)_{L^2(\mathbb{R}^N)} \to 0$.

Integrating by parts, we get

$$(w_n - w, \nabla \beta)_{L^2(\mathbb{R}^N)} = -(\nabla w_n - \nabla w, \beta)_{L^2(\mathbb{R}^N)} = (\nabla w, \beta)_{L^2(\mathbb{R}^N)} = 0,$$

$\forall \beta \in C_0^\infty (\mathbb{R}^N)$.

This shows that $\text{div} w = 0$ in $L^2(\mathbb{R}^N)$.

(d) Show that $w \perp \alpha$ in $L^2((\mathbb{R}^N))$. Let $n, m \in \mathbb{Z}^+$. Applying integration by parts again, we have

$$(w_n, \nabla q_m)_{L^2} = -(\nabla w_n, q_m)_{L^2} = 0 \text{ since } \text{div} w_n = 0.$$ 

Then $(w_n, \nabla q_m)_{L^2} \to (w, \alpha)_{L^2} = 0$ as $n, m \to \infty$.

(e) Need to point out that

$$(w, \alpha)_{L^2} = 0, \forall w \text{ such that } \text{div} w = 0,$$

is a sufficient condition for there to exist $q \in L^2$ such that $\nabla q = \alpha$ [9, Lemma 2.2.1].

Thus, this proposition has proved for every vector field $v \in L^2(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$.

Next theorem generalizes Proposition 2.6 and surveys some useful properties of the Leray’s projection operator defined in Chapter 1.

**Theorem 2.6 (The Hodge Decomposition in $H^m$).** Every vector field $v \in H^m(\mathbb{R}^N)$, $m \in \mathbb{Z}^+ \cup \{0\}$, has the unique orthogonal decomposition

$$v = w + \nabla \varphi$$

where $w$ is divergence-free vector field in $\mathbb{R}^N$, and $\varphi$ is a scalar in $\mathbb{R}^N$, such that the Leray’s projection operator $\mathcal{P} v = w$ satisfies

(i) $\mathcal{P} v, \nabla \varphi \in H^m$, $\int_{\mathbb{R}^N} \mathcal{P} v \cdot \nabla \varphi \, dx = 0$, $\text{div} \mathcal{P} v = 0$, and

$$\|\mathcal{P} v\|_m^2 + \|\nabla \varphi\|_m^2 = \|v\|_m^2.$$  

(2.53)
ii) $P$ commutes with weak derivatives,

$$ P D^\alpha v = D^\alpha P v, \quad \forall v \in H^m, \quad |\alpha| \leq m, \quad (2.54) $$

(iii) $P$ commutes with mollifiers $J_\epsilon$,

$$ P(J_\epsilon v) = J_\epsilon(P v), \quad \forall v \in H^m, \quad \epsilon > 0, \quad (2.55) $$

(iv) $P$ is symmetric,

$$ (P u, v)_m = (u, P v)_m. \quad (2.56) $$

Proof.

(a) By the part (iii) of the Proposition 2.6, $D^\beta w$ and $D^\beta(\nabla \varphi)$ are orthogonal in $L^2(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ for any multi-index $\beta$, $|\beta| \leq m$.

So, $w$ and $\nabla \varphi$ are orthogonal in $H^m(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$. Hence this decomposition of $v$ is unique.

(b) (1) Extend the previous to all $v \in H^m(\mathbb{R}^N)$ using the argument that $H^m(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ is dense in $H^m(\mathbb{R}^N)$ [3, p.251].

So, for given $v \in H^m(\mathbb{R}^N)$, $\exists \{v_n\} \subset H^m(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ such that $\|v_n - v\|_m \to 0$ as $n \to \infty$.

By Proposition 2.6, $\forall n$, $\exists w_n, q_n$ such that $w_n, \nabla q_n \in L^2(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N), \quad w_n \perp \nabla q_n$ in $L^2(\mathbb{R}^N), \quad v_n = w_n + \nabla q_n \quad \|v_n\|_m^2 = \|w_n\|_m^2 + \|\nabla q_n\|_m^2$. 

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Since \( \left\| v_n \right\|_m \to \left\| v \right\|_m \) as \( n \to \infty \), we have
\[
\lim_{n \to \infty} \left( \left\| w_n \right\|_m^2 + \left\| \nabla q_n \right\|_m^2 \right) = \left\| v \right\|_m^2.
\]

(2) Similarly as we did in Proposition 2.6, we can show that \( \exists w, \alpha \in H^m(\mathbb{R}^N), w_n \to w, \text{ and } \nabla q_n \to \alpha \) in \( H^m(\mathbb{R}^N) \); \( w \perp \alpha \) in \( L^2(\mathbb{R}^N) \); and that \( w \) is divergence-free, i.e., \( \text{div } w = 0 \) in sense of weak derivatives.

Also, since \( \alpha \in H^m(\mathbb{R}^N) \Rightarrow \alpha \in L^2(\mathbb{R}^N) \), it follows from previous argument in Proposition 2.6 that \( \exists \phi \in L^2(\mathbb{R}^N) \) such that \( \nabla \phi = \alpha \).

(3) Need to clarify that \( P v = w \).

\( \forall n \in \mathbb{Z}^+, \ P v_n = w_n, \text{ and } w_n \to w \) in \( H^m(\mathbb{R}^N) \) as \( n \to \infty \).

Consider
\[
\left\| w_n - P v \right\|_m = \left\| P v_n - P v \right\|_m = \left\| (P (v - v_n)) \right\|_m \leq 1 \cdot \left\| v_n - v \right\|_m \to 0 \text{ as } n \to \infty.
\]

So,
\[
w = \lim_{n \to \infty} w_n = P v.
\]

(4) Thus,
\[
v = P v + \nabla \phi \quad \text{where } P v, \nabla \phi \in H^m(\mathbb{R}^N),
\]
\[
\text{div } P v = 0, \text{ and } (P v, \nabla \phi)_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} P v \cdot \nabla \phi \, dx = 0.
\]

The orthogonality of \( P v \) and \( \nabla \phi \) implies uniqueness of the decomposition and that
\[
\left\| v \right\|^2_m = \left\| P v + \nabla \phi \right\|^2_m = \left\| P v \right\|^2_m + \left\| \nabla \phi \right\|^2_m. \tag{2.53}
\]

(ii) Note that \( \text{div}(D^\alpha w) = D^\alpha(\text{div } w) = 0 \), \( |\alpha| \leq m \), since \( \text{div } w = 0 \). Then \( P (D^\alpha w) = D^\alpha w \).

By Proposition 2.6, part (iii),
\[
D^\alpha v = D^\alpha w + D^\alpha \nabla \phi \quad \text{and } D^\alpha w \perp D^\alpha (\nabla \phi) \text{ in } L^2(\mathbb{R}^N).
\]

Hence
\[
P(D^\alpha \nabla \phi) = 0 \text{ and } \quad P(D^\alpha v) = P(D^\alpha w) + P(D^\alpha \nabla \phi) = D^\alpha w + 0 = D^\alpha (P w). \tag{2.54}
\]

(iii) \( w \in H^m(\mathbb{R}^N) \) and \( \text{div } w = 0 \). Since by the Proposition 2.6, part (ii), mollifiers commute with weak derivatives, then
\[
\text{div}(J_\epsilon w) = J_\epsilon(\text{div } w) = 0.
\]

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So, $P(Jw) = Jw$.

Show that $P(J\nabla \varphi) = 0$.

Let $u \in H^m(\mathbb{R}^N)$ and $\text{div} u = 0$ that implies $\text{div}(J_\epsilon u) = 0$.

Applying the symmetry property of mollifiers (Th.2.5(iii)) and then integrating by parts, we have

$$\langle u, J_\epsilon \nabla \varphi \rangle_{L^2(\mathbb{R}^N)} = \langle J_\epsilon u, \nabla \varphi \rangle_{L^2(\mathbb{R}^N)} = -\langle \nabla (J_\epsilon u), \varphi \rangle_{L^2(\mathbb{R}^N)} = 0.$$

So, $J_\epsilon (\nabla \varphi) \perp u$, $\forall u$ from the divergence-free space in $H^m(\mathbb{R}^N)$.

Hence $P(J_\epsilon \nabla \varphi) = 0$.

Now we get

$$P(J_\epsilon v) = P[J_\epsilon (w + \nabla \varphi)] = P(J_\epsilon w + J_\epsilon \nabla \varphi) = P(J_\epsilon w) + P(J_\epsilon \nabla \varphi) = J_\epsilon w + 0 = J_\epsilon (Pv). \tag{2.55}$$

Thus, the Leray’s projection operator commutes with mollifiers.

(iv) Let $u, v \in H^m(\mathbb{R}^N)$, $u = Pu + \nabla q$, $v = Pv + \nabla \varphi$, $\nabla Pu = 0$, $\nabla Pv = 0$.

Using results from the Proposition 2.6, part (iii), we have for any multi-index $\beta$, $|\beta| \leq m$,

$$D^\beta u = D^\beta (Pu) + D^\beta (\nabla q), \quad D^\beta v = D^\beta (Pv) + D^\beta (\nabla \varphi),$$

$$\nabla D^\beta (Pu) = 0, \quad \nabla D^\beta (Pv) = 0.$$

Applying an integration by parts, we get

$$\left(D^\beta (Pu), D^\beta (\nabla \varphi)\right)_{L^2(\mathbb{R}^N)} = \left(D^\beta (Pu), \nabla (D^\beta \varphi)\right)_{L^2(\mathbb{R}^N)} = -\left(\nabla D^\beta (Pu), D^\beta \varphi\right)_{L^2(\mathbb{R}^N)} = 0.$$

Hence

$$(Pu, \nabla \varphi)_m := \sum_{0 \leq |\beta| \leq m} (D^\beta (Pu), D^\beta (\nabla \varphi))_{L^2(\mathbb{R}^N)} = 0.$$

Similarly, we can obtain

$$(Pv, \nabla q)_m = (\nabla q, Pv)_m = 0.$$

Now we have

$$(Pu, v)_m = (Pu, Pv + \nabla \varphi)_m = (Pu, Pv)_m + (Pu, \nabla \varphi)_m = (Pu, Pv)_m + 0 = (Pu, Pv)_m + (\nabla q, Pv)_m = (Pu + \nabla q, Pv)_m = (u, Pv)_m.$$
So, we get

\[(PU, V)_m = (u, PV)_m.\]  \hspace{1cm} (2.56)

This theorem concludes our review of the technical background necessary to work on the initial value problems for the Euler and the Navier-Stokes equations.
Chapter 3


In this chapter, we prove existence and uniqueness of a smooth solution of the initial value problems for the Euler and the Navier-Stokes equations on some time interval $[0, T)$ applying classical energy methods. These methods use an approximation scheme and various a priori estimates on Sobolev norms of velocity $v$ allowing to replace the unbounded operators $\nabla$ and $\Delta$ by some operators bounded in a Banach space.


The kinetic energy of the fluid, $E = \frac{1}{2} \int_{\mathbb{R}^N} |v|^2 \, dx$, is a half of the square of $L^2$ norm of the velocity.

In Section 1.7, we got that

$$\frac{dE(t)}{dt} = -\nu \int_{\mathbb{R}^N} |\nabla v|^2 \, dx < 0. \tag{3.1}$$

So, the kinetic energy dissipates for viscous flows (case $\nu > 0$). In the case $\nu = 0$, for inviscid flows, kinetic energy is conserved. Thus, a priori, we have boundness of the velocity in $L^2$ norm.

3.1.1 A Basic Energy Estimate.

Here we will derive a basic energy estimate for a smooth solution of the Euler or the Navier-Stokes equations.

Proposition 3.1 (Basic Energy Estimate). Let $v_1$ and $v_2$ be two smooth solutions to the Navier-Stokes equations with external forces $F_1$ and $F_2$ correspondingly and the same viscosity $\nu \geq 0$. Suppose that these solutions exist on a common time interval $[0, T]$ and, for fixed time, decay fast enough at infinity to belong to $L^2(\mathbb{R}^N)$. Then

$$\sup_{0 \leq t \leq T} \|v_1 - v_2\|_0 \leq \left[ \| (v_1 - v_2)|_{t=0} \|_0 + \int_{\mathbb{R}^N} \|F_1 - F_2\|\, dt \right] \exp\left(\int_{\mathbb{R}^N} \|\nabla v_2\|_{L^\infty} \, dt \right). \tag{3.7}$$

Proof. Since $v_1$ and $v_2$ are solutions to the Navier-Stokes equations with respective external forces $F_1$ and $F_2$, then we have

$$\frac{Dv_i}{Dt} = -\nabla p_i + \nu \Delta v_i + F_i,$$

$$\text{div } v_i = 0,$$

$$v_i|_{t=0} = v_{0i} (i = 1, 2). \tag{3.2}$$

We assume that these solutions exist on a common time interval $[0, T]$ and that they vanish
sufficiently rapidly as \(|x| \to \infty\), so that \(v_i \in L^2(\mathbb{R}^N)\).

Take a difference of corresponding equations (3.2) and add and subtract \(v_1 \cdot \nabla v_2\) to and from respectively the left-hand sides of the first equations.

\[
\begin{align*}
  v_{1t} + v_1 \cdot \nabla v_1 &= -\nabla p_1 + \nu \Delta v_1 + F_1, \\
  v_{2t} + v_2 \cdot \nabla v_2 &= -\nabla p_2 + \nu \Delta v_2 + F_2, \\
  (v_1 - v_2)_t + v_1 \cdot \nabla v_1 - v_1 \cdot \nabla v_2 + v_1 \cdot \nabla v_2 - v_2 \cdot \nabla v_2 &= -\nabla (p_1 - p_2) + \nu \Delta (v_1 - v_2) + (F_1 - F_2).
\end{align*}
\]

So, we get

\[
\tilde{v}_t + v_1 \cdot \nabla \tilde{v} + \tilde{v} \cdot \nabla v_2 = -\nabla \tilde{p} + \nu \Delta \tilde{v} + \tilde{F}, \quad \text{div} \tilde{v} = 0, \quad \tilde{v}|_{t=0} = \tilde{v}_0 \quad (3.2a)
\]

where \(\tilde{v} = v_1 - v_2\), \(\tilde{p} = p_1 - p_2\), \(\tilde{F} = F_1 - F_2\), and \(\tilde{v}_0 = v_{01} - v_{02}\).

Take the \(L^2\) inner product \((\cdot , \cdot)\) of both sides of the first equation (3.2a) with \(\tilde{v}\).

\[
(\tilde{v}_t, \tilde{v}) + (v_1 \cdot \nabla \tilde{v}, \tilde{v}) + (\tilde{v}, \nabla v_2, \tilde{v}) = -(\nabla \tilde{p}, \tilde{v}) + \nu (\Delta \tilde{v}, \tilde{v}) + (\tilde{F}, \tilde{v}). \quad (3.3)
\]

Next, integrate by parts and use the divergence theorem and the fact that \(v_i, \ i = 1, 2, \) and so \(\tilde{v}\), are divergence free and vanish sufficiently rapidly at infinity.

Note that a multiplication by the unknown quantity (or a function of it) and an integration by parts are common for all energy methods.

Show that \((v_1 \cdot \nabla \tilde{v}, \tilde{v}) = -(\nabla \tilde{p}, \tilde{v}) = 0\).

1) \((v_1 \cdot \nabla \tilde{v}, \tilde{v}) = \int_{\mathbb{R}^N} (v_1 \cdot \nabla \tilde{v}) \cdot \tilde{v} \, dx = \int_{\mathbb{R}^N} \sum_{j=1}^{N} \left( \sum_{i=1}^{N} v_{1i} \frac{\partial}{\partial x_j} \tilde{v}_i \right) \tilde{v}_i \, dx
\]

\[
= \int_{\mathbb{R}^N} \sum_{j=1}^{N} v_{1j} \sum_{i=1}^{N} \frac{\partial v_j}{\partial x_i} \tilde{v}_i \, dx = \int_{\mathbb{R}^N} \sum_{j=1}^{N} v_{1j} \sum_{i=1}^{N} \frac{1}{2} \frac{\partial}{\partial x_j} (\tilde{v}_i^2) \, dx = \int_{\mathbb{R}^N} \sum_{j=1}^{N} v_{1j} \frac{1}{2} \frac{\partial}{\partial x_j} (\tilde{v}_i^2) \, dx
\]

\[
= \int_{\mathbb{R}^N} v_1 \cdot \nabla \left( \frac{1}{2} \tilde{v}^2 \right) \, dx = \int_{\mathbb{R}^N} \left\{ \nabla \cdot \left[ \frac{1}{2} \tilde{v}^2 v_1 \right] - \frac{1}{2} \tilde{v}^2 (\nabla \cdot v_1) \right\} \, dx
\]

\[
= \int_{\mathbb{R}^N} \left\{ \text{div} \left[ \frac{1}{2} \tilde{v}^2 v_1 \right] - \left( \frac{1}{2} \tilde{v}^2 \right) \text{div} v_1 \right\} \, dx = \int_{\mathbb{R}^N} \left\{ \text{div} \left[ \frac{1}{2} \tilde{v}^2 v_1 \right] - 0 \right\} \, dx
\]

\[
= \lim_{R \to \infty} \int_{|x|=R} \left( \frac{1}{2} \tilde{v}^2 \right) v_1 \cdot n \, dS = 0
\]

where \(n\) is the outward normal unit vector of the sphere \(|x| = R\).
2) \(- (\nabla \tilde{p}, \tilde{v}) = - \int_{\mathbb{R}^N} \nabla \tilde{p} \cdot \tilde{v} dx = - \sum_{i=1}^{N} \int_{\mathbb{R}^N} \tilde{p}_{x_i} \tilde{v}_i dx \)

\[= - \sum_{i=1}^{N} \left( - \int_{\mathbb{R}^N} \tilde{p}(\tilde{v}_i)_{x_i} dx + \lim_{R \to \infty} \int_{|x|=R} \tilde{p} \tilde{v}_i n_i dS \right) \] (after integration by parts)

\[= \int_{\mathbb{R}^N} \tilde{p} \sum_{i=1}^{N} (\tilde{v}_i)_{x_i} dx + 0 = \int_{\mathbb{R}^N} \tilde{p} \nabla \cdot \tilde{v} dx = \int_{\mathbb{R}^N} \tilde{p} \text{div} \tilde{v} dx = 0. \]

Also

\[ - (\Delta \tilde{v}, \tilde{v}) = - \sum_{i=1}^{N} \int_{\mathbb{R}^N} \Delta \tilde{v}_i \tilde{v}_i dx = - \sum_{i=1}^{N} \int_{\mathbb{R}^N} (\nabla \cdot \nabla \tilde{v}_i) \tilde{v}_i dx \]

\[= \sum_{i=1}^{N} \int_{\mathbb{R}^N} \left( \nabla \tilde{v}_i \cdot \nabla \tilde{v}_i \right) dx - \sum_{i=1}^{N} \lim_{R \to \infty} \int_{|x|=R} (\nabla \tilde{v}_i \tilde{v}_i) \cdot n_i dS \] (after integration by parts)

\[= \sum_{i=1}^{N} (\nabla \tilde{v}_i, \nabla \tilde{v}_i) - 0 = (\nabla \tilde{v}, \nabla \tilde{v}) = \int_{\mathbb{R}^N} \nabla \tilde{v} \cdot \nabla \tilde{v} dx = \int_{\mathbb{R}^N} |\nabla \tilde{v}|^2 dx = \|\nabla \tilde{v}(., t)\|_0^2 \geq 0. \]

Then from equation (3.3), we get the following basic energy identity:

\[(\tilde{v}_t, \tilde{v}) + \nu \|\nabla \tilde{v}\|_0^2 = - (\tilde{v} \cdot v_2, \tilde{v}) + (\tilde{F}, \tilde{v}). \quad (3.4) \]

Note that

\[(\tilde{v}_t, \tilde{v}) = \int_{\mathbb{R}^N} \tilde{v}_t \cdot \tilde{v} dx = \int_{\mathbb{R}^N} \sum_{i=1}^{N} (\tilde{v}_i)_t \tilde{v}_i dx = \sum_{i=1}^{N} \int_{\mathbb{R}^N} \left( \frac{1}{2} \tilde{v}_i^2 \right)_t dx = \]

\[= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \sum_{i=1}^{N} \tilde{v}_i^2 dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |\tilde{v}|^2 dx = \frac{1}{2} \frac{d}{dt} \|\tilde{v}(., t)\|_0^2 = \frac{1}{2} \mathcal{Z} \|\tilde{v}(., t)\|_0 \frac{d}{dt} \|\tilde{v}(., t)\|_0. \]

Using the Cauchy-Schwartz and Schwartz inequalities, find an upper estimate for the right-hand side of the equation (3.4).

\[- (\tilde{v} \cdot \nabla v_2, \tilde{v}) = - \int_{\mathbb{R}^N} (\tilde{v} \cdot \nabla v_2) \cdot \tilde{v} dx \leq \int_{\mathbb{R}^N} (\tilde{v} \cdot \nabla v_2) \cdot \tilde{v} dx \leq \int_{\mathbb{R}^N} |(\tilde{v} \cdot \nabla v_2) \cdot \tilde{v}| dx \leq \int_{\mathbb{R}^N} |\tilde{v} \cdot \nabla v_2| \cdot \tilde{v}| dx \leq \int_{\mathbb{R}^N} |\tilde{v} \cdot \nabla v_2| \cdot dx \leq \|\nabla v_2\|_{L^\infty} \int_{\mathbb{R}^N} |\tilde{v}|^2 dx = \|\nabla v_2\|_{L^\infty} \|\tilde{v}(., t)\|_0^2; \]

\[(\tilde{F}, \tilde{v}) = \int_{\mathbb{R}^N} \tilde{F} \cdot \tilde{v} dx \leq \int_{\mathbb{R}^N} \tilde{F} \cdot \tilde{v} dx \leq \|\tilde{F}(., t)\|_0 \|\tilde{v}(., t)\|_0. \]
Then from equation (3.4), we get the following inequality:

$$
\|\tilde{v}(\cdot, t)\|_0 \frac{d}{dt}(\|\tilde{v}(\cdot, t)\|_0) + \nu \|\nabla \tilde{v}(\cdot, t)\|_0^2 \leq \|\nabla v_2\|_{L^\infty} \|\tilde{v}(\cdot, t)\|_0^2 + \|\tilde{F}(\cdot, t)\|_0 \|\tilde{v}(\cdot, t)\|_0.
$$

(3.5)

Drop $\nu \|\nabla \tilde{v}(\cdot, t)\|_0^2$ in the left-hand side of the last inequality and divide this inequality by $\|\tilde{v}(\cdot, t)\|_0$. Then we have

$$
\frac{d}{dt}(\|\tilde{v}(\cdot, t)\|_0) \leq \|\nabla v_2\|_{L^\infty} \|\tilde{v}(\cdot, t)\|_0 + \|\tilde{F}(\cdot, t)\|_0.
$$

(3.6)

Recall Grönwall’s lemma:

If $\eta(t), \varphi(t), \psi(t) \geq 0$, $\eta(t)$ is absolutely continuous and $\varphi(t), \psi(t)$ are integrable on $[0, T]$ for some $T > 0$, and $\eta'(t) \leq \varphi(t)\eta(t) + \psi(t)$, then

$$
\forall t \in [0, T], \quad \eta(t) \leq e^{\int_0^T \varphi(s) \, ds}[\eta(0) + \int_0^T \psi(s) \, ds].
$$

Suppose that

$$
\eta(t) = \|\tilde{v}(\cdot, t)\|_0, \quad \varphi(t) = \|\nabla v_2\|_{L^\infty}, \quad \text{and} \quad \psi(t) = \|\tilde{F}(\cdot, t)\|_0.
$$

Hence from the inequality (3.6), applying the Grönwall’s lemma, we get

$$
\|\tilde{v}(\cdot, t)\|_0 \leq e^{\int_0^T \|\nabla v_2\|_{L^\infty} \, dt}[\|\tilde{v}(\cdot, t)\|_0 + \int_0^T \|\tilde{F}(\cdot, t)\|_0 \, dt].
$$

Thus, we derived the Basic Energy estimate.

$$
\sup_{0 \leq t \leq T} \|v_1 - v_2\|_0 \leq

\left[ \| (v_1 - v_2)_{t=0}\|_0 + \int_{\mathbb{R}^N} \|F_1 - F_2\|_0 \, dt \right] \exp\left( \int_0^T \|\nabla v_2\|_{L^\infty} \, dt \right).
$$

(3.7)

Note that the Basic Energy estimate (3.7) does not depend explicitly on the viscosity. So, the same estimate is applicable for solutions to both the Euler and the Navier-Stokes equations.

The Basic Energy estimate (3.7) immediately implies uniqueness of smooth solutions to the Euler and to the Navier-Stokes equations.

**Corollary 3.1 (Uniqueness of Solutions).** Let $v_1$ and $v_2$ be two smooth $L^2$ solutions to the Navier-Stokes equations on $[0, T]$ with the same initial data and forcing $F$. Then $v_1 = v_2$.

**Proof.** Indeed, with $v_1|_{t=0} = v_2|_{t=0}$ and $F_1 = F_2$, the Basic Energy estimate (3.7) gives
that \( v_1 = v_2, \forall t \in [0, T] \).

Another consequence of the Basic Energy estimate is the following useful estimate on the gradient of \( \tilde{v} = v_1 - v_2 \) that we will apply in the next subsection.

**Corollary 3.2.** Under the assumptions of Proposition 3.1,

\[
\nu \int_0^T \| \nabla \tilde{v}(\cdot, t) \|_0^2 dt \leq C(v_2, T) \left[ \| \tilde{v}(\cdot, 0) \|_0^2 + \left( \int_0^T \| \tilde{F}(\cdot, t) \|_0 dt \right)^2 \right] \quad (3.9)
\]

where the constant \( C(v_2, T) \) depends on \( \int_0^T \| \nabla v_2 \|_{L^\infty} dt \).

**Proof.** Integrate the inequality (3.5) in time on \([0, T]\) taking into account that

\[
\int_0^T \| \tilde{v}(\cdot, t) \|_0 \frac{dt}{dt} (\| \tilde{v}(\cdot, t) \|_0) dt = \frac{1}{2} \| \tilde{v}(\cdot, 0) \|_0^2.
\]

We get

\[
\frac{1}{2} \| \tilde{v}(\cdot, T) \|_0^2 - \frac{1}{2} \| \tilde{v}(\cdot, 0) \|_0^2 + \nu \int_0^T \| \nabla \tilde{v}(\cdot, t) \|_0^2 dt \leq \sup_{0 \leq t \leq T} \| \tilde{v}(\cdot, t) \|_0^2 \int_0^T \| \nabla v_2 \|_{L^\infty} dt + \sup_{0 \leq t \leq T} \| \tilde{v}(\cdot, t) \|_0 \int_0^T \| \tilde{F}(\cdot, t) \|_0 dt. \quad (3.8)
\]

Drop \( \frac{1}{2} \| \tilde{v}(\cdot, T) \|_0^2 \geq 0 \) at the left-hand side of the last inequality and use the estimate (3.7) for \( \sup_{0 \leq t \leq T} \| \tilde{v}(\cdot, t) \|_0 \) at the right-hand side of this inequality.

Denote \( a = \int_0^T \| \nabla v_2 \|_{L^\infty} dt \). Then we have

\[
\nu \int_0^T \| \nabla \tilde{v}(\cdot, t) \|_0^2 dt \leq \frac{1}{2} \| \tilde{v}(\cdot, 0) \|_0^2 + \nu \int_0^T \| \nabla \tilde{v}(\cdot, t) \|_0^2 dt \leq \frac{1}{2} \| \tilde{v}(\cdot, 0) \|_0^2 + a e^{2a} \left[ \| \tilde{v}(\cdot, 0) \|_0^2 + 2 \| \tilde{v}(\cdot, 0) \|_0 \int_0^T \| \tilde{F}(\cdot, t) \|_0 dt + \int_0^T \| \tilde{F}(\cdot, t) \|_0 dt \right]^2 \]

or

\[
\nu \int_0^T \| \nabla \tilde{v}(\cdot, t) \|_0^2 dt \leq \| \tilde{v}(\cdot, 0) \|_0 \int_0^T \| \tilde{F}(\cdot, t) \|_0 dt e^{a(2ae^a + 1) + \left( \int_0^T \| \tilde{F}(\cdot, t) \|_0 dt \right)^2 e^a(ae^a + 1).}
\]

Note that \( a = \int_0^T \| \nabla v_2 \|_{L^\infty} dt \geq 0 \). So, \( e^a \geq 1 > \frac{1}{2} \).
Also, by the elementary inequality
\[ \|\tilde{v}(\cdot, 0)\|_0 \int_0^T \|\tilde{F}(\cdot, t)\|_0 \, dt \leq \frac{1}{2} \left[ \|\tilde{v}(\cdot, 0)\|_0^2 + \left( \int_0^T \|\tilde{F}(\cdot, t)\|_0 \, dt \right)^2 \right]. \]

Applying that to the last estimate, we finally get
\[ \nu \int_0^T \|\nabla \tilde{v}(\cdot, t)\|_0^2 \, dt \leq C(v_2, T) \left[ \|\tilde{v}(\cdot, 0)\|_0^2 + \left( \int_0^T \|\tilde{F}(\cdot, t)\|_0 \, dt \right)^2 \right] \quad (3.9) \]
where \( C(v_2, T) = e^{a(2ae^a + \frac{3}{2})} \) with \( a = \int_0^T \|\nabla v_2\|_{L^\infty} \, dt. \)

### 3.1.2 Approximation of Inviscid Flows by High Reynold’s Number Viscous Flows.

We continue to assume here that both the smooth inviscid solution \( v^0 \) to the Euler equation and the smooth viscous solution \( v^\nu \) to the Navier-Stokes equations exist on a common time interval. The Basic Energy estimate allows to show that for the same initial data, the solution \( v^\nu \) converges to the solution \( v^0 \) as the viscosity \( \nu \to 0 \).

**Proposition 3.2** (Estimate of Closeness of Smooth Solutions to the Euler and to the Navier-Stokes Equations). Given fixed initial data, let \( v^\nu, \nu \geq 0 \), be a smooth solution to the Euler (for \( \nu = 0 \)) or the Navier-Stokes (for \( \nu > 0 \)) equations. Suppose that for \( 0 \leq \nu \leq \nu_0 \) these solutions exist on a common time interval \([0, T]\) and vanish sufficiently rapidly as \( |x| \to \infty \). Then

\[ \sup_{0 \leq t \leq T} \| v^\nu - v^0 \|_0 \leq C_1(v^0, T) \nu T \quad (3.10) \]

and

\[ \int_0^T \|\nabla (v^\nu - v^0)\|_0 \, dt \leq C_2(v^0, T) \nu^{1/2} T^{3/2} \quad (3.11) \]

where \( C_1(v^0, T) = \sup_{0 \leq t \leq T} \|\Delta v^0(\cdot, t)\|_0 \, e^{\int_0^T \|\nabla v^0\|_{L^\infty} \, dt} \), \( C_2(v^0, T) = C^{1/2}(v^0, T) \sup_{0 \leq t \leq T} \|\Delta v^0(\cdot, t)\|_0 \) with \( C(v^0, T) \) the same as in (3.9).

**Proof.** Suppose that in the Navier-Stokes equation

\[ \frac{Dv^\nu}{Dt} = -\nabla p + \nu \Delta v^\nu + F, \]

the external force \( F = -\nu \Delta v^\nu \). Then we get

\[ \frac{Dv^\nu}{Dt} = -\nabla p, \]
the Euler equation. So, \( v^{\nu} = v^{0} \) in this case.

Consider now two smooth solutions to the Navier-Stokes equation: \( v_{1} = v^{\nu} \) for external force \( F_{1} = 0 \) and \( v_{2} = v^{0} \) for \( F_{2} = -\nu \Delta v^{0} \). Note that \( (v^{\nu} - v^{0})|_{t=0} = 0 \).

Applying to these solutions the Basic Energy estimate (3.7), we get

\[
\sup_{0 \leq t \leq T} \| v^{\nu} - v^{0} \|_{0} \leq \int_{0}^{T} \| \nu \Delta v^{0} \|_{0} dt \leq \nu \sup_{0 \leq t \leq T} \| \Delta v^{0} \|_{0} T e^{\int_{0}^{T} \| \nu v \|_{L^{\infty}} dt}
\]

or

\[
\sup_{0 \leq t \leq T} \| v^{\nu} - v^{0} \|_{0} \leq C_{1}(v^{0}, T) \nu T
\]

where \( C_{1}(v^{0}, T) = \sup_{0 \leq t \leq T} \| \Delta v^{0}(\cdot, t) \|_{0} e^{\int_{0}^{T} \| \nu v \|_{L^{\infty}} dt} \).

From the estimate (3.9), we obtain

\[
\nu \int_{0}^{T} \| \nabla (v^{\nu} - v^{0}) \|_{0}^{2} dt \leq C(v^{0}, T) \left( \int_{0}^{T} \| \nu \Delta v^{0}(\cdot, t) \|_{0} dt \right)^{2}.
\]

To derive the second estimate of this proposition, apply the Schwartz inequality to the left-hand side of the previous inequality.

\[
\int_{0}^{T} \| \nabla (v^{\nu} - v^{0}) \|_{0} dt = \int_{0}^{T} 1 \| \nabla (v^{\nu} - v^{0}) \|_{0} dt \leq \left( \int_{0}^{T} 1^{2} dt \right)^{1/2} \left( \int_{0}^{T} \| \nabla (v^{\nu} - v^{0}) \|_{0}^{2} dt \right)^{1/2} = \left( \frac{T}{\nu} \right)^{1/2} \left( \nu \int_{0}^{T} \| \nabla (v^{\nu} - v^{0}) \|_{0}^{2} dt \right)^{1/2}.
\]

Now upper bound the last term using the previous estimate. We get

\[
\int_{0}^{T} \nabla (v^{\nu} - v^{0}) \|_{0} dt \leq \left( \frac{T}{\nu} C(v^{0}, T) \right)^{1/2} \left[ \left( \int_{0}^{T} \| \nu \Delta v^{0}(\cdot, t) \|_{0} dt \right)^{2} \right]^{1/2} \leq \left( \frac{T}{\nu} C(v^{0}, T) \right)^{1/2} \nu \sup_{0 \leq t \leq T} \| \Delta v^{0}(\cdot, t) \|_{0} T
\]

or

\[
\int_{0}^{T} \| \nabla (v^{\nu} - v^{0}) \|_{0} dt \leq C_{2}(v^{0}, T) \nu^{1/2} T^{3/2}
\]

where \( C_{2}(v^{0}, T) = \sqrt{C(v^{0}, T)} \sup_{0 \leq t \leq T} \| \Delta v^{0}(\cdot, t) \|_{0} \).

\[
\square
\]
The estimates (3.10) and (3.11) give the following corollary.

**Corollary 3.3.** *For the assumptions of Proposition 3.2, \( v^{\nu} \to v^0 \) as \( \nu \to 0 \) in the \( L^\infty([0,T]) \times L^2(\mathbb{R}^N) \) norm with rate first order of \( \nu \), and the gradients \( \nabla v^{\nu} \to \nabla v^0 \) with rate square root of \( \nu \).*

Thus, for small viscosity (high Reynold’s number), solutions to the Euler equation \( v^0 \) and to the Navier-Stokes equation \( v^{\nu} \) are close, and the viscous solution \( v^{\nu} \) can be a good approximation for the inviscid solution \( v^0 \).

Recall that we assumed that both \( v^{\nu} \) and \( v^0 \) are smooth solutions on a common time interval. It is not necessarily true for solutions that are not smooth. It is worth mention, that for small viscosity, the time interval of existence of solution to the Navier-Stokes equation \( v^{\nu} \) includes the time interval of existence of solution to the Euler equation[2].


Here we will prove the local-in-time existence of a solution to the Navier-Stokes equation in 3-D case without forcing. Note that the results of this section can be extended to the case of sufficiently smooth \( L^2 \) forcing.

We will use independent of viscosity \( \nu \) estimates. So, the proof holds for both the Navier-Stokes and Euler equations.

At first, we will find an approximate equation whose solution satisfies an energy estimate and exists for all time. Then we obtain the solution to the Navier-Stokes equation as a limit in the approximation scheme.

#### 3.2.1 Global Existence of Solution to a Regularization of the Euler and the Navier-Stokes Equations.

In this subsection, we will use the Picard theorem for ODEs on a Banach space.

**Theorem 3.1** *(Picard Theorem on a Banach Space)*. *Let \( O \subseteq B \) be an open subset of a Banach space \( B \) and let \( F : O \to B \) be a locally Lipschitz continuous mapping, i.e., for any \( X \in O \) there exists \( L > 0 \) and an open neighborhood \( U_X \subset O \) of \( X \) such that*

\[
\|F(\tilde{X}) - F(\hat{X})\|_B \leq L\|\tilde{X} - \hat{X}\|_B\quad \text{for all } \tilde{X}, \hat{X} \in U_X.
\]

*Then for any \( X_0 \in O \), there exists a time \( T \) such that the ODE*

\[
\frac{dX}{dt} = F(X), \quad X|_{t=0} = X_0
\]

*has a unique (local) solution \( X \in C^1([-T, T); O] \).*

The Navier-Stokes equations contain operators \( \nabla \) and \( \Delta \) unbounded in Banach space, that does not permit to apply directly the existence theory for ODEs. Therefore we will regu-
larize the Navier-Stokes equations in such way that regularized equations will satisfy the conditions of the Picard theorem.

To construct the approximate equation, we will use a mollification that allows us to convert unbounded differential operators into bounded operators.

We regularize the Navier-Stokes equations with initial data

\[ \begin{align*}
  v_t + v \cdot \nabla v &= -\nabla p + \nu \Delta v, \\
  \text{div} v &= 0, \\
  v|_{t=0} &= v_0,
\end{align*} \]

replacing them with the following approximate equations with unknowns \( v^\epsilon \) and \( p^\epsilon \) for some parameter \( \epsilon > 0 \):

\[ \begin{align*}
  v^\epsilon_t + J_\epsilon \left[ (J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon) \right] &= -\nabla p^\epsilon + \nu J_\epsilon (J_\epsilon \Delta v^\epsilon), \\
  \text{div} v^\epsilon &= 0, \\
  v^\epsilon|_{t=0} &= v_0,
\end{align*} \]

where \( J_\epsilon \) is defined in the equations (2.41)-(2.42).

To exclude the pressure \( p^\epsilon \) and the incompressibility condition \( \text{div} v^\epsilon = 0 \), apply to the equations (3.13) Leray’s operator \( P \) projecting onto the space of divergence-free functions:

\[ V^s = \{ v \in H^s(\mathbb{R}^N) : \text{div} v = 0 \}. \]

If \( v^\epsilon \in V^s \), then \( P v^\epsilon = v^\epsilon \) and \( P(v^\epsilon_t) = (P v^\epsilon)_t = v^\epsilon_t \). Also, \( P(\nabla p^\epsilon) = 0 \).

Since \( P \) commutes with derivatives and mollifiers, then

\[ P \left[ \nu J_\epsilon (J_\epsilon \Delta v^\epsilon) \right] = \nu J_\epsilon^2 \left[ \Delta (P v^\epsilon) \right] = \nu J_\epsilon^2 (\Delta v^\epsilon). \]

Hence we have

\[ v^\epsilon_t + P J_\epsilon \left[ (J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon) \right] = \nu J_\epsilon^2 (\Delta v^\epsilon) \]

or

\[ \frac{dv^\epsilon}{dt} = F_\epsilon(v^\epsilon), \]

\[ v^\epsilon|_{t=0} = v_0 \]

where

\[ F_\epsilon(v^\epsilon) = F_\epsilon^{(1)}(v^\epsilon) - F_\epsilon^{(2)}(v^\epsilon), \quad F_\epsilon^{(1)}(v^\epsilon) = \nu J_\epsilon^2 (\Delta v^\epsilon), \]

\[ F_\epsilon^{(2)}(v^\epsilon) = P J_\epsilon \left[ (J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon) \right]. \]

We reduced the regularized Navier-Stokes and Euler equations (3.13) to the ODE (3.17) in the Banach space \( V^s \). Here \( v^\epsilon \in C^1([0, T_\epsilon); V^s) \), i.e., \( v^\epsilon(t) \) is in the space of \( C^1 \) functions on the interval \( [0, T_\epsilon) \) for some \( T_\epsilon > 0 \) with values in \( V^s \)-Banach space equipped with the Sobolev norm \( \| \cdot \|_s \).
At first, prove that \( C^1([0, T_f); V^*) \) is a Banach space.

**Lemma 3.1.** Let \( B \) be a Banach space and \( C^1([0, T); B) := \{ f, \frac{df}{dt} : [0, T) \rightarrow B \mid f \in C^1([0, T)) \} \) for some \( T > 0 \). Then \( C^1([0, T); B) \) is a Banach space with the norm

\[
\|f\|_{C^1} := \sup_{0 \leq t < T} \|f(t)\|_B + \sup_{0 \leq t < T} \|\frac{df(t)}{dt}\|_B.
\]

**Proof.**

1) Show that \( \| \cdot \|_{C^1} \) is a norm.

(a) \( \forall t \in [0, T), \|f(t)\|_B \geq 0 \) and \( \|\frac{df(t)}{dt}\|_B \geq 0 \) as \( \| \cdot \|_B \) is a norm. Then \( \|f\|_{C^1} \geq 0 \).

(b) If \( f(t) \equiv 0 \) on \([0, T)\), then \( \frac{df(t)}{dt} \equiv 0 \) on \([0, T]\). Hence \( \|f(t)\|_B = \|\frac{df(t)}{dt}\|_B = 0 \) \( \forall t \in [0, T) \) since \( \| \cdot \|_B \) is a norm. So, \( \|f\|_{C^1} = 0 \).

Let \( \|f\|_{C^1} = 0 \). Then \( \|f\|_{C^1} \geq \|f(t)\|_B \geq 0 \) and \( \|f\|_{C^1} \geq \|\frac{df(t)}{dt}\|_B \geq 0 \). \( \forall t \in [0, T) \) imply \( \|f(t)\|_B = \|\frac{df(t)}{dt}\|_B = 0 \). \( \forall t \in [0, T) \).

So, \( f(t) \equiv 0 \) \( \forall t \in [0, T) \) since \( \| \cdot \|_B \) is a norm.

Thus, \( \|f\|_{C^1} = 0 \) iff \( f(t) \equiv 0 \) on \([0, T)\).

(c) \( \forall \alpha \in \mathbb{R}, \|\alpha f\|_{C^1} = \sup_{0 \leq t < T} \|\alpha f(t)\|_B + \sup_{0 \leq t < T} \|\alpha \frac{df(t)}{dt}\|_B = \sup_{0 \leq t < T} (|\alpha| \|f(t)\|_B) + \sup_{0 \leq t < T} (|\alpha| \|\frac{df(t)}{dt}\|_B) = |\alpha| (\sup_{0 \leq t < T} \|f(t)\|_B + \sup_{0 \leq t < T} \|\frac{df(t)}{dt}\|_B) = |\alpha| \|f\|_{C^1}. \)

(d) By the Triangle inequality for norm \( \| \cdot \|_B \), we have

\[
\forall f(t), g(t) \in B, \|f(t) + g(t)\|_B \leq \|f(t)\|_B + \|g(t)\|_B, \forall t \in [0, T).
\]

Then

\[
\sup_{0 \leq t < T} \|f(t) + g(t)\|_B \leq \sup_{0 \leq t < T} \|f(t)\|_B + \sup_{0 \leq t < T} \|g(t)\|_B.
\]

Similarly, we can get the following inequality for derivatives:

\[
\sup_{0 \leq t < T} \|\frac{d}{dt}[f(t) + g(t)]\|_B \leq \sup_{0 \leq t < T} \|\frac{df(t)}{dt}\|_B + \sup_{0 \leq t < T} \|\frac{dg(t)}{dt}\|_B.
\]

Adding the last two inequalities, we obtain the Triangle inequality for \( \| \cdot \|_{C^1} \):

\[
\|f + g\|_{C^1} \leq \|f\|_{C^1} + \|g\|_{C^1}.
\]

Thus, \( \| \cdot \|_{C^1} \) is a norm on \( C^1([0, T); B) \).
2) Prove that $C^1([0,T); B)$ is complete in $\| \cdot \|_{C^1}$ norm.

(a) Assume \( \{u_n\}_{n=1}^\infty \subset C^1([0,T); B) \) is a Cauchy sequence. So, given \( \varepsilon > 0, \exists N \in \mathbb{N} \) such that \( \forall m > n > N, \|u_n - u_m\|_{C^1} < \varepsilon, \) that implies 
\[
\|u_n(t) - u_m(t)\|_B \leq \|u_n - u_m\|_{C^1} < \varepsilon, \quad \forall t \in [0,T).
\]

Then \( \{u_n(t)\}_{n=1}^\infty \) is a Cauchy in \( B, \forall t \in [0,T). \)

Since \( B \) is complete, then \( \exists u(t) \in B \), such that \( \forall t \in [0,T), \|u_n(t) - u(t)\|_B \to 0 \) as \( n \to \infty. \)

(b) Show that \( \{u_n(t)\} \) converges to \( u(t) \) uniformly on \( [0,T) \).

Since \( \{u_n(t)\} \) is a Cauchy sequence in \( \| \cdot \|_{C^1} \) norm, so given \( \varepsilon > 0, \|u_n - u_m\|_{C^1} < \frac{\varepsilon}{3}, \forall m, n > N. \)

Now consider 
\[
\|u_m(t) - u_n(t)\|_B \leq \sup_{0 \leq t < T} \|u_m(t) - u_n(t)\|_B \leq \|u_m - u_n\|_{C^1} < \frac{\varepsilon}{3}, \quad \forall m, n > N,
\]
so \( \{u_n(t)\} \) is Cauchy in \( \| \cdot \|_B \) norm for each \( t \in [0,T) \). Since \( B \) is complete, \( \exists u(t) \in B \) such that \( \|u_n(t) - u(t)\|_B \to 0, \) as \( n \to \infty, \ t \in [0,T). \)

Now for any \( t_0 \in [0,T) \),
\[
\|u_m(t_0) - u_n(t_0)\|_B \leq \sup_{0 \leq t < T} \|u_m(t) - u_n(t)\|_B < \frac{\varepsilon}{3}.
\]

Let \( n \to \infty \), so
\[
\|u_m(t_0) - u(t_0)\|_B \leq \frac{\varepsilon}{3}.
\]

Since this is true for arbitrary \( t_0 \in [0,T) \) for the some fixed \( \varepsilon \), so
\[
\sup_{0 \leq t < T} \|u_m(t) - u(t)\|_B \leq \frac{\varepsilon}{3}.
\]
\((*)\)

Therefore, \( u_n(t) \to u(t) \) uniformly on \( [0,T) \).

Similarly, it could be shown that Cauchy sequence \( \{\frac{du_n(t)}{dt}\}_{n=1}^\infty \) converges uniformly on \( [0,T). \)

Hence \( u(t) \) is differentiable and \( \lim_{n \to \infty} \frac{du_n(t)}{dt} = \frac{du(t)}{dt} \) on \( [0,T). \)[8]

(c) Show that \( u(t) \in C^1([0,T); B). \)

Given any \( s, t \in [0,T), \)
\[
\|u(t) - u(s)\|_B \leq \|u(t) - u_n(t)\|_B + \|u_n(t) - u_n(s)\|_B + \|u_n(s) - u(s)\|_B.
\]

Applying the inequality \((*)\) to the first and to the third terms at the right-hand side of the last inequality, we have
\[
\|u(t) - u(s)\|_B \leq \frac{2\varepsilon}{3} + \|u_n(t) - u_n(s)\|_B.
\]

\( u_n(t) \) is continuous on \( [0,T) \). So, for given \( \varepsilon > 0, \exists \delta > 0, \) such that \( \forall s, t \in [0,T) \) with \( |s - t| < \delta, \) we have \( \|u_n(t) - u_n(s)\|_B < \frac{\varepsilon}{3}. \)
Combine this inequality with the previous one to get
\[
\forall \varepsilon > 0, \quad \|u(t) - u(s)\|_B < \varepsilon \quad \forall s, t \in [0, T) \text{ such that } |s - t| < \delta.
\]
So, \( u: [0, T) \to B \) is continuous.

Similarly, we can show that \( \frac{du(t)}{dt} \) is continuous on \( [0, T) \).

Then \( u \in C^1([0, T); B) \), and so \( C^1([0, T); B) \) is complete.

Thus, \( C^1([0, T); B) \) is a Banach space with norm \( \| \cdot \|_{C^1} \).

Next, we prove the following proposition.

**Proposition 3.3.** Consider an initial condition \( v_0 \in V^m, \ m \in \mathbb{Z}^+ \cup \{0\} \). Then

i) for any \( \varepsilon > 0 \) there exists the unique solution \( v^\varepsilon \in C^1([0, T_\varepsilon); V^m) \) to the ODE (3.16), where \( T_\varepsilon = T(\|v_0\|_m, \varepsilon) \);

ii) on any time interval \( [0, T] \) on which this solution belongs to \( C^1([0, T]; V^0) \),

\[
\sup_{0 \leq t \leq T} \|v^\varepsilon\|_0 \leq \|v_0\|_0.
\] (3.18)

**Proof.**

i) Our goal is to show that the right-hand side of the ODE (3.16) \( F_\varepsilon(v^\varepsilon) = F_\varepsilon^{(1)}(v^\varepsilon) - F_\varepsilon^{(2)}(v^\varepsilon) \) satisfies the conditions of the Picard theorem (Theorem 3.1).

(a) At first, clarify that \( F_\varepsilon \) maps \( V^m \) to \( V^m \).

Let \( v^\varepsilon \in V^m \), so \( \text{div} v^\varepsilon = 0 \). Since the mollifier \( J_\varepsilon \) commutes with derivatives, then

\[
\text{div} F_\varepsilon^{(1)}(v^\varepsilon) = \text{div}(\nu J_\varepsilon^2 \Delta v^\varepsilon) = \nu J_\varepsilon^2 \Delta (\text{div} v^\varepsilon) = 0.
\]

Also, \( \mathcal{P} \) is a projection into divergence-free vector fields, so

\[
F_\varepsilon^{(2)}(v^\varepsilon) = \mathcal{P} J_\varepsilon [(J_\varepsilon v^\varepsilon) \cdot (\nabla J_\varepsilon v^\varepsilon)]
\]

implies

\[
\text{div} F_\varepsilon^{(2)}(v^\varepsilon) = 0.
\]

Then

\[
\text{div} F_\varepsilon(v^\varepsilon) = \text{div} F_\varepsilon^{(1)}(v^\varepsilon) - \text{div} F_\varepsilon^{(2)}(v^\varepsilon) = 0.
\]

Thus, \( F_\varepsilon: V^m \to V^m \).
(b) Prove that \( F^{(1)}_\epsilon (v^\epsilon) \) is Lipschitz continuous.

Let \( v_1, v_2 \in C^1([0, T_\epsilon]; V^m) \). Consider

\[
\| F^{(1)}_\epsilon (v_1) - F^{(1)}_\epsilon (v_2) \|_m = \nu \| J^2_\epsilon (v_1 - v_2) \|_m = \nu \| \Delta J^2_\epsilon (v_1 - v_2) \|_m
\]

since the mollifier \( J_\epsilon \) commutes with derivatives.

Then by the definition of Sobolev norm, we have

\[
\| F^{(1)}_\epsilon (v_1) - F^{(1)}_\epsilon (v_2) \|_m = \nu \left[ \int_{\mathbb{R}^N} (1 + |\xi|^2)^m \left| \left[ \Delta J^2_\epsilon (v_1 - v_2) \right] \hat{f}(\xi) \right|^2 d\xi \right]^{1/2}
\]

\[
\nu \left[ \int_{\mathbb{R}^N} (1 + |\xi|^2)^m (2\pi)^{2m} |\xi|^4 \left| \left[ J^2_\epsilon (v_1 - v_2) \right] \hat{f}(\xi) \right|^2 d\xi \right]^{1/2} \leq 16\pi^4 \nu \left[ \int_{\mathbb{R}^N} (1 + |\xi|^2)^m \left[ \left[ J^2_\epsilon (v_1 - v_2) \right] \hat{f}(\xi) \right] \right]^{1/2}
\]

\[
16\pi^4 \nu \| J^2_\epsilon (v_1 - v_2) \|_{m+2} \leq 16\pi^4 \nu \frac{C_{m2}}{\epsilon^2} \| J^2_\epsilon (v_1 - v_2) \|_m \leq 16\pi^4 \nu \frac{C_{m2}}{\epsilon^2} \| v_1 - v_2 \|_m = \frac{C_{m2}}{\epsilon^2} \| v_1 - v_2 \|_m \quad \text{with} \quad C_3 = 16\pi^4 \nu C_{m2} C_{m0}.
\]

Here we consequently used the property of the Fourier transform

\[
\hat{\partial^\beta f}(\xi) = (2\pi i \xi)^\beta \hat{f}(\xi) \quad \text{with multi-index } \beta, \quad (2.11)
\]

the elementary fact that \( |\xi|^4 \leq (1 + |\xi|^2)^2 \), and twice the estimate for mollifiers

\[
\| J_\epsilon v \|_{m+k} \leq \frac{C_{mk}}{\epsilon^k} \| v \|_m \quad \forall v \in H^m, \quad \forall \epsilon > 0, \quad \forall k \in \mathbb{Z}^+ \cup \{0\},
\]

for \( k = 2 \) and \( k = 0 \).

So, \( F^{(1)}_\epsilon (v^\epsilon) \) is Lipschitz continuous.

(c) Now show that \( F^{(2)}_\epsilon (v^\epsilon) \) is Lipschitz continuous too.

Let again \( v_1, v_2 \in C^1([0, T_\epsilon]; V^m) \). Consider

\[
\| F^{(2)}_\epsilon (v_1) - F^{(2)}_\epsilon (v_2) \|_m = \| F^{(2)}_\epsilon (v_1) - \mathcal{P} J_\epsilon \left( (J_\epsilon v_1) \cdot \nabla (J_\epsilon v_2) \right) - \mathcal{P} J_\epsilon \left( (J_\epsilon v_1) \cdot \nabla (J_\epsilon v_2) \right) - F^{(2)}_\epsilon (v_2) \|_m
\]

\[
= \| \mathcal{P} J_\epsilon \left( (J_\epsilon v_1) \cdot \nabla (J_\epsilon v_2) \right) - \mathcal{P} J_\epsilon \left( (J_\epsilon v_1) \cdot \nabla (J_\epsilon v_2) \right) \|_m \leq \| \mathcal{P} J_\epsilon \left( (J_\epsilon v_1) \cdot \nabla (J_\epsilon (v_1 - v_2)) \right) \|_m + \| \mathcal{P} J_\epsilon \left( (J_\epsilon (v_1 - v_2)) \cdot \nabla (J_\epsilon v_2) \right) \|_m.
\]

By the Hodge decomposition in \( H^m \), we have

\[
\| \mathcal{P} v \|_m^2 + \| \nabla \varphi \|_m^2 = \| v \|_m^2.
\]

So, omitting the Leray’s projection operator \( \mathcal{P} \), we will use that \( \| \mathcal{P} v \|_m \leq \| v \|_m \) in the next inequality.
We will also continue using the estimates for mollifiers (2.48) and (2.49), the Calculus inequality in the Sobolev space (2.38), and the property of mollifiers (2.44). Then we get

$$\| F_{\epsilon}^{(2)}(v_1) - F_{\epsilon}^{(2)}(v_2) \|_m \leq$$

$$\| J_\epsilon \left[ (J_\epsilon v_1) \cdot \nabla (J_\epsilon (v_1 - v_2)) \right] \|_m + \| J_\epsilon \left[ (J_\epsilon (v_1 - v_2)) \cdot \nabla (J_\epsilon v_2) \right] \|_m \leq$$

$$C_{m0} \left[ \| J_\epsilon v_1 \|_{L^\infty} \| D^m J_\epsilon \nabla (v_1 - v_2) \|_0 + \| D^m J_\epsilon (v_1 - v_2) \|_0 \right] \| J_\epsilon \nabla (v_1 - v_2) \|_{L^\infty} \quad (2.48), \quad (2.49)$$

$$CC_{m0} \left[ \| J_\epsilon v_1 \| \| J_\epsilon (v_1 - v_2) \|_{m+1} + \| J_\epsilon v_1 \| \frac{C_1}{\epsilon^{(N/2)+m}} \| v_1 - v_2 \|_0 + \right.$$

$$\left. \frac{C_0}{\epsilon^{N/2}} \| v_1 - v_2 \| \| J_\epsilon v_2 \|_{m+1} + \| J_\epsilon (v_1 - v_2) \| \frac{C_1}{\epsilon^{(N/2)+m}} \| v_2 \|_0 \right] \leq$$

$$CC_{m0} \left[ \frac{C_0}{\epsilon^{N/2}} \| v_1 \| \| J_\epsilon (v_1 - v_2) \| \frac{C_1}{\epsilon^{(N/2)+m}} \| v_1 - v_2 \|_m + \right.$$

$$\left. \frac{C_0}{\epsilon^{(N/2)+m}} \| v_1 - v_2 \| \| J_\epsilon v_2 \| \frac{C_1}{\epsilon^{(N/2)+m}} \| v_1 - v_2 \|_m \right] \leq$$

$$CC_{m0} \frac{\| v_1 - v_2 \|_m}{\epsilon^{(N/2)+m+1}} \left( C_0 C_m \| v_1 \|_0 \epsilon^m + C_1 C_{m0} \| v_1 \|_0 + C_0 C_{(m+1)0} \| v_2 \|_0 + ight.$$

$$\left. C_1 \epsilon \| v_2 \|_0 \right) \leq \frac{C_2}{\epsilon^{(N/2)+m+1}} \left( \| v_1 \|_0 + \| v_2 \|_0 \right)$$

with $C_2 = CC_{m0} \max \{(C_0 C_1 + C_1 C_{m0}), (C_0 C_{(m+1)0} + C_1 C_{m0})\}$ and $\epsilon \in (0, 1)$.

Here we used the facts that

$$\| D^m \nabla f \|_0 \leq \| f \|_{m+1}, \quad \| D^m f \|_0 \leq \| f \|_m, \quad \| v_1 - v_2 \|_0 \leq \| v_1 - v_2 \|_m, \quad \epsilon^m < 1.$$

Also, we twice applied the estimate (2.48) not for $k = 0$ and $k = 1$ as earlier, but for $k = m$ and $k = m + 1$, i.e.

$$\| J_\epsilon v_1 \|_{0+m} \leq \frac{C_{m0}}{\epsilon^m} \| v_1 \|_0$$

and

$$\| J_\epsilon (v_1 - v_2) \|_{m+1} \leq \frac{C_{(m+1)0}}{\epsilon^{m+1}} \| v_1 - v_2 \|_0.$$

So, both $F_{\epsilon}^{(1)}(v^\epsilon)$ and $F_{\epsilon}^{(2)}(v^\epsilon)$ are Lipschitz continuous.
Then for $F_{\epsilon}(v^\epsilon) = F_{\epsilon}^{(1)}(v^\epsilon) - F_{\epsilon}^{(2)}(v^\epsilon)$, we get
\[
\|F_{\epsilon}(v_1) - F_{\epsilon}(v_2)\|_m \leq \|F_{\epsilon}^{(1)}(v_1) - F_{\epsilon}^{(1)}(v_2)\|_m + \|F_{\epsilon}^{(2)}(v_1) - F_{\epsilon}^{(2)}(v_2)\|_m \\
\leq \frac{\nu C_3}{\epsilon^2} \|v_1 - v_2\|_m + \frac{C_2}{\epsilon^{(N/2)+m+1}} \|v_1 - v_2\|_m = \bar{C} \|v_1 - v_2\|_m
\]
(3.19)

where $\bar{C} = \bar{C}(\|v_1\|_0, \|v_2\|_0, \epsilon, N, m)$.

Let $O^M = \{v \in V^m | \|\nu\|_m < M\}$.

So, $F_{\epsilon}$ is locally Lipschitz continuous on any such open set $O^M$.

Hence the Picard theorem (Theorem 3.1) gives us that for any given initial condition $v_0 \in H^m$, there exists the unique solution to the IVP (3.16) $v^\epsilon \in C^1([0, T_\epsilon); O^M)$, $m \in \mathbb{Z}^+ \cup \{0\}$, for some $T_\epsilon > 0$ (locally in time).

ii) Next, prove the energy bound (3.18) for any $v^\epsilon \in C^1([0, T_\epsilon); O^M)$.

Take the $L^2$ inner product of the equation (3.16) with $v^\epsilon$.
\[
(v^\epsilon, \frac{d}{dt}v^\epsilon)_{L^2(\mathbb{R}^N)} = (v^\epsilon, F_{\epsilon}(v^\epsilon))_{L^2(\mathbb{R}^N)},
\]
i.e.,
\[
\int_{\mathbb{R}^N} v^\epsilon \cdot \frac{dv^\epsilon}{dt} \, dx = \int_{\mathbb{R}^N} v^\epsilon \cdot F_{\epsilon}(v^\epsilon) \, dx
\]
or
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} (v^\epsilon)^2 \, dx = \nu \int_{\mathbb{R}^N} v^\epsilon \cdot J_{\epsilon}^2(\Delta v^\epsilon) \, dx - \int_{\mathbb{R}^N} v^\epsilon \cdot \mathcal{P} J_{\epsilon} \left[(J_{\epsilon} v^\epsilon) \cdot \nabla (J_{\epsilon} v^\epsilon)\right] \, dx.
\]
(3.20)

Now use the properties of mollifiers and the properties of the projection operator $\mathcal{P}$ from Theorem 2.5(iii) and Theorem 2.6(iii) respectively and after that, integrate by parts both of the integrals on the right-hand side of the last equation. Then we have
\[
\nu \int_{\mathbb{R}^N} v^\epsilon \cdot J_{\epsilon}^2(\Delta v^\epsilon) \, dx \overset{Th.2.5}{=} \nu \int_{\mathbb{R}^N} (J_{\epsilon} v^\epsilon) \cdot J_{\epsilon}(\nabla^2 v^\epsilon) \, dx \overset{int.by parts}{=}
\]
\[
- \nu \int_{\mathbb{R}^N} \nabla (J_{\epsilon} v^\epsilon) \cdot \nabla (J_{\epsilon} v^\epsilon) \, dx = -\nu \int_{\mathbb{R}^N} (\nabla J_{\epsilon} v^\epsilon)^2 \, dx = -\nu \int_{\mathbb{R}^N} \left[J_{\epsilon}(\nabla v^\epsilon)^2\right] \, dx
\]
and
\[
- \int_{\mathbb{R}^N} v^\epsilon \cdot \mathcal{P} J_{\epsilon} \left[(J_{\epsilon} v^\epsilon) \cdot \nabla (J_{\epsilon} v^\epsilon)\right] \, dx \overset{Th.2.6}{=} - \int_{\mathbb{R}^N} \frac{v^\epsilon}{\|v^\epsilon\|} \cdot J_{\epsilon} \left[(J_{\epsilon} v^\epsilon) \cdot \nabla (J_{\epsilon} v^\epsilon)\right] \, dx
\]
\[
\overset{Th.2.5}{=} - \int_{\mathbb{R}^N} (J_{\epsilon} v^\epsilon) \cdot \left[(J_{\epsilon} v^\epsilon) \cdot \nabla (J_{\epsilon} v^\epsilon)\right] \, dx = - \int_{\mathbb{R}^N} (J_{\epsilon} v^\epsilon) \cdot \frac{1}{2} \nabla (J_{\epsilon} v^\epsilon)^2 \, dx \overset{int.by parts}{=}
\]
\[
\frac{1}{2} \int_{\mathbb{R}^N} \nabla \cdot (J_{\epsilon} v^\epsilon) (J_{\epsilon} v^\epsilon)^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^N} J_{\epsilon} \left[\nabla \cdot v^\epsilon\right] (J_{\epsilon} v^\epsilon)^2 \, dx = 0.
\]
Combining these results with the equation (3.20), we obtain

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} (v^\epsilon)^2 \, dx = -\nu \int_{\mathbb{R}^N} (\nabla J_\epsilon v^\epsilon)^2 \, dx \]

or

\[ \frac{d}{dt} \| v^\epsilon \|_0^2 = -2\nu \| \nabla J_\epsilon v^\epsilon \|_0^2 \leq 0 \text{ for } \nu \geq 0. \]

Thus, \( \| v^\epsilon \|_0 \) is not increasing in time, i.e.,

\[ \| v^\epsilon \|_0 \leq \| v_0 \|_0, \quad \forall t \geq 0 \]

or

\[ \sup_{0 \leq t \leq T} \| v^\epsilon \|_0 \leq \| v_0 \|_0. \tag{3.18} \]

Note that the energy estimate (3.18) does not contain explicitly \( \epsilon \) and \( \nu \). So, it could be applied for any positive \( \epsilon \) to regularize both the Navier-Stokes or the Euler equations.

We emphasize here the importance of the special choice of the regularization in the equation (3.17) that makes possible to obtain this valuable estimate.

We have proved the existence and the uniqueness of solution \( v^\epsilon \) to the regularized Navier-Stokes and Euler equations locally in time. Now we want to show that this equation has global-in-time solutions. For this purpose, using the fact that the functional \( F \) in the regularized equation (3.16) is time independent, we will apply the following continuation property of ODEs on a Banach space.

**Theorem 3.2 (Continuation of an Autonomous ODE on a Banach Space).** Let \( \mathcal{O} \subset B \) be an open set of Banach space \( B \), and let \( F: \mathcal{O} \rightarrow B \) be a locally Lipschitz continuous operator. Then the unique solution \( X \in C^1([0, T); \mathcal{O}) \) to the autonomous ODE,

\[ \frac{dX}{dt} = F(X), \quad X|_{t=0} = X_0 \in \mathcal{O}, \]

either exists globally in time, or \( T < \infty \) and \( X(t) \) leaves the open space \( \mathcal{O} \) as \( t \rightarrow T^- \).[4]

Now we ready to get the main result of this subsection.

**Theorem 3.3 (Global Existence of Regularized Solutions).** Given an initial condition \( v_0 \in V^m \), \( m \in \mathbb{Z}^+ \cap \{0\} \), for any \( \epsilon > 0 \), there exists for all time a unique solution \( v^\epsilon \in C^1([0, \infty); V^m) \) to the regularized equation (3.16).

**Proof.** By the Proposition 3.6, there exists the unique solution \( v^\epsilon \in C^1([0, T_\epsilon); V^m) \) to the regularized equation (3.16). Now show that there exists an a priori bound on \( \| v^\epsilon(\cdot, t) \|_m \).
Take the weak derivative $D^\alpha$ with multi-index $\alpha$ in the spatial variables of equation (3.16). We have

$$D^\alpha \left( \frac{dv^\epsilon}{dt} \right) = D^\alpha \left( F^\epsilon(v^\epsilon) \right).$$

Then take the $L^2$ inner product of the last equation with $D^\alpha(v^\epsilon)$. We obtain

$$\int_{\mathbb{R}^N} D^\alpha(v^\epsilon) \cdot \frac{d}{dt} \left( D^\alpha(v^\epsilon) \right) dx = \int_{\mathbb{R}^N} D^\alpha(v^\epsilon) \cdot D^\alpha(F^\epsilon(v^\epsilon)) dx.$$

For the left-hand side of this equality, we have

$$\int_{\mathbb{R}^N} D^\alpha(v^\epsilon) \cdot \frac{d}{dt} \left( D^\alpha(v^\epsilon) \right) dx = \frac{1}{2} \int_{\mathbb{R}^N} D^\alpha(v^\epsilon) \cdot \frac{d}{dt} \left( D^\alpha(v^\epsilon) \right) dx = \frac{1}{2} \int_{\mathbb{R}^N} \left| D^\alpha(v^\epsilon) \right|^2 dx = \frac{1}{2} \frac{d}{dt} \left\| D^\alpha(v^\epsilon) \right\|_0^2 = \left\| D^\alpha(v^\epsilon) \right\|_0 \frac{d}{dt} \left( \left\| D^\alpha(v^\epsilon) \right\|_0 \right).$$

Bound the right-hand side of that equality using the Schwartz inequality,

$$\int_{\mathbb{R}^N} D^\alpha(v^\epsilon) \cdot D^\alpha(F^\epsilon(v^\epsilon)) dx \leq \int_{\mathbb{R}^N} \left| D^\alpha(v^\epsilon) \cdot D^\alpha(F^\epsilon(v^\epsilon)) \right| dx \leq \left\| D^\alpha(v^\epsilon) \right\|_0 \left\| D^\alpha(F^\epsilon(v^\epsilon)) \right\|_0.$$

Then dividing both sides by $\left\| D^\alpha(v^\epsilon) \right\|_0$, we get

$$\frac{d}{dt} \left\| D^\alpha(v^\epsilon) \right\|_0 \leq \left\| D^\alpha(F^\epsilon(v^\epsilon)) \right\|_0.$$

Take a sum by $0 \leq |\alpha| \leq m$ and use the Sobolev norm $\|v^\epsilon\|_m = \sum_{0 \leq |\alpha| \leq m} \left\| D^\alpha(v^\epsilon) \right\|_0$. We have

$$\frac{d}{dt} \|v^\epsilon(\cdot, t)\|_m \leq \|F^\epsilon(v^\epsilon(\cdot, t))\|_m$$

(\*)

Recall now the relation (3.19).

$$\|F^\epsilon(v_1) - F^\epsilon(v_2)\|_m \leq \bar{C}(\|v_1\|_0, \|v_2\|_0, \epsilon, N, m) \|v_1 - v_2\|_m$$

(3.19)

where $\bar{C} = \frac{\nu^2 C_4}{\epsilon^2} + \frac{C_2(\|v_1\|_0 + \|v_2\|_0)}{\epsilon^{(N/2)+m+1}}$ and $C_2, C_4$ are positive constants.

Choose $v_1 = v^\epsilon$ and $v_2 = 0$. We get

$$\|F^\epsilon(v^\epsilon)\|_m \leq \bar{C}(\|v^\epsilon\|_0, \epsilon, N, m) \|v^\epsilon\|_m.$$

So, applying this inequality to the inequality (\*), we obtain

$$\frac{d}{dt} \|v^\epsilon(\cdot, t)\|_m \leq \bar{C}(\|v^\epsilon\|_0, \epsilon, N, m) \|v^\epsilon\|_m.$$
By the energy estimate (3.18), we have

\[ \| v^\epsilon (\cdot, t) \|_0 \leq \| v_0 \|_0 \quad \text{on } [0, T]. \]

Applying that to the inequality (***) and taking into account that \( \bar{C} \) is monotone increasing in \( \| v^\epsilon \|_0 \), we get the following bound:

\[ \frac{d}{dt} \| v^\epsilon (\cdot, t) \|_m \leq \bar{C}(\| v_0 \|_0, \epsilon, N, m) \| v^\epsilon \|_m. \]

Now use Grönwall’s inequality in differential form:

if \( \eta(t) \geq 0 \) and absolutely continuous on \([0, T]\), \( \varphi(t), \psi(t) \geq 0 \) and summable on \([0, T]\), and almost everywhere \( \eta'(t) \leq \varphi(t)\eta(t) + \psi(t) \), then

\[ \eta(t) \leq e^{\int_0^t \varphi(s) \, ds} \left[ \eta(0) + \int_0^t \psi(s) \, ds \right], \quad t \in [0, T]. \]

Take \( \eta(t) = \| v^\epsilon (\cdot, t) \|_m \), \( \varphi(t) \equiv \bar{C} \), and \( \psi(t) \equiv 0 \). We get the following bound:

\[ \| v^\epsilon (\cdot, t) \|_m \leq \| v_0 \|_m e^{\bar{C}t} \leq \| v_0 \|_m e^{\bar{C}T}. \]

So, \( \forall T \geq 0 \), \( \| v^\epsilon \|_m \) is bounded.

Hence \( v^\epsilon \) never leaves set \( \mathcal{O} \) where it is bounded, and \( F_\epsilon \) is Lipschitz continuous.

Therefore by the Theorem 3.2, there exists a unique solution \( v^\epsilon \) to the regularized equation (3.16) for all time \( t \in [0, \infty) \).

Thus, the solution \( v^\epsilon \) exists and is unique globally-in-time. \( \square \)

3.2.2 Local-in-Time Existence of Solutions to the Euler and the Navier-Stokes Equations.

The goal of this subsection is to show that for any viscosity \( 0 \leq \nu < \infty \), there exists a time interval \([0, T]\) of existence of the unique solution \( v^\epsilon \in C([0, T]; C^1(\mathbb{R}^3)) \) to the Euler and the Navier-Stokes equations, and that a subsequence of regularized solutions \( (v^\epsilon) \) converges to \( v^\nu \).

At first, we need to prove the following a priori higher-order energy estimate that is independent of \( \epsilon \).

**Proposition 3.4 (The \( H^m \) Energy Estimate).** Let \( v_0 \in V^m \). Then the unique regularized solution \( v^\epsilon \in C^1([0, \infty); V^m) \) to the equation (3.16) satisfies

\[
\frac{1}{2} \frac{d}{dt} \| v^\epsilon \|_m^2 + \nu \| J_\epsilon \nabla v^\epsilon \|_m^2 \leq C_m \| \nabla J_\epsilon v^\epsilon \|_{L^\infty} \| v^\epsilon \|_m^2. \quad (3.21)
\]

**Proof.** Let \( v^\epsilon \in C^1([0, \infty); V^m) \) be a smooth solution to the equation (3.16), i.e.,

\[
v^\epsilon_t = \nu J_\epsilon^2 \Delta v^\epsilon - \mathcal{P} J_\epsilon [ (J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon) ].
\]

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Take the derivative $D^\alpha$, $|\alpha| \leq m$ of this equation.

$$D^\alpha v^\epsilon_t = \nu D^\alpha J_\varepsilon^2 \Delta v^\epsilon - D^\alpha \mathcal{P} J_\varepsilon \left[ (J_\varepsilon v^\epsilon) \cdot \nabla (J_\varepsilon v^\epsilon) \right].$$

Consider the $L^2$ inner product of both sides of this equation with $D^\alpha v^\epsilon$.

$$\langle D^\alpha v^\epsilon_t, D^\alpha v^\epsilon \rangle_{L^2} = \nu \langle D^\alpha J_\varepsilon^2 \Delta v^\epsilon, D^\alpha v^\epsilon \rangle_{L^2} - \langle D^\alpha \mathcal{P} J_\varepsilon \left[ (J_\varepsilon v^\epsilon) \cdot \nabla (J_\varepsilon v^\epsilon) \right], D^\alpha v^\epsilon \rangle_{L^2}. \quad (*)$$

Note that the left-hand side of this equation can be written in the following form:

$$\langle D^\alpha v^\epsilon_t, D^\alpha v^\epsilon \rangle_{L^2} = \frac{1}{2} \int_{\mathbb{R}^N} \frac{d}{dt} (D^\alpha V^\epsilon)^2 \, dx = \frac{1}{2} \frac{d}{dt} \| D^\alpha v^\epsilon \|^2_0.$$

To the first term at the right-hand side of the equation (*), apply the properties of the mollifiers (ii) and (iii) from Theorem 2.5 and an integration by parts with the divergence theorem. Then we get

$$\langle D^\alpha J_\varepsilon^2 \Delta v^\epsilon, D^\alpha v^\epsilon \rangle_{L^2} = \frac{\text{Th.} 2.5 \ (\ii)}{\text{Th.} 2.5 \ (\ii)} = \langle J_\varepsilon^2 D^\alpha \Delta v^\epsilon, D^\alpha v^\epsilon \rangle_{L^2} = \langle J_\varepsilon^2 \Delta D^\alpha v^\epsilon, D^\alpha v^\epsilon \rangle_{L^2} \quad (*)$$

Consider now the $L^2$ inner product $\left( \mathcal{P} J_\varepsilon \left[ (J_\varepsilon v^\epsilon) \cdot \nabla (D^\alpha J_\varepsilon v^\epsilon) \right], D^\alpha v^\epsilon \right)_{L^2}$ and apply the properties of Leray’s projection operator (ii) and (iv) from Theorem 2.6 and then all operations we just have mentioned above.

$$\left( \mathcal{P} J_\varepsilon \left[ (J_\varepsilon v^\epsilon) \cdot \nabla (D^\alpha J_\varepsilon v^\epsilon) \right], D^\alpha v^\epsilon \right)_{L^2} = \frac{\text{Th.} 2.6 \ (\ii)}{\text{Th.} 2.6 \ (\ii)} = \left( J_\varepsilon \left[ (J_\varepsilon v^\epsilon) \cdot \nabla (D^\alpha J_\varepsilon v^\epsilon) \right], \mathcal{P} D^\alpha v^\epsilon \right)_{L^2} \quad (*)$$

Add this inner product (zero value) to the right-hand side of the equation (*) and then take
a sum of both sides of the resulting equation by \( \alpha \), \( 0 \leq |\alpha| \leq m \). We have

\[
\frac{1}{2} \frac{d}{dt} \| v^\epsilon \|_m^2 = -\nu \| J_\epsilon \nabla v^\epsilon \|_m^2 - \sum_{0 \leq |\alpha| \leq m} \left( D^\alpha \mathcal{P} J_\epsilon \left[ (J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon) \right], D^\alpha v^\epsilon \right)_{L^2} + \sum_{0 \leq |\alpha| \leq m} \left( \mathcal{P} J_\epsilon \left[ (J_\epsilon v^\epsilon) \cdot \nabla (D^\alpha J_\epsilon v^\epsilon) \right], D^\alpha v^\epsilon \right)_{L^2}
\]
or

\[
\frac{1}{2} \frac{d}{dt} \| v^\epsilon \|_m^2 + \nu \| J_\epsilon \nabla v^\epsilon \|_m^2 = - \sum_{0 \leq |\alpha| \leq m} \left( \left\{ D^\alpha \mathcal{P} J_\epsilon \left[ (J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon) \right] - \mathcal{P} J_\epsilon \left[ (J_\epsilon V^\epsilon) \cdot \nabla (D^\alpha J_\epsilon v^\epsilon) \right] \right\}, D^\alpha v^\epsilon \right)_{L^2} \quad (**)
\]

Work now to find a bound for the right-hand side of this equation.

\[
- \sum_{0 \leq |\alpha| \leq m} \left( \left\{ D^\alpha \mathcal{P} J_\epsilon \left[ (J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon) \right] - \mathcal{P} J_\epsilon \left[ (J_\epsilon V^\epsilon) \cdot \nabla (D^\alpha J_\epsilon v^\epsilon) \right] \right\}, D^\alpha v^\epsilon \right)_{L^2} \leq \text{Th. 2.5(ii)} \quad \text{Th. 2.6(iv)}
\]

\[
\left| \sum_{0 \leq |\alpha| \leq m} \mathcal{P} J_\epsilon \left[ D^\alpha [(J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)] - [(J_\epsilon v^\epsilon) \cdot \nabla (D^\alpha J_\epsilon v^\epsilon)] \right], D^\alpha v^\epsilon \right|_{L^2} \leq \begin{array}{l}
\text{Th. 2.6(iv)} \\
\text{Th. 2.5(iii)}
\end{array}
\]

\[
\sum_{0 \leq |\alpha| \leq m} \| D^\alpha [(J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)] - [(J_\epsilon v^\epsilon) \cdot \nabla (D^\alpha J_\epsilon v^\epsilon)] \|_0 \| D^\alpha J_\epsilon v^\epsilon \|_0 \leq \left( \text{for } 0 \leq |\alpha| \leq m, \| D^\alpha J_\epsilon v^\epsilon \|_0 \leq \| J_\epsilon v^\epsilon \|_m \right) \leq C_{m0} \| v^\epsilon \|_m \quad \text{Th. 2.5(v)} \quad \text{for } k = 0
\]

\[
C_{m0} \| v^\epsilon \|_0 \sum_{0 \leq |\alpha| \leq m} \| D^\alpha [(J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)] - [(J_\epsilon v^\epsilon) \cdot D^\alpha (\nabla J_\epsilon v^\epsilon)] \|_0 \leq (2.39)
\]

( Applying the Calculus inequality (2.39), we choose \( u = J_\epsilon v^\epsilon \) and \( v = \nabla (J_\epsilon V^\epsilon) \).)

\[
C_{m0} C \| v^\epsilon \|_m \| \nabla J_\epsilon v^\epsilon \|_{L^\infty} \| D^{m-1} \nabla J_\epsilon v^\epsilon \|_0 + \| D^m J_\epsilon v^\epsilon \|_0 \| \nabla J_\epsilon v^\epsilon \|_{L^\infty} \leq \left( \text{Use that } \| D^{m-1} \nabla J_\epsilon v^\epsilon \|_0 \leq \| D^m J_\epsilon v^\epsilon \|_0 \right)
\]

\[
2C_{m0} C \| v^\epsilon \|_m \| \nabla J_\epsilon v^\epsilon \|_{L^\infty} \| D^m J_\epsilon v^\epsilon \|_0 \leq 2C_{m0} C \| v^\epsilon \|_m \| \nabla J_\epsilon v^\epsilon \|_{L^\infty} \| J_\epsilon v^\epsilon \|_m \leq C_{m0} C \| v^\epsilon \|_m \| J_\epsilon \nabla v^\epsilon \|_{L^\infty} \quad \text{Th. 2.5(iv)}
\]

Utilizing this bound, from the equation (**), we obtain the following estimate:

\[
\frac{1}{2} \frac{d}{dt} \| v^\epsilon \|_m^2 + \nu \| J_\epsilon \nabla v^\epsilon \|_m^2 \leq C_m \| J_\epsilon \nabla v^\epsilon \|_{L^\infty} \| v^\epsilon \|_m^2. \quad (3.21)
\]

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Now show that the solutions to the regularized equation (3.16) form a contraction in the low norm $C([0, T]; L^2(\mathbb{R}^3))$.

**Lemma 3.2.** The bounded family $v^\epsilon$ forms a Cauchy sequence in $C([0, T]; L^2(\mathbb{R}^3))$. In particular, there exists a constant $\tilde{C}$, such that, for all $\epsilon$ and $\epsilon'$,

$$\sup_{0 < t < T} \| v^\epsilon - v^{\epsilon'} \|_0 \leq \tilde{C} \max \{ \epsilon, \epsilon' \}. \quad (3.22)$$

**Proof.** Let $v^\epsilon$ and $v^{\epsilon'}$ be two solutions to the equation (3.16) with parameters $\epsilon$ and $\epsilon'$ respectively, i.e.,

$$\frac{d v^\epsilon}{dt} = \nu J^2_\epsilon \Delta v^\epsilon - \mathcal{P} J_\epsilon [(J_\epsilon v^\epsilon) \cdot \nabla(J_\epsilon v^\epsilon)], \quad v^\epsilon|_{t=0} = v_0$$

and

$$\frac{d v^{\epsilon'}}{dt} = \nu J^2_{\epsilon'} \Delta v^{\epsilon'} - \mathcal{P} J_{\epsilon'} [(J_{\epsilon'} v^{\epsilon'}) \cdot \nabla(J_{\epsilon'} v^{\epsilon'})], \quad v^{\epsilon'}|_{t=0} = v_0.$$  

Subtract these equations and take $L^2$ inner product from their difference with $(v^\epsilon - v^{\epsilon'})$. Take into account that

$$\left( \frac{d}{dt} (v^\epsilon - v^{\epsilon'}), v^\epsilon - v^{\epsilon'} \right)_{L^2} = \frac{1}{2} \int_{\mathbb{R}^3} \frac{d}{dt} (v^\epsilon - v^{\epsilon'})^2 \, dx = \frac{1}{2} \frac{d}{dt} \| v^\epsilon - v^{\epsilon'} \|^2_0.$$  

We will have

$$\frac{1}{2} \frac{d}{dt} \| v^\epsilon - v^{\epsilon'} \|^2_0 = T1 + T2$$

where $T1 = \nu (J^2_\epsilon \Delta v^\epsilon - J^2_{\epsilon'} \Delta v^{\epsilon'}, v^\epsilon - v^{\epsilon'})_{L^2}$ and

$$T2 = (\mathcal{P} J_{\epsilon'} [(J_{\epsilon'} v^{\epsilon'}) \cdot \nabla(J_{\epsilon'} v^{\epsilon'})] - \mathcal{P} J_\epsilon [(J_\epsilon v^\epsilon) \cdot \nabla(J_\epsilon v^\epsilon)], v^\epsilon - v^{\epsilon'})_{L^2}.$$  

At first, find a bound for $T1$.

After adding and subtracting $J^2_\epsilon \Delta v^\epsilon$ and grouping, we get

$$T1 = \nu (J^2_\epsilon \Delta v^\epsilon - J^2_{\epsilon'} \Delta v^{\epsilon'} + J^2_\epsilon \Delta v^\epsilon - J^2_{\epsilon'} \Delta v^{\epsilon'}, v^\epsilon - v^{\epsilon'})_{L^2} = T11 + T12$$

with $T11 = \nu ((J^2_\epsilon - J^2_{\epsilon'}) \Delta v^\epsilon, v^\epsilon - v^{\epsilon'})_{L^2}$ and $T12 = \nu (J^2_\epsilon \Delta (v^\epsilon - v^{\epsilon'}), v^\epsilon - v^{\epsilon'})_{L^2}$.

For $T11$, consequently apply the Schwartz inequality, add and subtract $J_\epsilon \Delta v^\epsilon$, $J_{\epsilon'} \Delta v^{\epsilon'}$, and $\Delta v^\epsilon$, apply the Triangle inequality, and the properties of mollifiers (iv, v) from Theorem 2.5.
Then we have

\[
T_{11} \leq \|(J^2 - J_\epsilon^2)\Delta v^\epsilon\|_0 \|v^\epsilon - v'^\epsilon\|_0 = \\
\|(J^2 \Delta v^\epsilon - J_\epsilon \Delta v^\epsilon) + (J_\epsilon \Delta v^\epsilon - \Delta v^\epsilon) - (J^2_\epsilon \Delta v^\epsilon - J_\epsilon \Delta v^\epsilon)\|_0 \|v^\epsilon - v'^\epsilon\|_0 \\
\leq \|J_\epsilon (J_\epsilon \Delta v^\epsilon) - (J_\epsilon \Delta v^\epsilon)\|_0 + \|J_\epsilon \Delta v^\epsilon - \Delta v^\epsilon\|_0 + \|J_\epsilon (J_\epsilon \Delta v^\epsilon) - (J_\epsilon \Delta v^\epsilon)\|_0 + \\
\|J_\epsilon \Delta v^\epsilon - \Delta v^\epsilon\|_0 \|v^\epsilon - v'^\epsilon\|_0 \overset{\text{Th.2.5 (iv)}}{\leq} \\
(C_1 \epsilon \|J_\epsilon \Delta v^\epsilon\|_1 + C_2 \epsilon \|\Delta v^\epsilon\|_1 + C_3 \epsilon' \|J_\epsilon \Delta v^\epsilon\|_1 + C_4 \epsilon' \|\Delta v^\epsilon\|_1) \|v^\epsilon - v'^\epsilon\|_0 \overset{\text{Th.2.5 (v)}}{\leq} \\
(C_1 C_{10} \epsilon \|\Delta v^\epsilon\|_1 + C_2 \epsilon \|\Delta v^\epsilon\|_1 + C_3 C_{10} \epsilon' \|\Delta v^\epsilon\|_1 + C_4 \epsilon' \|\Delta v^\epsilon\|_1) \|v^\epsilon - v'^\epsilon\|_0 \leq \\
\bar{C} \max\{\epsilon, \epsilon'\} \|\Delta v^\epsilon\|_1 \|v^\epsilon - v'^\epsilon\|_0 \leq \bar{C} \max\{\epsilon, \epsilon'\} \|v^\epsilon\|_3 \|v^\epsilon - v'^\epsilon\|_0.
\]

Here \(\bar{C} = \max\{C_1 C_{10}, C_2, C_3 C_{10}, C_4\}\), and we used that \(\|\Delta v^\epsilon\|_1 \leq \|v^\epsilon\|_3\).

For \(T_{12}\), apply the properties of mollifiers (ii,iii) from the Theorem 2.5 and then integrate by parts.

\[
T_{12} = \nu \left(\Delta J^2_\epsilon(v^\epsilon - v'^\epsilon), v^\epsilon - v'^\epsilon\right)_{L^2} \overset{\text{Th.2.5 (ii,iii)}}{=} \nu \left(\nabla^2 J_\epsilon(v^\epsilon - v'^\epsilon), J_\epsilon(v^\epsilon - v'^\epsilon)\right)_{L^2} \\
- \nu \left(\nabla J_\epsilon(v^\epsilon - v'^\epsilon), \nabla J_\epsilon(v^\epsilon - v'^\epsilon)\right)_{L^2} = -\nu \|\nabla J_\epsilon(v^\epsilon - v'^\epsilon)\|^2_0 \leq 0.
\]

So,

\[
T_1 = T_{11} + T_{12} \leq T_{11} \leq \bar{C} \max\{\epsilon, \epsilon'\} \|v^\epsilon\|_3 \|v^\epsilon - v'^\epsilon\|_0.
\]

Now estimate the second term, \(T_{2}\). Begin from applying the symmetry property of the projection operator \(P\) (Th.2.6(iv)).

\[
T_2 = \left(\mathcal{P}\left[J_\epsilon \left[(J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)\right] - J_\epsilon \left[(J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)\right]\right] ; v^\epsilon - v'^\epsilon\right)_{L^2} = \\
\left(J_\epsilon \left[(J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)\right] - J_\epsilon \left[(J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)\right] ; \mathcal{P}v^\epsilon - \mathcal{P}v'^\epsilon\right)_{L^2}.
\]

Recall that \(\mathcal{P}v^\epsilon = v^\epsilon\) and \(\mathcal{P}v'^\epsilon = v'^\epsilon\) since \(v^\epsilon\) and \(v'^\epsilon\) are divergence-free.

In the following grouping, subtract and add \(J_\epsilon \left[(J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)\right], J_\epsilon \left[(J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)\right],\)

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\[ J'_e[(J_e v)^\epsilon) \cdot \nabla (J_e v^\epsilon)] \] and \[ J'_e[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] \]. Then we have

\[ T2 = \left( J'_e[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] - J'_e[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] + J'_e[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] - J'_e[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] + J'_e[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] - J'_e[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)], v^\epsilon - v^{\epsilon'} \right)_{L^2} = R1 + R2 + R3 + R4 + R5 \]

where

\[ R1 = \left( (J'_e - J_e)[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)], v^\epsilon - v^{\epsilon'} \right)_{L^2}, \]
\[ R2 = \left( J'_e[(J_e v^\epsilon - J_e) \cdot \nabla (J_e v^\epsilon)], v^\epsilon - v^{\epsilon'} \right)_{L^2}, \]
\[ R3 = \left( J'_e[(J_e v^\epsilon - v^\epsilon) \cdot \nabla (J_e v^\epsilon)], v^\epsilon - v^{\epsilon'} \right)_{L^2}, \]
\[ R4 = \left( J'_e[(J_e v^\epsilon') \cdot \nabla (J_e v^\epsilon)], v^{\epsilon'} - v^{\epsilon'} \right)_{L^2}, \]
\[ R5 = \left( J'_e[(J_e v^\epsilon') \cdot \nabla (J_e v^\epsilon)], v^{\epsilon'} - v^{\epsilon'} \right)_{L^2}. \]

Estimate separately each part of \( T2 \). To do that, we will use the Schwartz and triangle inequalities for \( L^2 \) norm, properties of mollifiers (Th.2.5), the Calculus inequality in Sobolev space (Th.2.4(i)), and the Sobolev inequality. Also we will prove a small auxiliary claim that we will apply to \( R3 \), and integrate by parts evaluating \( R5 \).

\[ \begin{align*}
|R1| & \leq \| (J'_e - J_e)[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] \|_0 \| v^\epsilon - v^{\epsilon'} \|_0 = \\
& \| J'_e[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] - [(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] + [(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] - J'_e[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] \|_0 \| v^\epsilon - v^{\epsilon'} \|_0 \leq \\
& \| J'_e[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] - [(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] \|_0 + \\
& \| J'_e[(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] - [(J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon)] \|_0 \| v^\epsilon - v^{\epsilon'} \|_0 \leq \\
& (C_1 \epsilon' \| (J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon) \|_1 + C_2 \epsilon \| (J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon) \|_1) \| v^\epsilon - v^{\epsilon'} \|_0 \leq \\
& (C_1 + C_2) \max\{ \epsilon, \epsilon' \} \| (J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon) \|_1 \| v^\epsilon - v^{\epsilon'} \|_0 \leq \\
& C_3 (C_1 + C_2) \max\{ \epsilon, \epsilon' \} \left( \| J_e v^\epsilon \|_{L^\infty} \| \nabla^2 (J_e v^\epsilon) \|_0 + \| \nabla (J_e v^\epsilon) \|_0 \| \nabla (J_e v^\epsilon) \|_{L^\infty} \right) \leq \| v^\epsilon - v^{\epsilon'} \|_0 \leq 
\end{align*} \]
\[ C_3 (1 + C_2) \max \{\epsilon, \epsilon'\} (C_4 \| v^\epsilon \|_{L^\infty} \| J_\epsilon v^\epsilon \|_2 + \| J_\epsilon (\nabla v^\epsilon) \|_{L^\infty}) \\]
\[ \leq \| v^\epsilon - v^{\epsilon'} \|_0 \]

\[ C_3 (1 + C_2) \max \{\epsilon, \epsilon'\} (C_4 C_5 \| v^\epsilon \|_s C_{20} \| v^{\epsilon'} \|_2 + C_{10} \| v^{\epsilon'} \|_1 \| \nabla v^{\epsilon'} \|_{L^\infty}) \| v^\epsilon - v^{\epsilon'} \|_0 \leq \]

\[ C_3 (1 + C_2) \max \{\epsilon, \epsilon'\} (C_4 C_5 C_{20} \| v^\epsilon \|_s \| v^{\epsilon'} \|_2 + C_{10} \| v^{\epsilon'} \|_1 C_6 \| v^\epsilon \|_{s+1}) \| v^\epsilon - v^{\epsilon'} \|_0 \leq \]

\[ \bar{C}_1 \max \{\epsilon, \epsilon'\} \| v^\epsilon \|_m \| v^{\epsilon'} - v^\epsilon \|_0 \]

with \( \bar{C}_1 = (1 + C_2) C_3 \max \{C_4 C_5 C_{20}, C_{10} C_6\} \), and we choose \( s = m - 1 \) so that \( m > \frac{N}{2} + 1 \).

Here we used that \( \| v^\epsilon \|_1 \leq \| v^\epsilon \|_2 \leq \| v^\epsilon \|_{s+1} \) where \( s > \frac{N}{2} \).

Now estimate \( R2 \).

\[ |R2| = \left| (J_\epsilon' [(J_\epsilon' - J_\epsilon) v^\epsilon \cdot \nabla (J_\epsilon v^\epsilon)], v^\epsilon - v^{\epsilon'}) \right|_{L^2} \]

\[ \leq C_1 \max \{\epsilon, \epsilon'\} \|v^\epsilon\|_s \|J_\epsilon (\nabla v^\epsilon)\|_0 + \|J_\epsilon v^\epsilon - J_\epsilon v^{\epsilon'} + v^{\epsilon'}\|_0 C_5 \|J_\epsilon v^\epsilon\|_{s+1} \]

\[ \leq C_1 C_00 \left( C_2 \| (J_\epsilon' - J_\epsilon) v^\epsilon \|_s \| J_\epsilon (\nabla v^\epsilon) \|_0 + \| J_\epsilon v^\epsilon - J_\epsilon v^{\epsilon'} + v^{\epsilon'} \|_0 C_5 \| J_\epsilon v^\epsilon \|_{s+1} \right) \]

\[ \leq C_1 C_00 \left[ C_2 C_00 (\| J_\epsilon v^\epsilon - v^\epsilon \|_s + \| J_\epsilon v^\epsilon - v^{\epsilon'} \|_s) \| v^{\epsilon'} \|_{s+1} \right] \]

\[ \leq C_1 C_00 \left[ C_2 C_00 (\| J_\epsilon v^\epsilon - v^\epsilon \|_s + \| J_\epsilon v^\epsilon - v^{\epsilon'} \|_s) \| v^{\epsilon'} \|_1 \right] \]

\[ \leq (C_4 \epsilon' \| v^{\epsilon'} \|_1 + C_5 \epsilon \| v^{\epsilon'} \|_1) C_3 C_{(s+1)0} \| v^\epsilon \|_{s+1} \]

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\[
C_1 C_0 \left[ C_2 C_0 (C_6 + C_7) + C_3 C_{(s+1)0} (C_4 + C_5) \right] \max \{ \epsilon, \epsilon' \} \| v^\epsilon \|_{s+1} \| v^\epsilon - v'^\epsilon \|_0 \leq \tilde{C}_2 \max \{ \epsilon, \epsilon' \} \| v^\epsilon \|_m^2 \| v^\epsilon - v'^\epsilon \|_0
\]

where \( \tilde{C}_2 = C_1 C_0 \max \{ C_2 C_0 (C_6 + C_7), C_3 C_{(s+1)0} (C_4 + C_5) \} \), and we choose \( s = m - 1 \) so that \( m > \frac{N}{2} + 1 \).

Also, we used here that \( \| v^\epsilon \|_1 \leq \| v^\epsilon \|_{s+1} \) with \( s > \frac{N}{2} \).

To estimate \( R3 \), we need the following claim.

**Claim.** Let \( a, b \colon \mathbb{R}^N \to \mathbb{R}^N \) and \( a_i, b_i \in H^m(\mathbb{R}^N) \), \( i = 1, 2, \ldots, N \), \( m \in \mathbb{Z}^+ \cup \{ 0 \} \). Then

\[
| (b \cdot \nabla a, b)_{L^2} | \leq N^3 \| b \|_0^2 \| \nabla a \|_{L^\infty}.
\]

**Proof.**

\[
| (b \cdot \nabla a, b)_{L^2} | = \left| \int_{\mathbb{R}^N} (b \cdot \nabla a) \cdot b \, dx \right| \leq \int_{\mathbb{R}^N} \left| (b \cdot \nabla a) \right| \cdot b \, dx
\]

\[
= \int_{\mathbb{R}^N} \left| \sum_{i=1}^N \left( \sum_{j=1}^N b_j \frac{\partial a_i}{\partial x_j} \right) b_i \right| \, dx \leq \int_{\mathbb{R}^N} \left| \sum_{i,j=1}^N b_i b_j \left| \frac{\partial a_i}{\partial x_j} \right| \right| \, dx \leq \sum_{i,j=1}^N \sup_{1 \leq i,j \leq N} \left| \frac{\partial a_i}{\partial x_j} \right| \int_{\mathbb{R}^N} |b_i b_j| \, dx
\]

\[
= \sum_{i,j=1}^N \left\| \frac{\partial a_i}{\partial x_j} \right\|_{L^\infty} \int_{\mathbb{R}^N} |b_i b_j| \, dx \leq N^2 \max_{1 \leq i,j \leq N} \left\| \frac{\partial a_i}{\partial x_j} \right\|_{L^\infty} \int_{\mathbb{R}^N} \sum_{i,j=1}^N |b_i b_j| \, dx
\]

\[
= N^2 \| \nabla a \|_{L^\infty} \int_{\mathbb{R}^N} \left( \sum_{i=1}^N b_i^2 + 2 \sum_{i<j} b_i b_j \right) \, dx
\]

\[
= N^2 \| \nabla a \|_{L^\infty} \int_{\mathbb{R}^N} \left( N \sum_{i=1}^N b_i^2 - \sum_{i<j} (|b_i| - |b_j|)^2 \right) \, dx
\]

\[
= N^3 \| \nabla a \|_{L^\infty} \| b \|_0^2.
\]

We used the following definition of \( L^\infty \)-norm:

If \( a = (a_1, a_2, \ldots, a_N) \), \( a_i \in C^1(X) \) for some open set \( X \),

then \( \| \nabla a \|_{L^\infty(X)} := \max_{1 \leq i,j \leq N} \left\| \frac{\partial a_i}{\partial x_j} \right\|_{L^\infty(X)} \).

Alternatively, we could use an equivalent norm

\[
\| \nabla a \|_{L^\infty} := \sum_{i,j=1}^N \left\| \frac{\partial a_i}{\partial x_j} \right\|_{L^\infty}.
\]
They are equivalent since
\[
\max_{1 \leq i, j \leq N} \| \frac{\partial a_i}{\partial x_j} \|_{L^\infty} \leq \sum_{i, j = 1}^{N} \| \frac{\partial a_i}{\partial x_j} \|_{L^\infty} \leq N^2 \max_{1 \leq i, j \leq N} \| \frac{\partial a_i}{\partial x_j} \|_{L^\infty}.
\]

Return to our estimate of R3. Choose \( a = J_\epsilon v^\epsilon \) and \( b = J_\epsilon (v^\epsilon' - v^\epsilon) \).

\[
\tag*{(i)} ||v^\epsilon - v^\epsilon'||_{L^\infty} \leq N^2 \sup_{1 \leq i, j \leq N} \| \frac{\partial a_i}{\partial x_j} \|_{L^\infty} \leq \frac{N^2}{2} S_0
\]

with \( C_3 = N^2 S_0^{-1} C_1 \), and we choose \( s = m - 1 \) so that \( m > \frac{N}{2} + 1 \).

\[
\begin{align*}
|R4| &= |(J_\epsilon'[J_\epsilon'(v^\epsilon' - v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)], v^\epsilon - v^\epsilon')_{L^2}| & \text{Th. 2.5} \\
&\leq |[(J_\epsilon'(v^\epsilon' - v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)], -J_\epsilon'(v^\epsilon' - v^\epsilon)_{L^2}| & \text{Claim} \\
&\leq |(b \cdot \nabla a, b)_{L^2}| & \text{Sobolev Ineq.} \\
&\leq N^2 C_0 \| v^\epsilon' - v^\epsilon \|_{L^\infty}^2 \leq N^2 C_0 \| v^\epsilon' - v^\epsilon \|_{L^\infty}^2 & \text{Sobolev Ineq.} \tag*{(ii)} \\
&\leq N^2 \| v^\epsilon - v^\epsilon' \|_{L^\infty}^2 C_1 \| V^\epsilon \|_{s+1} = C_3 \| v^\epsilon' \|_{m} \| v^\epsilon - v^\epsilon' \|_{L^\infty}^2 & \text{Sobolev Ineq.} \tag*{(iii)}
\end{align*}
\]
\[ C_1C_{00} \left[ C_2 \| v^\epsilon \|_s (C_3 \epsilon + C_4 \epsilon) \| v^\epsilon \|_2 + C_{00} \| v^\epsilon \|_0 (C_5 \epsilon + C_6 \epsilon) \| v^\epsilon \|_{s+2} \right] \| v^\epsilon - v^{\epsilon'} \|_0 \leq \\
C_1C_{00} \max \{ \epsilon, \epsilon' \} \left[ C_2 (C_3 + C_4) \| v^\epsilon \|_s \| v^{\epsilon'} \|_2 + C_{00} (C_5 + C_6) \| v^\epsilon \|_0 \| v^{\epsilon'} \|_{s+2} \right] \| v^\epsilon - v^{\epsilon'} \|_0 \leq \\
\bar{C}_4 \max \{ \epsilon, \epsilon' \} \| v^\epsilon \|_s \| v^{\epsilon'} \|_{s+2} \| v^\epsilon - v^{\epsilon'} \|_0 \leq \bar{C}_4 \max \{ \epsilon, \epsilon' \} \| v^\epsilon \|_m \| v^{\epsilon'} \|_m \| v^\epsilon - v^{\epsilon'} \|_0 \]  
where \( \bar{C}_4 = C_1C_{00} \max \{ C_2 (C_3+C_4), C_{00} (C_5+C_6) \} \), and we choose \( s = m - 2 \) so that \( m > \frac{N}{2} + 2 \).

\[
R5 = (J_{\epsilon'} (\{J_{\epsilon'} v^{\epsilon'}\} \cdot \nabla J_{\epsilon'} (v^\epsilon - v^{\epsilon'})), v^\epsilon - v^{\epsilon'})_{L^2}^{Th.2.5} = (\{J_{\epsilon'} v^{\epsilon'}\} \cdot \nabla J_{\epsilon'} (v^\epsilon - v^{\epsilon'}), J_{\epsilon'} (v^\epsilon - v^{\epsilon'}))_{L^2}^{Th.2.5} = \\
= \int_{\mathbb{R}^N} \left[(J_{\epsilon'} v^{\epsilon'}) \cdot \nabla J_{\epsilon'} (v^\epsilon - v^{\epsilon'}) \right] \cdot [-J_{\epsilon'} (v^{\epsilon'} - v^\epsilon)] \, dx.
\]

Denote \( J_{\epsilon'} v^{\epsilon'} = a \) and \( J_{\epsilon'} (v^{\epsilon'} - v^\epsilon) = b \) where \( a, b : \mathbb{R}^N \to \mathbb{R}^N \).

Then

\[
R5 = -\int_{\mathbb{R}^N} (a \cdot \nabla b) \cdot b \, dx = -\int_{\mathbb{R}^N} \sum_{j=1}^N \sum_{i=1}^N a_i \frac{\partial b_j}{\partial x_i} b_j \, dx = -\int_{\mathbb{R}^N} \sum_{j=1}^N \sum_{i=1}^N a_i \frac{\partial b_j}{\partial x_i} b_j \, dx = \\
- \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i=1}^N a_i \sum_{j=1}^N \frac{\partial |b_j|^2}{\partial x_i} \, dx = - \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i=1}^N a_i \frac{\partial |b|^2}{\partial x_i} \, dx = \\
- \frac{1}{2} \int_{\mathbb{R}^N} a \cdot \nabla (|b|^2) \, dx \overset{\text{Int. by parts}}{=} \frac{1}{2} \int_{\mathbb{R}^N} (\nabla \cdot a) |b|^2 \, dx = 0
\]

since \( v^\epsilon, v^{\epsilon'} \) and so \( a, b \) vanish at infinity, and

\[
\nabla \cdot a = \text{div} a = \text{div} (J_{\epsilon'} v^{\epsilon'}) \overset{Th.2.5 (ii)}{=} J_{\epsilon'} (\text{div} v^{\epsilon'}) = 0
\]
because \( v^{\epsilon'} \in V^m (\mathbb{R}^N) \).

Putting \( T1 \) and \( T2 \) together, we get for \( m > \frac{N}{2} + 2 \).

\[
\frac{d}{dt} \| v^\epsilon - v^{\epsilon'} \|_0^2 = T1 + T2 = T1 + \sum_{i=1}^5 |R_i| \leq T1 + \sum_{i=1}^5 |R_i| \leq \\
\bar{C}_1 \max \{ \epsilon, \epsilon' \} \| v^\epsilon \|_3 \| v^\epsilon - v^{\epsilon'} \|_0 + \bar{C}_1 \max \{ \epsilon, \epsilon' \} \| v^{\epsilon'} \|_m^2 \| v^\epsilon - v^{\epsilon'} \|_0 + \\
\bar{C}_2 \max \{ \epsilon, \epsilon' \} \| v^\epsilon \|_m^2 \| v^\epsilon - v^{\epsilon'} \|_0 + \bar{C}_5 \| v^\epsilon \|_m \| v^\epsilon - v^{\epsilon'} \|_0 + \\
\bar{C}_4 \max \{ \epsilon, \epsilon' \} \| v^{\epsilon'} \|_m \| v^\epsilon - v^{\epsilon'} \|_0 + 0 \leq \\
\bar{C}_5 \| v^\epsilon \|_m \max \{ \epsilon, \epsilon' \} (1 + 2 \| v^\epsilon \|_m + \| v^{\epsilon'} \|_m) + \| v^\epsilon - v^{\epsilon'} \|_0 \| v^\epsilon - v^{\epsilon'} \|_0 \leq \\
\bar{C}_5 \| v^\epsilon \|_m (1 + 2 \| v^\epsilon \|_m + \| v^{\epsilon'} \|_m) \max \{ \epsilon, \epsilon' \} + \| v^\epsilon - v^{\epsilon'} \|_0 \| v^\epsilon - v^{\epsilon'} \|_0
\]

where \( \bar{C}_5 = \max \{ \bar{C}, \bar{C}_1 + \bar{C}_2, \bar{C}_3, \bar{C}_4 \} \) and \( \| v^\epsilon \|_3 \leq \| v^{\epsilon'} \|_m \) for \( m > \frac{N}{2} + 2 \).
Now use that the family \((v^\epsilon)\) is bounded. This will be proved as part 1) of the proof of Theorem 3.4.

Let \(v^\epsilon, v'^\epsilon \in \mathcal{O}_M = \{v \in V^m(\mathbb{R}^N) \mid \|v\|_m < M\}\).

Then \(1 + 2\|v^\epsilon\|_m + \|v'^\epsilon\|_m < 1 + 3M\). Denote \(C(M) = \bar{C}_5M(1 + 3M) > 0\).

We obtain the following IVP for linear OD inequality:

\[
\begin{cases}
 \frac{1}{2} 2 \|v^\epsilon - v'^\epsilon\|_0 \frac{dt}{dt} \leq C(M) \max\{\epsilon, \epsilon'\} + \|v^\epsilon - v'^\epsilon\|_0 \|v^\epsilon - v'^\epsilon\|_0, \\
\|v^\epsilon - v'^\epsilon\|_0 \big|_{t=0} = \|v_0^\epsilon - v_0'^\epsilon\|_0 = \|v_0 - v_0\|_0 = 0.
\end{cases}
\]

Take

\[
\|v^\epsilon - v'^\epsilon\|_0 = \varphi(t)e^{C(M)t} \quad \text{with} \; \varphi(0) = 0.
\]

Then we have

\[
\frac{d\varphi}{dt} e^{C(M)t} \leq C(M) \max\{\epsilon, \epsilon'\}.
\]

Integrating in \(t\), we get

\[
\int_0^t d\varphi \leq C(M) \max\{\epsilon, \epsilon'\} \int_0^t e^{-C(M)t} \, dt
\]

or

\[
\varphi(t) \leq \max\{\epsilon, \epsilon'\} (1 - e^{-C(M)t}).
\]

So,

\[
\|v^\epsilon - v'^\epsilon\|_0 \leq \max\{\epsilon, \epsilon'\} (e^{C(M)t} - 1) \leq \max\{\epsilon, \epsilon'\} e^{C(M)t}.
\]

Therefore

\[
\sup_{0 \leq t \leq T} \|v^\epsilon - v'^\epsilon\|_0 \leq \tilde{C} \max\{\epsilon, \epsilon'\}
\]

(3.22)

where \(\tilde{C} = e^{C(M)T}\) depends on some selected \(M > 0\) and \(T\).

Thus, \((v^\epsilon)\) is a Cauchy sequence in \(C([0, T]; L^2(\mathbb{R}^3))\).

We now prove the main theorem of this section (and entire chapter).

**Theorem 3.4** (Local-in-Time Existence of Solutions to the Euler and the Navier-Stokes Equations). *Given an initial condition \(v_0 \in V^m\), \(m \geq \|\frac{\partial}{\partial t}\| + 2\), then the following results are true.*

1) **There exists a time** \(T\) **with the rough upper bound**

\[
T < \frac{1}{C_m \|v_0\|_m}
\]

(3.23)

**such that for any viscosity** \(0 \leq \nu < \infty\) **there exists the unique solution** \(v^\nu \in C([0, T]; C^2(\mathbb{R}^3)) \cap C^1([0, T]; C(\mathbb{R}^3))\) **to the Euler or the Navier-Stokes equations.**
The solution \( v^\nu \) is the limit of subsequence of approximate solutions \( v^\epsilon \) to the equations (3.16) and (3.17).

ii) The approximate solution \( v^\epsilon \) and the limit \( v^\nu \) satisfy the higher-order energy estimates

\[
\begin{align*}
\sup_{0 \leq t \leq T} \| v^\epsilon \|_m & \leq \frac{\| v_0 \|_m}{1 - C'_m T \| v_0 \|_m}, \\
\sup_{0 \leq t \leq T} \| v^\nu \|_m & \leq \frac{\| v_0 \|_m}{1 - C'_m T \| v_0 \|_m}.
\end{align*}
\]

Proof.

1) Show that \( (v^\epsilon) \), family of solutions of the regularized equation, is uniformly bounded in \( H^m \), \( \forall m > \frac{N}{2} + 1 \).

Apply the energy estimate (3.21) and the Sobolev inequality (2.21) to bound the time derivative of \( \| v^\epsilon \|_m \).

\[
\frac{d}{dt} \| v^\epsilon \|_m \leq C_m \| J_{\epsilon} \nabla v^\epsilon \|_{L^\infty} \| v^\epsilon \|_m - v \| J_{\epsilon} \nabla v^\epsilon \|_m^2 \leq C_m \| J_{\epsilon} \nabla v^\epsilon \|_{L^\infty} \| v^\epsilon \|_m - C'_m \| \nabla v^\epsilon \|_{L^\infty} \| v^\epsilon \|_m \leq C_m \| v^\epsilon \|_m^2.
\]

Thus, \[
\frac{d}{dt} \| v^\epsilon \|_m \leq C'_m \| v^\epsilon \|_m^2.
\]

Here we applied the Sobolev inequality to \( \| \nabla v^\epsilon \|_{L^\infty} \) \( (k = 1) \). So, it must be that \( m > \frac{N}{2} + 1 \).

Integrate the inequality (3.25) in time.

\[
\int_{\| v_0 \|_m}^{\| v^\epsilon \|_m} \frac{d\| v^\epsilon \|_m}{\| v^\epsilon \|_m^2} \leq \int_0^t C'_m \, dt,
\]

\[
- \frac{1}{\| v^\epsilon \|_m} + \frac{1}{\| v_0 \|_m} \leq C'_m t \text{ or } \| v^\epsilon \|_m \leq \frac{1}{\| v_0 \|_m - C'_m t} \quad \forall t \in [0, T].
\]

The last upper bound makes sense when \( \| v_0 \|_m^{-1} - C'_m t > 0 \) only. Therefore \( T < \frac{1}{C'_m \| v_0 \|_m} \).

The bound \( \frac{1}{\| v_0 \|_m - C'_m t} \) is monotonically increasing on \([0, T]\). Then, for all \( \epsilon \),

\[
\sup_{0 \leq t \leq T} \| v^\epsilon \|_m \leq \frac{\| v_0 \|_m}{1 - C'_m T \| v_0 \|_m}
\]

(3.24)

Thus, for \( T < \frac{1}{C'_m \| v_0 \|_m} \), \( \forall m > \frac{N}{2} + 1 \), the family \( (v^\epsilon) \) is uniformly bounded in \( C([0, T]; H^m(\mathbb{R}^N)) \).

2) Show that the family of time derivatives \( (\frac{dv^\epsilon}{dt}) \) is uniformly bounded in \( H^{m-2}(\mathbb{R}^N) \).
From the regularized equation (3.16), we have
\[
\left\| \frac{dv^\epsilon}{dt} \right\|_{m-2} \leq \left\| F^{(1)}_{\epsilon}(v^\epsilon) \right\|_{m-2} + \left\| F^{(2)}_{\epsilon}(v^\epsilon) \right\|_{m-2}.
\]
(a) \[
\left\| F^{(1)}_{\epsilon}(v^\epsilon) \right\|_{m-2} = \nu \left\| J^2 \Delta v^\epsilon \right\|_{m-2} \leq \nu \left\| J^2 v^\epsilon \right\|_{m} \leq \nu \left\| J^2 v^\epsilon \right\|_{m} \leq \nu C_m^2 \left\| v^\epsilon \right\|_{m} = \nu C^{(1)}(m) \left\| v^\epsilon \right\|_{m} \quad \text{with} \quad C^{(1)}(m) = C_m^2.
\]
(b) \[
\left\| F^{(2)}_{\epsilon}(v^\epsilon) \right\|_{m-2} = \left\| \mathcal{P} J_{\epsilon}[(J_{\epsilon} v^\epsilon) \cdot \nabla (J_{\epsilon} v^\epsilon)] \right\|_{m-2} \leq \left\| J_{\epsilon}[(J_{\epsilon} v^\epsilon) \cdot \nabla (J_{\epsilon} v^\epsilon)] \right\|_{m-2} \leq C_{m0} \left\| (J_{\epsilon} v^\epsilon) \cdot \nabla (J_{\epsilon} v^\epsilon) \right\|_{m-2} \leq C_{m0} \left\| (J_{\epsilon} v^\epsilon) \cdot \nabla (J_{\epsilon} v^\epsilon) \right\|_{m-2} \leq C_{m0} \left\| (J_{\epsilon} v^\epsilon) \cdot \nabla (J_{\epsilon} v^\epsilon) \right\|_{m-2} \leq C_{m0} C \left[ \left\| J_{\epsilon} v^\epsilon \right\|_{L^\infty} \left( \left\| D^{m-2} \nabla (J_{\epsilon} v^\epsilon) \right\|_{0} + \left\| D^{m-2} (J_{\epsilon} v^\epsilon) \right\|_{0} \right) \right] \leq C_{m0} \left[ \frac{C_{m0}}{\epsilon^{(N/2)+1}} \left\| v^\epsilon \right\|_{0} \left\| D^{m-1} (J_{\epsilon} v^\epsilon) \right\|_{0} + \left\| J_{\epsilon} v^\epsilon \right\|_{m-2} \frac{C_{m0}}{\epsilon^{(N/2)+1}} \left\| v^\epsilon \right\|_{0} \right] \leq C_{m0} \left[ \frac{C_{m0}}{\epsilon^{(N/2)+1}} \left( C_0 C_{(m-1)0} \left\| v^\epsilon \right\|_{m-1} + \frac{C_{(m-2)0} C_{1}}{\epsilon} \left\| v^\epsilon \right\|_{m} \right) \right] \leq \frac{C_{m0} C}{\epsilon^{(N/2)+1}} \left( C_0 C_{(m-1)0} \epsilon + C_{(m-2)0} C_{1} \right) \left\| v^\epsilon \right\|_{m}^2 = C^{(2)}(\epsilon, N, m) \left\| v^\epsilon \right\|_{m}^2
\]
with \( C^{(2)}(\epsilon, N, m) = \frac{C_{m0} C}{\epsilon^{(N/2)+1}} \left( C_0 C_{(m-1)0} \epsilon + C_{(m-2)0} C_{1} \right) \).

Here we used that \( \left\| v^\epsilon \right\|_{0} \leq \left\| v^\epsilon \right\|_{m} \) and \( \left\| v^\epsilon \right\|_{m-2} \leq \left\| v^\epsilon \right\|_{m-1} \leq \left\| v^\epsilon \right\|_{m} \).

Also, we assumed here that \( m-2 > \frac{N}{2} \). So, \( m > \frac{N}{2} + 2 \).

(c) Putting together the estimates in \( H^{m-2} \) of \( F^{(1)}_{\epsilon} \) and \( F^{(2)}_{\epsilon} \), we obtain
\[
\left\| \frac{dv^\epsilon}{dt} \right\|_{m-2} \leq \nu C^{(1)}(m) \left\| v^\epsilon \right\|_{m} + C^{(2)}(\epsilon, N, m) \left\| v^\epsilon \right\|_{m}^2
\]
Since the family \( v^\epsilon \) is uniformly bounded in \( H^m(\mathbb{R}^N) \) for \( m > \frac{N}{2} + 1 \), then the family \( \left\{ \frac{dv^\epsilon}{dt} \right\} \) is uniformly bounded in \( H^{m-2}(\mathbb{R}^N) \) for \( m > \frac{N}{2} + 2 \).

3) Let \( v^\epsilon, v'^\epsilon \in \mathcal{O}_M = \{ v \in V^m \mid \left\| v \right\|_{m} < M \} \) and \( \mathcal{M} = \frac{\left\| v_0 \right\|_{m}}{1-C_m^2 \left\| v_0 \right\|_{m}}. \)

By the Lemma 3.2, \( \sup_{0 \leq t \leq T} \left\| v^\epsilon - v'^\epsilon \right\|_{0} \leq C \left( \left\| v_0 \right\|_{m}, T \right) \max \{ \epsilon, \epsilon' \} \), i.e., the family \( v^\epsilon \subset \mathcal{O}_M \) forms a Cauchy sequence in \( C([0, T]; L^2(\mathbb{R}^3)) \), that is a Banach space (Lemma 3.1). Then as \( \epsilon \to 0^+ \), \( (v^\epsilon) \) strongly converges to some \( v \in C([0, T]; L^2(\mathbb{R}^3)) \), i.e., \( \exists v \in C([0, T]; L^2(\mathbb{R}^3)) \) such that
\[
\sup_{0 \leq t \leq T} \left\| v^\epsilon - v \right\|_{0} \leq C \epsilon, \quad \forall \epsilon > 0. \quad (3.26)
\]
4) To this point, we proved that \( (\nu^\epsilon) \) is uniformly bounded in a high norm, and it strongly converges in \( L^2 \)-norm (in \( H^0 \)).

Next, show that \( (\nu^\epsilon) \) has strong convergence in all intermediate norms. For this purpose, we will use the following interpolation lemma for Sobolev spaces.

**Lemma 3.3** (Interpolation in Sobolev Spaces). [1] Given \( \epsilon > 0 \), there exists a constant \( C_s \) so that for all \( \nu \in H^s(\mathbb{R}^N) \) and \( 0 < s' < s \),

\[
\|\nu\|_{s'} \leq C_s \|\nu\|_0^{1-(s'/s)} \|\nu\|_s^{s'/s}. \tag{3.27}
\]

Let \( s = m \) and \( 0 < m' < m \). Apply the Interpolation lemma to the difference \( \nu^\epsilon - \nu \) and take the supremum by \( t \in [0, T] \). We have

\[
\sup_{0 \leq t \leq T} \|\nu^\epsilon - \nu\|_{m'} \leq C_m \sup_{0 \leq t \leq T} \|\nu^\epsilon - \nu\|_0^{1-(m'/m)} \sup_{0 \leq t \leq T} \|\nu^\epsilon - \nu\|_{m/m}.
\]

Since \( (\nu^\epsilon) \) is uniformly bounded in \( H^m(\mathbb{R}^N) \) by (3.24), we know \( \exists \nu \in H^m(\mathbb{R}^N) \) and a subsequence \( (\nu^\epsilon_j) \) such that \( \nu^\epsilon_j \to \nu \) weakly in \( H^m(\mathbb{R}^N) \) [5, Theorem 3.7]. It follows that [5, Proposition 3.6]

\[
\|\nu\|_m \leq \lim_{j \to \infty} \|\nu^\epsilon_j\|_m \leq \frac{\|\nu_0\|_m}{1 - C_m' T \|\nu_0\|_m}.
\]

Hence using the triangle inequality, we obtain

\[
\sup_{0 \leq t \leq T} \|\nu^\epsilon - \nu\|_m \leq \sup_{0 \leq t \leq T} \|\nu^\epsilon\|_m + \sup_{0 \leq t \leq T} \|\nu\|_m \leq \frac{2\|\nu_0\|_m}{1 - C_m' T \|\nu_0\|_m}.
\]

Putting this estimate together with the bound (3.26), we have

\[
\sup_{0 \leq t \leq T} \|\nu^\epsilon - \nu\|_{m'} \leq C_m [C(\|\nu_0\|_m, T) \epsilon]^{1-(m'/m)} \left( \frac{2\|\nu_0\|_m}{1 - C_m' T \|\nu_0\|_m} \right)^{m'/m} = C_1(\|\nu_0\|_m, T) \epsilon^{1-(m'/m)}.
\]

Therefore \( (\nu^\epsilon) \) strongly converges to \( \nu \) as \( \epsilon \to 0 \) in \( C([0, T]; H^{m'}(\mathbb{R}^N)) \), \( \forall 0 \leq m' < m \).

5) Employing the Sobolev inequality for \( (\nu^\epsilon - \nu) \) with \( m' = s + k, \ k = 2 \), and \( s > \frac{N}{2} \), we get

\[
\|\nu^\epsilon - \nu\|_{C^2(\mathbb{R}^N)} \leq C \|\nu^\epsilon - \nu\|_{m'} \text{ with some constant } C.
\]

So,

\[
\sup_{0 \leq t \leq T} \|\nu^\epsilon - \nu\|_{C^2(\mathbb{R}^N)} \leq C_2 \epsilon^{1-(m'/m)} \text{ with } C_2 = CC_1.
\]
Thus, \((v^\epsilon)\) strongly converges to \(v\) in \(C([0, T]; C^2(\mathbb{R}^N))\) with \(\frac{N}{2} + 2 < m' < m\) or \(\left\lceil \frac{N}{2} \right\rceil + 2 \leq m' < m\) where \(\left\lceil \frac{N}{2} \right\rceil\) is a ceiling function of \(\frac{N}{2}\), the least integer greater or equal to \(\frac{N}{2}\).

6) Note that strong convergence in \(C([0, T]; C^2(\mathbb{R}^N))\) implies an uniform convergence for the sequence \((v^\epsilon)\) and for sequences of spatial derivatives of \(v^\epsilon\) up to the second order since from the last inequality, we have

\[
\sup_{0 \leq t \leq T} \sum_{|\alpha| \leq 2} \sup_{x \in \mathbb{R}^N} |D_\alpha^2 (v^\epsilon - v)| < C_2 \epsilon^{1-(m'/m)} \quad \text{with} \quad \left\lceil \frac{N}{2} \right\rceil + 2 \leq m' < m.
\]

Alternatively, we can apply the following corollary of the Arzela-Ascoli theorem:

If a sequence of differentiable functions and sequence of their derivatives are uniformly bounded, then there exists a subsequence that converges uniformly.

Hence subsequences of \((\nabla v^\epsilon)\) and \((\Delta v^\epsilon)\) converge uniformly to \(\nabla v\) and \(\Delta v\) respectively as \(\epsilon \to 0^+\). The projection operator \(P\) is uniformly continuous. Also, using the properties of mollifiers (i,iv) from the Theorem 2.5, we get

\[
\lim_{\epsilon \to 0^+} J_\epsilon v^\epsilon = \lim_{\epsilon \to 0^+} v^\epsilon = v \quad \text{(strong and uniform convergence)}.
\]

Then \(F_\epsilon(v^\epsilon) = \nu J_\epsilon^2 \Delta v^\epsilon - PJ_\epsilon [(J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon)]\) converges uniformly in \(C([0, T]; C(\mathbb{R}^3))\) to \(w = [\nu \Delta v - P(v \cdot \nabla v)] \in C([0, T]; C(\mathbb{R}^3))\) as \(\epsilon \to 0^+\).

Note that \(v^\epsilon \to v\) in \(C([0, T]; C^2(\mathbb{R}^3))\) implies that this is true in \(C([0, T]; C(\mathbb{R}^3))\) since \(C^2(\mathbb{R}^3) \subseteq C(\mathbb{R}^3)\).

By the equation (3.16), \(v^\epsilon_t = F_\epsilon(v^\epsilon)\).

Hence there exists subsequence \((v^\epsilon_j)\) (induced by uniformly convergent subsequence \((v^\epsilon_j)\)) that uniformly converges to \(w\) in \(C([0, T]; C(\mathbb{R}^N))\).

Justify that \(w = v_t\) in the Banach space \(C([0, T]; C(\mathbb{R}^3))\).

Using Fundamental Theorem of Calculus, we can write

\[
v^\epsilon(t, x) = \int_0^t v^\epsilon_s(s, x) \, ds
\]

with \(v^\epsilon_s(s, x)\) continuous in both variables.

Now, as \(\epsilon \to 0^+\), \(v^\epsilon_s(s, x) \to w(s, x)\) uniformly in \(s\) by previous, and so

\[
\int_0^t v^\epsilon_s(s, x) \, ds \to \int_0^t w(s, x) \, ds \quad \text{as} \quad \epsilon \to 0^+.
\]

Also, we know that as \(\epsilon \to 0^+\), \(v^\epsilon(t, x) \to v(t, x)\) uniformly in \(t\).
It follows that
\[ v(t, x) = \int_0^t w(s, x) \, ds \]

Consider
\[ \frac{v(t+h,x)-v(t,x)}{h} = \frac{1}{h} \left[ \int_0^{t+h} w(s, x) \, ds - \int_0^t w(s, x) \, ds \right] = \frac{1}{h} \int_t^{t+h} w(s, x) \, ds. \]

Then in \( C(\mathbb{R}^3) \) norm we have
\[
\left\| \frac{v(t+h,x)-v(t,x)}{h} - w(t, x) \right\|_{C(\mathbb{R}^3)} = \left\| \frac{1}{h} \int_t^{t+h} w(s, x) \, ds - \frac{1}{h} \int_t^{t+h} w(t, x) \, ds \right\|_{C(\mathbb{R}^3)} \\
= \frac{1}{h} \left\| \int_t^{t+h} [w(s, x) - w(t, x)] \, ds \right\|_{C(\mathbb{R}^3)} \leq \frac{1}{h} \int_t^{t+h} \left\| w(s, x) - w(t, x) \right\|_{C(\mathbb{R}^3)} \, ds \\
\leq \frac{1}{h} \mathcal{K} \sup_{t \leq s \leq t+h} \left\| w(s, x) - w(t, x) \right\|_{C(\mathbb{R}^3)}.
\]

For the last term, \( w(s, \cdot) \) continuous on \([0, T]\) implies \( w(s, \cdot) \) uniformly continuous on \([0, T]\), so given \( \epsilon > 0, \exists h = h(\epsilon) > 0 \) such that \( \forall s, t \in [0, h], \)
\[
\left\| w(s, x) - w(t, x) \right\|_{C(\mathbb{R}^3)} < \epsilon.
\]

Therefore,
\[
\sup_{t \leq s \leq t+h} \left\| w(s, x) - w(t, x) \right\|_{C(\mathbb{R}^3)} \leq \epsilon.
\]

So, \( v_t = w = \nu \Delta v - \mathcal{P}(v \cdot \nabla v) \).

Thus, \( v = v'^{\nu} \), a classical solution to the Euler or the Navier-Stokes equations.

Note that then \( v'^{\nu} \in C([0, T]; C^2(\mathbb{R}^N)) \cap C^1([0, T]; C(\mathbb{R}^N)) \).

This concludes chapter 3 and the present paper.
References


