NONCOMMUTATIVE RINGS: A SURVEY OF DIVISION RINGS AND SIMPLE RINGS

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by

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Dedications

To my Mom, Kazia and Turunj, and Eran

Thanks for all your support
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Abstract

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In this thesis we start with some important classical results in noncommutative ring theory. Namely, we classify all semisimple Artinian rings in terms of matrices with entries from division rings. In the second chapter we start with some natural constructions of division rings. We do this by taking a polynomial ring and ”skewing” the multiplication. Further in the chapter we show what conditions a ring must meet in order to have (or be imbedded in) a division ring of fractions. The remainder of the second chapter is devoted to more constructions of division rings. In the final chapter we move our focus to Simple rings. We give a construction of simple rings as well as many examples.
INTRODUCTION

The primary goal of this thesis is to give methods for constructing two classes of noncommutative rings called division rings and simple rings. A division ring is a ring in which every non-zero element has a multiplicative inverse. The key thing is to note that multiplication need not be commutative. The first example of a (noncommutative) division ring was discovered in 1843 by the English mathematician Sir William Hamilton. Hamilton was searching for a way to represent vectors in space in an analogous manner to how vectors in the plane are represented by complex numbers. It turns out such a construction is impossible, but in his failed attempt, Hamilton discovered the Quaternions. Hamilton’s discovery turned out to be a division ring as well as a four dimensional vector space over the real numbers. The next example of a division ring was found in 1903 by Hilbert. Hilbert started with the field of Laurent series \( F((x)) \), over the field \( F = R(t) \) (the rational function field over the real numbers. He then “skewed” the multiplication via the automorphism which maps \( t \) to \( 2t \). That is, the indeterminate \( x \) no longer commutes with the coefficients. Instead, we define \( xt = 2tx \). The resulting ring is denoted \( F((s; \sigma)) \), where \( \sigma(t) = 2t \). Hilbert showed that \( F((s; \sigma)) \) is a division ring, called the division ring of skew Laurent series. One interesting facet of Hilbert’s division ring is that it is infinite dimensional over its center.

In the 1920s and 30s ring theorists developed structure theories for large classes of noncommutative rings. It was discovered that, in some sense, division rings provide the underpinnings for many important classes of noncommutative rings and various division ring constructions were discovered. Some of these constructions are not
easily accessible to most people studying advanced math. However, skew Laurent
construction only requires a field and an automorphism. We will review the skew
Laurent series construction and we will determine the center in all cases (i.e. where the
automorphism has finite and infinite periods). We then show how these constructions
can be generalized to the case where the coefficient ring is simple (a ring $R$ is simple
if its only two-sided ideals are $(0)$ and $R$). That is, if $R$ is simple and $\sigma$ is an
automorphism of $R$, then we prove that the skew Laurent series ring $R((x; \sigma))$ is
simple and we determine its center in all cases.

We also show how certain classes on noncommutative domains (Ore domains) have
division rings of fractions and use this to give alternative methods for constructing
division rings. Finally, we construct simple rings by “skewing” the multiplication via
a derivation, instead of an automorphism. We begin the thesis by reviewing some
basic noncommutative ring theory so that the reader will get an idea how division
rings and simple rings figure into the basic structure theory of noncommutative rings.
Chapter 1: Classical Results in Noncommutative Ring Theory

We begin our discussion with some classical results in Noncommutative Ring Theory. We start with modules and these lead to our first example of a division ring. Next, our discussion takes us into the exploration of the Jacobson radical of a ring. We will try to find rings that have a “nice” Jacobson radical and, this will allow us to define a semisimple ring and from there to go further and define a simple ring. All of this serves two purposes: to prove the classical theorems of Wedderburn and Artin which state that every simple Artinian ring is isomorphic to a matrix ring over a division ring and that every semisimple Artinian ring is isomorphic to a finite direct product of matrix rings over division rings. These theorems indicate the significant role played by division rings.

Section 1: Modules

Definition 1.a: An \( R \)-module is a vector space over a ring \( R \).

Definition 1.b: Alternatively, we say that the additive abelian group \( M \) is said to be an \( R \)-module if there exists a mapping \( M \times R \rightarrow M \) defined by

\[
(m, r) \mapsto mr \ (m \in M, r \in R)
\]

such that:

1. \( m(a + b) = ma + mb \)
2. \( (m + n)a = ma + na \)
3. \( (ma)b = m(ab) \)

for all \( m, n \in M \) and all \( a, b \in R \).

We remark that we are omitting the module axiom which states that \( m1 = m \) for all
$m \in M$ where 1 is the unity of $R$. In this chapter, we are not assuming that our rings contain a unity. In fact, the goal of several of our results is to conclude certain classes of rings do contain a unity.

Since it will be necessary to check that certain objects are submodules, we give the Submodule Criterion as the definition for what a submodule is.

**Definition 2:** Let $R$ be a ring and $M$ an $R$-module. A subset $N$ of $M$ is a submodule of $M$ if

1. $N \neq \emptyset$
2. $x + yr \in N$ for all $r \in R$ and all $x, y \in N$.

**Definition 3:** An $R$-module $M$ is said to be faithful if $Mr = (0)$ implies $r = 0$

**Definition 4:** If $M$ is an $R$-module, then the set $A(M) = \{x \in R : Mx = (0)\}$.

With respect to the definition of faithful, we can now say that $M$ is faithful if $A(M) = (0)$

**Lemma 5:** $A(M)$ is a two-sided ideal of $R$. Moreover, $M$ is a faithful $R/A(M)$-module.

**Proof:** $A(M)$ is clearly an additive subgroup of $R$. Let $r \in R$ and $x \in A(M)$. Now $Mxr = (0)r = (0) \Rightarrow xr \in A(M)$. Thus $A(M)$ is a right ideal. To see that $A(M)$ is a left ideal, notice that $M(rx) = (Mr)x \subset Mx = (0) \Rightarrow rx \in A(M)$. Hence, $A(M)$ is a two-sided ideal of $R$.

To see that $M$ is an $R/A(M)$-module, let $m \in M$, $r + A(M) \in R/A(M)$. We define the action $m(r + A(M)) = mr$. To show this is well-defined:
\[ r + A(M) = r' + A(M) \Rightarrow r - r' \in A(M) \]

\[ \Rightarrow m(r - r') = 0 \text{ for all } m \in M \Rightarrow mr = mr'. \text{ Thus} \]

\[ m(r + A(M)) = mr = mr' = m(r' + A(M)). \] Now let \( m \in M, r_1, r_2 \in R \), then

\[
m(r_1 + A(M) + r_2 + A(M)) = m(r_1 + r_2 + A(M)) \]

\[
= m(r_1 + r_2) \]

\[
= mr_1 + mr_2 \]

\[
= m(r_1 + A(M)) + m(r_2 + A(M)). \]

The second and third axioms for modules follow just as easily.

To show that \( M \) is faithful, suppose for all \( m \in M \), \( m(r + A(M)) = 0 \). Then

\[ mr = 0 \Rightarrow r \in A(M). \text{ Hence } r + A(M) = A(M), \text{ the zero element of } R/A(M). \]

Let \( M \) be an \( R \)-module. For any \( a \in R \), define \( T_a : M \to M \) by \( mT_a = ma \) for all \( m \in M \). Notice we are writing \( Ta \) on the right. This is done in order to have

\[ T_{ab} = T_aT_b. \] Since \( M \) is an \( R \)-module, then \( T_a \in \text{End}(M) \); that is,

\[
(m + n)T_a = mT_a + nT_a = ma + na = mT_a + nT_a \text{ for all } m, n \in M. \]

From this point on, \( E(M) \) will denote the set of all endomorphisms of the additive group of \( M \).

\( E(M) \) is a ring using pointwise addition for the sum and composition for the product.

**Lemma 6:** \( R/A(M) \) is isomorphic to a subring of \( E(M) \).

**Proof:** Consider \( \Phi : R \to E(M) \) defined by \( \Phi(a) = T_a \). Then

\[
\Phi(a + b) = \Phi(a) + \Phi(b) \text{ since } mT_{a+b} = mT_a + mT_b = m(T_a + T_b). \]

Also
\[ \Phi(ab) = \Phi(a)\Phi(b) \] since \( mT_{ab} = m(ab) = (ma)b = (mT_a)b = m(T_aT_b) \). Thus \( \Phi \) is a ring homomorphism of \( R \) into \( E(M) \). We need to find the kernel of \( \Phi \). Now \( a \in A(M) \iff Ma = (0) \iff ma = 0 \) for all \( m \in M \iff \Phi(a) = T_a = 0 \iff a \in \ker \Phi \).

Thus we get that \( A(M) = \ker \Phi \). Since the image of \( R \) under \( \Phi \) is a subring of \( E(M) \), then the First Isomorphism Theorem for rings gives our desired result. \[ \square \]

**Definition 7:** The commuting ring of \( R \) on \( M \) is

\[ C(M) = \{ \phi \in E(M) : T_a\phi = \phi T_a \ \forall \ a \in R \} \]

**Remark:** \( C(M) \) is clearly a subring of \( E(M) \). Note that if \( \phi \in C(M) \), then for all \( m \in M, a \in R \) we have \( (m\phi)(a) = (m\phi)T_a = m(\phi T_a) = m(T_a\phi) = (mT_a)\phi = (ma)\phi \).

So, \( \phi \) is an \( R \)-module homomorphism of \( M \) into \( M \). Thus \( C(M) \) is the ring of all \( R \)-module endomorphisms of \( M \).

**Definition 8:** \( M \) is said to be an irreducible \( R \)-module if \( MR \neq (0) \) and if the only submodules of \( M \) are \((0)\) and \( M \).

We conclude this section with our first construction of a division ring.

**Schur’s Lemma:** If \( M \) is an irreducible \( R \)-module, then \( C(M) \) is a division ring.

**Proof:** It suffices to show that if \( \phi \neq 0 \) in \( C(M) \), then \( \phi^{-1} \in C(M) \). In fact, we need only show \( \phi^{-1} \in E(M) \) because then \( \phi T_a = T_a\phi \) will imply \( T_a\phi^{-1} = \phi^{-1}T_a \) giving \( \phi^{-1} \in C(M) \). If \( W = M\phi \), we claim \( W \) is in fact a submodule of \( M \). To see this, let \( r \in R, x, y \in W \). Then there exists \( m, n \in M \) such that
\[ x + yr = m\phi + (n\phi)r = m\phi + (nr)\phi = (m + nr)\phi \in W. \] Since \( \phi \neq 0 \) and \( M \) is irreducible, we get that \( W = M\phi = M \) giving us that \( \phi \) is onto. Also \( \ker \phi \) is a
submodule of $M$ and $\ker \phi = M$ (since $\phi \neq 0$); thus $\ker \phi = (0)$ and $\phi$ is one-to-one. This shows $\phi^{-1}$ exists in $E(M)$. Thus, every non-zero element of the commuting ring is invertible \qed

Section 2: Jacobson Radical

As we will never mention any other radical in this Thesis, we will refer to the Jacobson radical as simply the radical. The radical helps us identify “bad” elements in a ring and when we mod out by it, the ring we obtain is more amenable to analysis.

**Definition 9:** For any ring $R$, the right Jacobson radical of $R$, denoted $J(R)$, is the set of all elements of $R$ which annihilate all the irreducible right $R$-modules. If $R$ has no irreducible modules, we say that $J(R) = R$.

Note that the left Jacobson radical is defined similarly. These two turn out to be the same so we shall drop the right-left distinction.

**Definition 10:** A right ideal $\rho$ of a ring $R$ is said to be **regular** if there exists $a \in R$ such that $x - ax \in \rho$ for all $x \in R$.

**Lemma 11:** If $M$ is an irreducible $R$-module, then $M \cong R/\rho$ for some maximal right ideal $\rho$ of $R$. Moreover, $\rho$ is regular. Conversely, for every such maximal right ideal $\rho$, $R/\rho$ is an irreducible $R$-module.

**Proof:** Since $M$ is irreducible, $MR \neq (0)$. Let $S = \{ u \in M : uR = (0) \}$. We first show that $S$ is a submodule of $M$. Clearly $S \neq \emptyset$ as $0R = (0)$. For $x, y \in S$ and $r \in R$, we have $xR = (0)$ and $yrR \subset yR = (0)$, so $yr \in S$ and hence $x + yr \in S$. 

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Since $S \neq M$, we must have that $S = (0)$. Equivalently, if $m \neq 0$ is in $M$, then $mR \neq (0)$. Since it is easy to see $mR$ is a submodule of $M$, we must have $mR = M$.

Define $\phi : R \rightarrow M$ by $\phi(r) = mr$ for all $r \in R$. If $r, s \in R$, then

$\phi(r + s) = m(r + s) = mr + ms = \phi(r) + \phi(s)$ and $\phi(rs) = (mr)s = \phi(r)s$, showing $\phi$ is an $R$-module homomorphism. Since $mR = M$, $\phi$ is onto. Let $\rho = \ker \phi$, a right ideal of $R$. By the First Isomorphism Theorem for modules, $M \cong R/\rho$.

We now show that $\rho$ is maximal and regular.

Maximal: If $\rho \subset \rho_0$ where $\rho_0$ is a right ideal of $R$, then $\rho_0/\rho$ is a nonzero submodule of the irreducible module $R/\rho$. Hence $\rho_0/\rho = R/\rho$ which implies (by the correspondence Theorem for modules), $\rho_0 = R$, proving $\rho$ is maximal.

Regular: Since $mR = M$, there exists $a \in R$ such that $ma = m$. Thus, for all $x \in R$, $max = mx \Rightarrow m(x - ax) = 0 \Rightarrow x - ax \in \rho$.

Conversely, let $\rho$ be a maximal right ideal of $R$. By the correspondence Theorem, every submodule of $R/\rho$ is of the form $\rho_0/\rho$ where $\rho_0$ is a right ideal of $R$ containing $\rho$. $\rho$ is maximal implies that $\rho_0 = \rho$ or $\rho_0 = R$, and thus $\rho_0/\rho = (0)$ or $\rho_0/\rho = R/\rho$.

Hence, $R/\rho$ is an irreducible $R$-module. □

Note the following two observations:

(1) $J(R) = \bigcap A(M)$ where the intersection runs over all irreducible $R$-modules.

Since each $A(M)$ is a two-sided ideal of $R$, $J(R)$ is indeed a two-sided ideal of $R$.

(2) If $R$ has a unity (even a left unity), then all its right ideals are regular. To see this, let $e$ be a left unity and $\rho$ a right ideal of $R$. Now $x - ex = x - x = 0 \in \rho$ for
all $x \in R$ (that is, in the definition of regular, simply take $a = e$).

Remark: If $R$ has an irreducible module then by Lemma 11 $R$ has a maximal regular right ideal $\rho$. In this case $R$ must have a proper regular right ideal. As we stated, if $R$ has no irreducible modules, then $J(R) = R$. Since our main goal is to prove $J(R/J(R)) = (0)$, and this will trivially be the case if $R = J(R)$, without loss of generality we may assume that the collection of irreducible $R$-modules is non-empty.

**Definition 12:** Let $R$ be a ring and $\rho$ a right ideal of $R$. Then,

$$\{x \in R : Rx \subseteq \rho\}.$$  

We now prove a few theorems involving maximal regular right ideals. It turns out that we will be able to characterize the radical in terms of these ideals.

**Theorem 13:** $J(R) = \bigcap(\rho : R)$ where $\rho$ runs over all the maximal regular right ideals of $R$, and where $(\rho : R)$ is the largest two-sided ideal of $R$ contained in $\rho$.

**Proof:** Let $\rho$ be a maximal regular right ideal of $R$ and let $M = R/\rho$. If $x \in A(M)$, then $Mx = (0)$; that is, $(R/\rho)x = \rho$, so $(r + \rho)x = \rho$ for all $r \in R$. Thus $Rx \subseteq \rho$ which implies $A(M) \subseteq (\rho : R)$. The other inclusion follows by going the opposite direction. Hence $A(M) = (\rho : R)$ which implies $(\rho : R)$ is a two-sided ideal and $J(R) = \bigcap(\rho : R)$. Since $\rho$ is regular there exists $a \in R$ such that $x - ax \in \rho$ for all $x \in R$. Hence, if $x \in (\rho : R)$, then $ax \in Rx \subseteq \rho$ implying $x \in \rho$. Thus $(\rho : R) \subseteq \rho$.

To see that $(\rho : R)$ is the largest two-sided ideal contained in $\rho$ suppose $J$ is a two-sided ideal contained in $\rho$. Then $x \in J \Rightarrow rx \in J\rho$ for all $r \in R \Rightarrow x \in (\rho : R)$; that is, $J \subseteq (\rho : R)$. □
Next we simplify the characterization of \( J(R) \) in terms of the maximal regular right ideals instead of \((\rho : R)\)'s. To do so we need:

**Lemma 14:** If \( \rho \) is a proper regular right ideal of \( R \), then \( \rho \) can be embedded in a maximal regular right ideal of \( R \).

**Proof:** \( \rho \) regular implies there exists \( a \in R \) such that \( x - ax \in \rho \) for all \( x \in R \). Let \( M \) be the set of all proper right ideals of \( R \) containing \( \rho \). Clearly \( \rho \in M \Rightarrow M \neq \emptyset \).

Let \( C \) be a chain in \( M \). Then \( \rho' = \bigcup_{I \in C} I \) is a right ideal of \( R \). If \( a \in \rho' \), then \( a \in I \) for some \( I \); but then, for all \( x \in R, x - ax \in \rho \subset I \) would imply \( I = R \), a contradiction. Thus, \( a \notin \rho' \) from which we conclude \( \rho' \in M \). By Zorn’s Lemma, \( M \) has a maximal element \( \rho_0 \). Since \( x - ax \in \rho \subset \rho_0 \) for all \( x \in R \), we have \( \rho_0 \) is regular. To see that \( \rho_0 \) is maximal, suppose \( \rho_0 \subsetneq J \) for some right ideal \( J \). Then \( J \) would also contain \( \rho \) and by the maximality of \( \rho_0 \), we must have \( J = R \). \( \square \)

**Definition 15:** An element \( a \in R \) is said to be **right-quasi-regular** if there exists an \( a' \in R \) such that \( a + a' + aa' = 0 \). We call \( a' \) a **right-quasi-inverse** of \( a \).

**Definition 16:** A right ideal of \( R \) is **right-quasi-regular** if each of its elements is right-quasi-regular.

We remark that if an element \( a \) is both left and right-quasi-regular, then the left and right-quasi-inverses are the same. If \( a + b + ba = 0 \) and \( a + c + ac = 0 \) for some \( b, c \in R \) then \( ba + bc + bac = ac + bc + bac \) which implies \( ba = ac \). Going back to our first pair of equations, we get \( b = c \).

**Theorem 17:** \( J(R) = \bigcap \rho \) where \( \rho \) runs over all the maximal regular right ideals of
\(R\). Furthermore, \(J(R)\) is a right-quasi-regular ideal of \(R\) and contains all the right-quasi-regular right ideals of \(R\); that is, \(J(R)\) is the unique maximal right-quasi-regular right ideal of \(R\).

**Proof:** \(\subset\) By Theorem 13 \(J(R) = \bigcap(\rho : R) \subset \rho\) and since \((\rho : R) \subset \rho\) we conclude \(J(R) \subset \bigcap\rho\) where \(\rho\) runs over all the maximal regular right ideals of \(R\).

\(\supset\) Let \(T = \bigcap\rho\) and let \(x \in T\). Consider \(S = \{xy + y : y \in R\}\). We claim that \(S = R\). Now \(S\) is clearly a right ideal of \(R\) and \(S\) is regular by taking \(a = -x\); that is, for all \(y \in R, y - (-x)y = xy + y \in S\). If \(S\) is proper, \(S\) is contained in a maximal regular right ideal \(\rho_0\) of \(R\). Now \(x \in \bigcap\rho \Rightarrow x \in \rho_0 \Rightarrow xy \in \rho_0\) and since \(xy + y \in \rho_0\), then \(y \in \rho_0\) for all \(y \in R\) and hence \(\rho_0 = R\), a contradiction.

Thus \(S = R\). In particular, there exists \(w \in R\) such that \(-x = w + xw\) or \(x + w + xw = 0\). If \(T \not\subseteq J(R)\), then for some irreducible \(R\)-module \(M\), we must have \(MT \neq (0)\) and so \(mT \neq (0)\) for some \(m \in M\). Thus \(mT\) is a nonzero submodule of \(M\), which gives us that \(mT = M\) by irreducibility. Thus, for some \(t \in T, mt = -m\).

Since \(t \in T\), the above argument implies that \(t + s + ts = 0\). So \(0 = mt + ms + mts = -m + ms - ms = -m\) which implies \(m = 0\), a contradiction.

Thus \(\bigcap \subset J(R)\).

In our proof of \(\supset\), we showed for any \(x \in \bigcap = J(R)\), there exists \(y \in R\) such that \(x + y + xy = 0\); that is, \(J(R)\) is a right-quasi-regular. Using the same argument that shows \(\bigcap \rho \subset J(R)\), we conclude \(\rho \subset J(R)\) for any right-quasi-regular right ideal \(\rho\) of \(R\).

\[\square\]

**Remark:** Suppose that \(a \in J(R)\). Then there exists an \(a' \in R\) such that \(a + a' + aa' = 0\). Now \(a' = -a - aa'\) and \(a \in J(R)\) gives \(a' \in J(R)\). Then there
exists $a'' \in J(R)$ such that $a' + a'' + a'a'' = 0$. Then $a'$ has $a$ as a left-quasi-inverse and $a''$ as a right-quasi-inverse. By our earlier remark we get, that $a = a''$. We have shown that $a$ is left-quasi-regular which means that $J(R)$ is a left-quasi-regular ideal of $R$. Using the left analog of the above result, the right radical of $R$ (as a left-quasi-regular ideal) is contained in the left radical. Similarly the left radical of $R$ is contained in the right radical. We have shown now what we meant earlier in the section when we stated that the left and right radicals are the same.

Another remark: In a commutative ring with unity, the radical is the intersection of all maximal ideals.

**Definition 18:** An element $a \in R$ is **nilpotent** if there exists $n \in \mathbb{N}$ such that $a^n = 0$

**Definition 19:** A right (left, two-sided) ideal is **nil** if each element is nilpotent.

**Definition 20:** A right (left, two-sided) ideal $\rho$ is **nilpotent** if there is an $m \in \mathbb{N}$ such that $a_1 \cdots a_m = 0$ for all $a_1, \ldots, a_m \in \rho$ (that is, $\rho$ is nilpotent if $\rho^m = (0)$ for some $m \in \mathbb{N}$).

Note that every nilpotent ideal is nil. (It is possible to construct a nil ideal that is not nilpotent.)

**Lemma 21:** Every nil right ideal of $R$ is contained in $J(R)$.

**Proof:** Suppose that $\rho$ is a nil right ideal and if $a \in \rho, a^m = 0$ for some $m \in \mathbb{N}$. If $b = -a + a^3 \cdots + (-1)^{m-1}a^{m-1}$, then $ab = -a^2 + a^3 + \cdots + (-1)^{m-1}a^m$ which implies $ab + b = -a$. Thus $a + b + ab = 0$. So $\rho$ is right-quasi-regular and hence
\[ \rho \subset J(R). \]

The last result in this section and the question: if we factor out the radical of a ring, what is the radical of the resulting quotient ring?

**Theorem 22:** \( J(R/J(R)) = (0) \)

**Proof:** Let \( \overline{R} = R/J(R) \) and let \( \rho \) be a maximal regular right ideal of \( R \). Since \( J(R) \subset \rho \), then (by the correspondence Theorem) \( \overline{\rho} = \rho/J(R) \) is a maximal right ideal of \( \overline{R} \). If \( x - ax \in \rho \) for all \( x \in R \), then \( \overline{x} - \overline{a} \overline{x} \in \overline{\rho} \) for all \( \overline{x} \in \overline{R} \) and so \( \overline{\rho} \) is regular. Since \( J(R) = \bigcap \rho \) (where \( \rho \) runs over all maximal regular right ideals of \( R \)), we get that \( \bigcap \overline{\rho} = (0) \). Thus, by Theorem 19, \( J(\overline{R}) = (\text{the intersection over all maximal regular right ideals of } \overline{R}) \subset \bigcap \overline{\rho} = (0) \).

**Section 3: Artinian Rings**

**Definition 23:** A ring is **Artinian** if any non-empty set of right ideals has a minimal element.

**Definition 24:** A ring satisfies the **descending chain condition** (DCC) on right ideals if whenever \( \rho_1 \supset \cdots \supset \rho_n \supset \cdots \) is a descending chain of right ideals of \( R \), then there is some \( n \in \mathbb{N} \) such that \( \rho_n = \rho_k \) for all \( k \geq n \) (that is, the descending chain is eventually stationary).

**Proposition 25:** A ring is Artinian if and only if it satisfies the DCC.

**Proof:** Suppose \( R \) is Artinian and let \( \rho_1 \supset \cdots \supset \rho_n \supset \cdots \) be a descending chain of right ideals of \( R \). If \( R \) is Artinian, then the set \( \{\rho_i\}_{i \in \mathbb{N}} \) of right ideals has a minimal
element, say $\rho_m$. Since $\rho_m \supset \rho_{m+1} \supset \cdots$ and $\rho_m$ is minimal, we must have that $\rho_m = \rho_k$ for all $k \geq m$.

Conversely, suppose that $R$ satisfies the DCC on right ideals. Let $S$ be a nonempty set of right ideals. Pick $\rho_1 \in S$. If $\rho_1$ is minimal, we are done. Otherwise, there is some $\rho_2 \in S$ such that $\rho_1 \supset \rho_2$. If $\rho_2$ is minimal, then again we are done. If not, then there is another ideal $\rho_3 \in S$ such that $\rho_2 \supset \rho_3$. Continuing in this fashion, we get a descending chain $\rho_1 \supset \cdots \supset \rho_n \supset \cdots$ and by DCC, this chain becomes stationary forcing us to have a minimal element in $S$. $\square$

Note: Given a nonzero right ideal $I$ of an Artinian ring, there is a minimal right ideal contained in $I$.

We look at some examples:

(1) Let $A$ be a ring which is an algebra over a field $F$ (that is, the elements of $F$ commute with the elements of $A$). Then $A$ is a vector space over $F$ and the right ideals of $A$ are also subspaces (with the scalar action on the right). If $A$ is finite dimensional as a vector space over $F$, then it would be impossible to have a non-terminating descending chain of right ideals. Thus $A$ is Artinian.

(2) We note that the polynomial ring $K[t]$, with coefficients in a field $K$, is not Artinian. For example,

$$(t) \supset (t^2) \supset (t^3) \supset \cdots$$

is a descending chain of ideals which never terminates. However, we claim that the quotient ring $K[t]/(t^n)$ is Artinian for all positive integers $n$. To see this, let $I/(t^n)$ be an arbitrary ideal of $K[t]/(t^n)$. Since $K[t]$ is a PID, $I = (f(t))$ for some monic
polynomial \( f(t) \). Since \( (t^n) \subset I \), it follows that \( f(t)|(t^n) \), and \( K[t] \) a UFD gives \( f(t) = t^j \) for some \( j \leq n \). Hence \( I/(t^n) = (t^j)/(t^n) \) from which it follows that 
\( K[t]/(t^n) \) is Artinian.

(3) We will show in Chapter 3 that if \( D \) is a division ring, then the matrix ring 
\( M_n(D) \) is Artinian. In fact, if \( R \) is Artinian, then it can be shown that \( M_n(R) \) is 
Artinian.

(4) A finite direct sum of Artinian rings is Artinian.

(5) A homomorphic image of an Artinian ring is Artinian.

**Theorem 26:** If \( R \) is Artinian, then \( J(R) \) is a nilpotent ideal.

**Proof:** Let \( J = J(R) \) and consider the descending chain of right ideals 
\( J \supset J^2 \supset \cdots \supset J^n \supset \cdots \). Since \( R \) is Artinian then there exists \( n \in \mathbb{N} \) such that 
\( J^n = J^k \) for all \( k \geq n \). Since \( J^{2n} = J^n \), if \( xJ^{2n} = (0) \) then \( xJ^n = (0) \). We want to 
prove \( J^n = (0) \). So suppose that \( J^n \neq (0) \) and define \( W = \{ x \in R : xJ^n = (0) \} \). It is 
easy to check \( W \) is an ideal of \( R \) (for the ideal conditions, if \( a \in W \) and \( r \in R \),
certainly \( ra \in W \) and \( arJ^n \subset aJ^n = (0) \) show \( ar \in W \)). To show the radical is 
nilpotent, we consider two cases. If \( J^n \subset W \) then \( J^nJ^n = (0) \) which implies 
\( J^{2n} = J^n = (0) \) and we’re done. Otherwise suppose \( J^n \not\subseteq W \). Letting \( \overline{R} = R/W \) we 
see that \( \overline{J^n} \neq (0) \). If \( \overline{xJ^n} = (0) \), then 
\( xJ^n \subset W \Rightarrow (0) = xJ^nJ^n = xJ^{2n} = xJ^n \Rightarrow x \in W \Rightarrow \overline{x} = 0 \). That is, if \( \overline{xJ^n} = (0) \),
this forces \( \overline{x} = (0) \). Since \( \overline{J^n} \neq (0) \), it is a nonzero ideal of the Artinian ring \( \overline{R} \) and 
hence must contain a minimal right ideal \( \overline{p} \neq (0) \) of \( \overline{R} \). This implies \( \overline{p} \) is an 
irreducible \( \overline{R} \)-module and thus is annihilated by \( J(\overline{R}) \). However \( \overline{J^n} \subset J(\overline{R}) \subset J(\overline{R}) \).
(J(R) is right-quasi-regular, so \(J(R)\) is a right-quasi-regular ideal of \(R\), so 
\(J(R) \subset J(R)\). Hence \(pJ^n = (0)\) and we have shown this implies \(p = (0)\), a 
contradiction. \(\square\)

**Corollary 27:** If \(R\) is Artinian, then any (right, left, or two-sided) nil ideal of \(R\) is 
nilpotent.

**Proof:** By Lemma 21, we showed that every nil right ideal of \(R\) is contained in 
\(J(R)\) and if \(R\) is Artinian, \(J(R)\) is nilpotent. So, in this case, any nil ideal is a 
nilpotent ideal. \(\square\)

Remark: If \(R\) contains a nonzero nilpotent right ideal, then \(R\) contains a nonzero 
nilpotent two-sided ideal. To see this let \(\rho \neq (0)\) be a nilpotent right ideal of \(R\). If 
\(R\rho = (0)\) then \(R\rho \subset \rho\) and hence \(\rho\) is a two-sided ideal of \(R\). Now if \(\rho^m = (0)\) for 
some \(m \in \mathbb{N}\), then 
\[(R\rho)^m = R\rho R\rho \cdots R\rho = R(\rho R)(\rho R) \cdots (\rho R) \subset R\rho^{m-1} \rho = R\rho^m = (0),\]
proving \(R\rho\) is 
nilpotent.

**Definition 28:** An element \(e \neq 0\) in \(R\) is an **idempotent** if \(e^2 = e\).

**Lemma 29:** Let \(R\) be a ring having no nonzero nilpotent ideals. Suppose \(\rho \neq (0)\) is 
a minimal right ideal of \(R\). Then \(\rho = eR\) for some idempotent \(e \in R\).

**Proof:** Let \(\rho \neq (0)\) be a minimal right ideal of \(R\). Since \(R\) has no nonzero nilpotent 
ideals, then by the preceding remark, \(\rho^2 \neq (0)\) and hence there is some \(x \in \rho\) such 
that \(x\rho \neq (0)\). However, \(x\rho \subset \rho\) is a right ideal of \(R\) and by the minimality of \(\rho\), we 
must have \(x\rho = \rho\). Thus there is some \(e \in \rho\) such that \(xe = x\) which implies
$xe^2 = xe$ and so $x(e^2 - e) = 0$. Let $\rho_0 = \{a \in \rho : xa = 0\}$. Clearly $\rho_0$ is a right ideal of $R$ contained in $\rho$ and $\rho_0 \neq \rho$ since $x\rho \neq (0)$. By the minimality of $\rho$, we get that $\rho_0 = (0)$. Since $e^2 - e \in \rho_0$, we can conclude $e^2 = e$ and since $xe = x \neq 0$, we have $e \neq 0$; that is, $e$ is an idempotent in $R$. Now $e \in \rho \Rightarrow eR \subset \rho$ is a right ideal of $R$ containing $e^2 = e \neq 0$ and so $eR \neq (0)$. Again using the minimality of $\rho$, we have $\rho = eR$. \qedhere

Lemma 30: Let $R$ be a ring and suppose that for some $a \in R$ $a^2 - a$ is nilpotent. Then, either $a$ is nilpotent or, for some $q(x) \in \mathbb{Z}[x], e = aq(a)$ is a nonzero idempotent.

Proof: Suppose $(a^2 - a)k = 0$ for some $k \in \mathbb{N}$. Expanding this we get

$a^k = a^{k+1}p(a)$ where $p(x) \in \mathbb{Z}[x]$. Now

$$a^k = a^{k+1}p(a)$$
$$= aap(a)$$
$$= a^{k+1}p(a)p(a)$$
$$= a^{k+2}p(a)^2$$

continuing we get

$$a^k = a^{2k}p(a)^k$$

If $a$ is nilpotent, we are done. Otherwise, $a^k \neq 0$ and so

$$0 \neq e = a^kp(a)^k = (a^{2k}p(a)^k)p(a)^k = a^{2k}p(a)^{2k} = e^2$$

(the last equality uses that $a^k$ commutes with $p(a)^k$). Rewriting,
\[ e = a^k p(a)^k = aq(a) \text{ for some } q(x) \in \mathbb{Z}[x]. \]

**Theorem 31:** If \( R \) is Artinian and \( \rho \neq (0) \) is a right ideal that is not nilpotent, then \( \rho \) contains a nonzero idempotent.

**Proof:** By Theorem 26, \( R \) Artinian implies \( J(R) \) is a nilpotent ideal. Since \( \rho \) is not nilpotent, \( \rho \) is not contained in \( J(R) \).

Now let \( \overline{R} = R/J(R) \). Since \( J(\overline{R}) = (0) \), then there are no nonzero nilpotent ideals in \( \overline{R} \). Let \( \rho = \{ a + J(R) : a \in \rho \} \); that is, \( \overline{\rho} \) is the image of \( \rho \) in \( \overline{R} \). Also \( \overline{\rho} \neq (0) \) and \( \overline{R} \) Artinian implies \( \overline{\rho} \) contains a minimal right ideal of \( R \), say \( \overline{\rho_0} \). By Lemma 29, \( \overline{\rho_0} \) contains and idempotent element \( \overline{e} \) such that \( \overline{\rho_0} = \overline{e} \overline{R} \). Let \( a \in \rho \) map onto \( \overline{e} \); that is, \( \overline{e} = a + J(R) \). Since \( \overline{e}^2 = \overline{e} \Rightarrow a^2 - a \) maps onto \( \overline{0} \) (that is, \( a^2 - a + J(R) = J(R) \)). Hence \( a^2 - a \in J(R) \) which implies \( a^2 - a \) is nilpotent. We claim \( a \) is not nilpotent. For if \( a^k = 0 \) for some \( k \in \mathbb{N} \),

\[ \overline{0} = \overline{a^k} = \overline{a^k} = \overline{e}, \]

a contradiction. By Lemma 30, there must exist a polynomial \( q(x) \) such that \( e = aq(a) \) is a nonzero idempotent. Finally \( a \in \rho \Rightarrow aq(a) \in \rho \Rightarrow e \in \rho \).

We will now consider the ring \( eRe \) where \( e \) is an idempotent in \( R \). This ring is closely related to \( R \).

**Theorem 32:** Let \( R \) be any ring and let \( e \) be an idempotent in \( R \). Then

\[ J(eRe) = eJ(R)e. \]

**Proof:** “\( \subset \):” Let \( M \) be an irreducible \( R \)-module. We will show that \( J(eRe) \) annihilates all such \( M \). We consider two cases:
Case 1: If $Me \neq (0)$, then there exists $m \in M$ such that $me \neq 0 \in Me$. Now

$$(me)(eRe) = me^2Re = meRe.$$  

Since $me \neq 0$ and $me = mee \in meR$, $M$ irreducible

$\Rightarrow meR = M \Rightarrow meRe = Me$. Thus $Me$ is an irreducible $eRe$-module (if $N \neq (0)$ is an $eRe$-submodule of $Me$ and $0 \neq me \in N$, then $(me)eRe = Me \subset N$ which implies $N = Me$) and consequently $MeJ(eRe) = (0)$. Since every element of $J(eRe)$ is of the form $eRe$, we have $MeJ(eRe) = MJ(eRe) = (0)$.

Case 2: If $Me = (0)$, then clearly $MeJ(eRe) = MJ(eRe) = (0)$.

Thus, in both cases, we get that $J(eRe)$ annihilates all irreducible $R$-modules and hence $J(eRe) \subset J(R)$. So $J(eRe) = eJ(eRe)e \subset eJ(R)e$.

“$\supset$” Let $a \in eJ(R)e \subset J(R)$ (since $J(R)$ is a two-sided ideal). Thus $a$ has a right and left quasi-inverse $a'$. So $a + a' + aa' = 0$. Multiplying both sides on the left and right by $e$ and using that $eae = a$ (since $a \in eJ(R)e$), we get

$$0 = a + ea'e + eaa'e$$

$$= a + ea'e + (eae)(ea'e)$$

$$= a + ea'e + a(ea'e)$$

This gives us that $ea'e$ is a right quasi-inverse of $a$. Since quasi-inverses are unique, then $a' = ea'e$. Thus every element in $eJ(R)e$ is quasi-regular in $eRe$. Moreover, $eJ(R)e$ is an ideal of $eRe$ and as a quasi-regular ideal of $eRe$ it must be contained in $J(eRe)$; that is, $eJ(R)e \subset J(eRe)$. □

**Theorem 33:** Let $R$ be a ring having no nonzero nilpotent ideals and suppose $e \neq 0$ is an idempotent in $R$. Then $eR$ is a minimal right ideal if and only if $eRe$ is a
division ring.

**Proof:** ($\Rightarrow$) Suppose $eRe$ is a minimal right ideal of $R$. If $eae \neq 0 \in eRe$, then $(0) \neq earR \subset eR$ which implies $eaeR = eR$ by the minimality of $eR$. Thus there is an element $y \in R$ such that $eay = e$. So $eaye = e^e \Rightarrow (eae)(eye) = e$. We must show $eae$ has a multiplicative inverse that works on both sides. Now $eR(eaeR) \subset eR$ and is nonzero since $eae \in eR(eaeR)$. Hence $eR(eaeR) = eR$. So there exists $x \in R$ such that $(exe)(eae) = e$. We can conclude $eye = exe$ (because in any ring with unity, if an element has a right multiplicative inverse and a left multiplicative inverse, it is easy to show these must be equal) and $eae$ is invertible. Hence $eRe$ with unity $e$ is a division ring.

($\Leftarrow$) Suppose $eRe$ is a division ring. We claim $\rho = eR$ is minimal. Let $\rho_0 \subset \rho$ be a nonzero right ideal of $R$. Then $\rho_0e \neq (0)$, otherwise $\rho_0^2 \subset \rho_0\rho = \rho_0Er = (0)$, but this is a contradiction as $R$ has no nilpotent ideals. Now let $0 \neq a \rho_0$ such that $ae \neq 0$.

Since $ea = a$, $0 \neq ae = eae \in \rho_0$. Since $eRe$ is a division ring, there exists and element $exe$ such that $(eae)(exe) = e$. Then $e = (eae)(exe) \in \rho_0 \Rightarrow \rho = eR \subset \rho_0 \Rightarrow \rho_0 = \rho$ and so $eR$ is minimal. \qed

Just interchanging left with right in the above Theorem, we immediately have:

**Corollary 34:** Let $R$ be a ring having no nonzero nilpotent ideals and suppose $e \neq 0$ is an idempotent in $R$. Then $eR$ is a minimal right ideal if and only if $Re$ is a minimal left ideal.

We conclude this section with an example.

Example: We will show in Chapter 3 Section 3 (Theorem 13) that the only
two-sided ideals of $M_n(F)$ (where $F$ is a field) are $(0)$ and $M_n(F)$ (and we will say that $M_n(F)$ is simple). Hence $M_n(F)$ has no nonzero nilpotent ideals. For $n = 2$, 

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is a nonzero idempotent. Now

$$eM_2(F)e = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in F \right\}$$

and is clearly ring isomorphic to $F$ (a field, so a division ring). By Theorem 33,

$$eM_2(F) = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in F \right\}$$

is a minimal right ideal. (This example easily generalizes to $M_n(F)$).

Section 4: Semisimple Artinian Rings

So far we have been studying the radical of a ring. We now switch gears and focus our interest on rings whose radicals are as trivial as possible; that is, we will study rings $R$ with $J(R) = (0)$.

Definition 35: A ring is called semisimple if $J(R) = (0)$.

Theorem 36: Let $R$ be a semisimple Artinian ring (SSAR) and let $\rho \neq (0)$ be a right ideal of $R$. Then $\rho = eR$ for some idempotent $e \in R$.

Proof: Since $J(R) = (0)$, by Lemma 21, $R$ has non nonzero nilpotent ideals. Since $\rho \neq (0)$, $\rho$ is not nilpotent and by Theorem 31, $\rho$ contains a nonzero idempotent element $e$. Let $A(e) = \{x \in \rho : ex = 0\}$. It is easy to check $A(e)$ is a right ideal. We claim that $A = \{A(e) : e^2 = e \neq 0 \in \rho\}$ is a non-empty set of right ideals. This is clear since $\rho$ does contain at least one idempotent.
Since $R$ Artinian, $A$ has a minimal element; that is, there is some idempotent $e_0 \in \rho$ such that $A(e_0)$ is minimal in $A$.

If $A(e_0) = (0)$, then for all $x \in \rho$,

$$e_0(x - e_0x) = 0 \Rightarrow x - e_0x \in A(e_0) = (0) \Rightarrow x = e_0x$$

for all $x \in \rho$. This gives that $\rho = e_0\rho \subset e_0R \subset \rho$ (since $e_0 \subset \rho$). Hence $\rho = e_0R$ and we are done.

If $A(e_0) \neq (0)$, then by Theorem 31, $A(e_0)$ must have an idempotent, say $e_1$. So $e_1 \in \rho$ and $e_0e_1 = 0$ (by definition of $A(e_0)$). Consider $e' = e_0 + e_1 - e_1e_0 \in \rho$ (since $e_0, e_1 \in \rho$). Now $(e')^2 = (e_0 + e_1 - e_1e_0)(e_0 + e_1 - e_1e_0) = e_0e_0 + e_0e_1 - e_0e_1e_0 + e_1e_0 + e_1e_1 - e_1e_1e_0 - e_1e_0e_0 - e_1e_0e_1e_0 + e_1e_0e_1e_0$. Combining this with the fact that $e_0^2 = e_0$, $e_1^2 = e_1$ and $e_0e_1 = 0$, we get that $(e')^2 = e_0 + e_1 - e_1e_0 = e'$; that is, $e'$ is an idempotent in $\rho$.

Also $e'e_1 = e_0e_1 + e_1^2 - e_1e_0 = e_1 \neq 0 \Rightarrow e' \neq 0$. Now if $e'x = 0$, then

$$(e_0 + e_1 - e_1e_0)x = 0 \Rightarrow e_0(e_0 + e_1 - e_1e_0)x = 0 \Rightarrow (e_0^2 + e_0e_1 - e_0e_1e_0)x = e_0x = 0.$$ 

Thus $A(e') \subset A(e_0)$. Since $e_1 \in A(e_0)$ and $e_1 \notin A(e')$, this inclusion must be proper.

Since $A(e_0)$ is minimal, we must have $A(e') = (0)$ which is impossible since $e_1 \in A(e_1)$. Thus $A(e_0) = (0)$.

**Corollary 37:** If $R$ is a semisimple Artinian ring and $A$ is an ideal of $R$, then $A = eR = Re$ for some idempotent $e$ in the center of $R$.

**Proof:** Since $A$ is a right ideal of $R$, then by the previous theorem, $A = eR$ for some idempotent $e$. Let $B = \{x - xe : x \in A\}$. Note that $ex = e$ for all $x \in A$ and also $Be = (0)$. Also $Be = \{xe - xe^2 : x \in A\} = \{xe - xe : x \in A\} = (0)$. $A = eR \Rightarrow eA = e^2R = eR \Rightarrow eA = A$. Thus $BA = BeA = (0)$. Since $A$ is also a left ideal, $B$
must be a left ideal of $R$. Also, since $B \subset A$ we have $B^2 \subset BA = (0)$ and hence $B$ is nilpotent. Since $R$ is semisimple we must have $B = (0)$. Thus $x = xe$ for all $x \in A \Rightarrow A \subset Re$ and $e \in A \Rightarrow Re \subset A$; that is, $A = Re$. In particular, $e$ is both a left and a right unity for $A$ (because for all $a \in A$, $a = er \Rightarrow ea = e^2r = er = a$ and $a = se = ae = se^2 = se = a$).

To see that $e$ is in the center of $R$, let $y \in R$. Since $ye \in A$ and $e$ is a left unity for $A$, then $eye = ye$. Since $ey \in A$ and $e$ is a right unity of $A$, then $eye = ey$. Hence $ey = ye$ for all $y \in R$, proving $e$ is in the center of $R$. □

**Corollary 38**: A semisimple Artinian has a two-sided unity.

**Proof**: Let $R$ be a SSAR. Then $R$ is an ideal of $R$ and by the previous corollary we have the desired result. □

**Theorem 39**: If $A$ is an ideal of $R$, then $J(A) = A \cap J(R)$.

**Proof**: If $a \in A \cap J(R)$, then $a \in J(R) \Rightarrow$ there is a $b \in R$ such that $a + b + ab = 0$. Since $A$ is an ideal, $a \in A \Rightarrow ab \in A \Rightarrow b \in A$. Thus $A \cap J(R)$ is a quasi-regular ideal of $A$ and so by Theorem 17, $A \cap J(R) \subset J(A)$.

To see the other inclusion, suppose $\rho$ is a maximal regular right ideal of $R$ and let $\rho_A = A \cap \rho$.

If $A \nsubseteq \rho$, then the maximality of $\rho$ implies $A + \rho = R$. Thus, by the Second Isomorphism Theorem for modules, we have that

$$R/\rho \cong (A + \rho)/\rho \cong A/A \cap \rho = A/\rho_A$$

By Lemma 11 $R/\rho$ is an irreducible $R$-module. Hence $A/\rho_A$ is irreducible and thus
$\rho_A$ is a maximal ideal of $A$. Now since $\rho$ is regular, there is a $k \in R$ such that $x - kx \in \rho$. Since $A + \rho = R$, then there is an $a \in A, r \in \rho$ such that $k = a + r$.

Thus $x - kx = x - (a + r)x = x - ax - rx \in \rho$, and $rx \in \rho \Rightarrow x - ax \in \rho$. Hence for all $c \in A, c - ac \in A \cap \rho = \rho_A$, showing $\rho_A$ is regular in $A$. Thus $\rho_A$ is a maximal regular right ideal of $A$ and therefore $J(A) \subset \rho_A$. This is true for all maximal regular right ideals $\rho$ of $R$ such that $A \notin \rho$ and certainly for those which do contain $A$ (in this case, $\rho_A = A \cap \rho = A$ and $J(A) = A$). Thus $J(A) \subset \cap \rho_A = \cap (A \cap \rho) = A \cap (\cap \rho) = A \cap J(R)$, and we are done. \qed

**Corollary 40:** If $R$ is semisimple, then so is every ideal of $R$.

**Proof:** For any ideal $A$ of $R$, $J(A) = A \cap J(R)$, and $J(R) = (0) \Rightarrow J(A) = A \cap (0) = (0)$. Thus $A$ is semisimple. \qed

**Lemma 41:** An ideal of a semisimple Artinian ring is a semisimple Artinian ring.

**Proof:** Let $R$ be a SSAR and $A \neq (0)$ be an ideal of $R$. By previous corollaries, we have that $A = eR = Re$ for some idempotent $e$ in the center of $R$, that $1 \in R$, and also that $A$ is semisimple. We claim that $R = Re \oplus R(1 - e)$; that is, $R$ is the direct sum of $Re$ with $R(1 - e)$ (called the Peirce decomposition of $R$ relative to $e$). Note that since $1 - e$ is also in the center of $R$, then $R(1 - e)$ is an ideal of $R$. For any $x \in R, x = xe + x(1 - e)$; so $R = Re + R(1 - e)$. We also wish to show that $Re \cap R(1 - e) = (0)$. If $x = re = s(1 - e) \in Re \cap R(1 - e)$, then $x = xe = s(1 - e)e = s(e - e) = 0$. Thus $R = A \oplus R(1 - e)$. Finally define $\phi : R \longrightarrow A$ by $\phi(a + b) = a$, an onto ring homomorphism with $\ker \phi = R(1 - e)$.

Hence $A \cong R/R(1 - e)$ as rings. Since $A$ is the homomorphic image of the Artinian
ring $R$, then $A$ must be Artinian. \hfill \Box

**Definition 42:** A ring $R$ is simple if $R^2 \neq 0$ and the only two sided ideals of $R$ are $(0)$ and $R$.

Remarks: (1) A simple ring $R$ with unity must be semisimple. Why? Since $J(R)$ is a two-sided ideal of $R$, $J(R) = (0)$ or $J(R) = R$. Since $1 \in R$, then $J(R) = R$ would imply $M1 = (0)$ for any irreducible $R$-module $M$ and hence $MR = (0)$, a contradiction. So we must have $J(R) = (0)$.

(2) A simple Artinian ring $R$ must be semisimple. Why? $R$ Artinian implies $J(R)$ is nilpotent and so, if $J(R) = R$, then $R$ is nilpotent. $R$ simple implies $R^2 \neq (0)$ and $R^2$ is a two-sided ideal of $R$, we must have $R^2 = R$. Now $R^n = (0)$ for some $n > 2$ imply $R^{n-2}R^2 = R^{n-1}R = R^{n-1} = (0)$. Continuing in this manner, we obtain $R^2 = (0)$, a contradiction. Thus $J(R) = (0)$.

Before proving our next result, we first show in an Artinian ring, we cannot have an infinite direct sum of right (or two-sided) ideals. Suppose $A_1 \oplus A_2 \oplus \cdots$ is such an infinite direct sum. Consider the descending chain

$$A_1 \oplus A_2 \oplus \cdots \supset A_2 \oplus A_3 \oplus \cdots \supset A_3 \oplus A_4 \oplus \cdots$$

$R$ Artinian implies there exists an $n$ such that

$A_n \oplus A_{n+1} \oplus \cdots = A_{n+1} \oplus A_{n+2} \oplus \cdots$. This would imply that, for all $a \neq 0 \in A_n$, we could write $a$ as a finite sum of elements in $A_{n+1} \oplus A_{n+2} \oplus \cdots$, but this violates the fact that the sum $A_n \oplus A_{n+1} \oplus \cdots$ is direct (specifically, the intersection of $A_n$ with a finite sum of any of the other must be $(0)$).
**Theorem 43:** A semisimple Artinian ring is the direct sum of a finite number of simple Artinian rings.

**Proof:** We first claim that we can choose a nonzero minimal ideal of $R$. We can assume $R$ is not simple (otherwise the theorem is automatic). So $R$ has a proper nontrivial ideal (hence a proper right ideal). Now $R$ Artinian implies we can choose a minimal two-sided ideal $A \neq (0)$. We have shown $A$ must be semisimple Artinian and has a unity. Thus $A^2 \neq (0)$ Suppose $B \neq (0)$ is an ideal contained in $A$. Then $ABA$ is an ideal of $R$ and is nonzero since $A$ has a unity. Now $ABA \subset A$ and by the minimality of $A$, we have $ABA = A$. However, $ABA$ is also contained in $B$, giving $B = A$. Thus $A$ is simple.

By the argument given in the proof of Lemma 40, we can write $R = A \oplus T_0$ where $T_0$ is an ideal of $R$ and hence is itself semisimple Artinian. If $T_0 = (0)$, we are done. Otherwise pick a minimal ideal $A_1$ of $R$ lying in $T_0$. $A_1$ is simple Artinian and, as above, we can write $T_0 = A_1 \oplus T_1$ where $T_1$ is an ideal of $R$ which is semisimple Artinian. So we have $R = A_0 \oplus A_1 \oplus T_1 \oplus (where \ A_0 = A)$. Continuing in this fashion, we get ideals $A = A_0, A_1, A_2, ..., A_k, ...$ of $R$ that are all simple Artinian with $R = A_0 \oplus A_1 \oplus \cdots \oplus A_k \oplus T_k (T_k$ semisimple Artinian). At some point, $T_k = (0)$ for some $k$; otherwise we would have an infinite direct sum of ideal of $R$, a contradiction. Hence $R = A_0 \oplus \cdots \oplus A_k$, a direct sum of simple Artinian rings. \[\square\]

**Section 5: Wedderburn-Artin**

The Wedderburn-Artin theorem gives a complete classification of semisimple Artinian rings. This result has numerous application in noncommutative ring theory.
and is of particular importance in the theory of group representations. We will see that division rings play a fundamental role in the Wedderburn-Artin result and, in fact, are the underpinnings of this structure theorem.

**Definition 44:** A ring \( R \) is called **primitive** if it has a faithful irreducible \( R \)-module.

Since all our \( R \)-modules are right modules, then such a ring should really be called right primitive.

**Theorem 45:** \( R \) is primitive if and only if there is a regular right ideal \( \rho \subset R \) such that \( (\rho : R) = (0) \). Moreover, \( R \) is semisimple.

**Proof:** Let \( R \) be primitive with a faithful irreducible module \( M \). By Lemma 11, \( M \cong R/\rho \) for some minimal regular right ideal \( \rho \). \( R \) faithful implies \( A(M) = (0) \).

On the other hand,

\[
A(M) = A(R/\rho)
\]

\[
= \{ x \in R : (R/\rho)x = (0) \}
\]

\[
= \{ x \in R : (R/\rho)x = \rho \}
\]

\[
= \{ x \in R : Rx \subseteq \rho \}
\]

\[
= (\rho : R)
\]

Thus \( (\rho : R) = (0) \). Conversely, let \( \rho \) be a maximal regular right ideal of \( R \) such that \( (\rho : R) = (0) \). Setting, \( M = R/\rho \), we see that \( M \) must be irreducible. Furthermore, \( A(M) = (\rho : R) = (0) \) implies \( M \) is a faithful \( R \)-module and we can conclude \( R \) is primitive.
Now $R$ primitive implies $(\rho : R) = (0)$ for some maximal regular right ideal and since $J(R) = \cap (\rho : R)$, $\rho$ running over all maximal regular right ideals of $R$, we have $J(R) = (0)$.

We obtain two immediate corollaries:

**Corollary 46:** A commutative primitive ring must be a field.

**Proof:** Suppose $R$ is a commutative primitive ring. By the preceding theorem there is a maximal regular ideal $\rho$ such that $(\rho : R) = (0)$. $R$ commutative implies $\rho = (\rho : R) = (0)$ (because if $x \in \rho \Rightarrow xr \in \rho$ for all $r \in R$), hence $(0)$ is a maximal ideal of $R$. Since $R$ is commutative having a maximal ideal $(0)$, it must be Artinian and thus is a SSAR. By Corollary 38, $1 \in R$. Finally, it is easy to show that any commutative ring with unity, in which $(0)$ is a maximal ideal, must be a field.

**Corollary 47:** (a) Any simple, semisimple ring is primitive. (b) Any simple ring with unity is primitive.

**Proof:** (a) We claim $(\rho : R) = (0)$ for some maximal regular right ideal of $R$. Otherwise, since $(\rho : R)$ is a two-sided ideal, we would have $(\rho : R) = R$, for all such $\rho$. This would imply $J(R) = R$, a contradiction.

(b) Let $1 \in R$ and let $\rho$ be a maximal regular right ideal of $R$. Since $1 \in R$, $\rho$ is automatically regular and $(\rho : R) = R$ would imply $R = \rho$, contrary to maximality of $\rho$. Thus $(\rho : R) = (0)$ and $R$ is hence primitive.
This corollary demonstrates that, with a possible few bizarre exceptions, we can view primitivity as a generalization of simplicity. We now give an example of a primitive ring which is not simple.

Let $V \neq 0$ be a countably infinite dimensional vector space over any field (e.g. $V = F^\infty$). Let $L = \text{Hom}_F(V,V)$, the ring of all $F$–linear transformations from $V$ to $V$. $L$ is a ring under ordinary addition and composition. We want $V$ to be an $L$–module and thus define a mapping

$$L \times V \longrightarrow V \text{ defined by } (T, v) = T(v), \ T \in L, v \in V$$

To see that this map is in fact a Module let $T, T_1, T_2 \in L, v, v_1, v_2 \in V$. Then,

1. $(T, v_1 + v_2) = T(v_1 + v_2) = T(v_1) + T(v_2) = (T, v_1) + (T, v_2)$,
2. $(T_1 + T_2, v) = (T_1 + T_2)(v) = T_1(v) + T_2(v) = (T_1, v) + (T_2, v)$,
3. $((T, v_1), v_2) = (T(v_1), v_2) = (T(v_1))(v_2) = T(v_1v_2) = (T, v_1v_2)$.

Now we will show that $V$ is a faithful irreducible $L$–module.

**Faithful:** Suppose $(T, V) = 0$. Then $T(V) = 0$. If $T \neq 0 \Rightarrow V = 0$.

**Irreducible:** Clearly $(L, V) \neq (0)$ since $V \neq 0$ and since there is some $T \in L$ such that $T(V) \neq 0$. Now assume that $W$ is a submodule of $V$. If $W = (0)$ then we are done. Thus, assume $W \neq (0)$. Since $W$ is a submodule then for all $v \in V, w_1, w_2 \in W$ we have $w_1 + w_2 v \in W$. Therefore, as a vector space $W = V$.

To see that $L$ is not simple let

$$I = \{T \in L : \text{ rank } T < \infty\} = \{T \in L : \dim_F T(V) < \infty\}$$

We wish to show that $I$ is an proper two sided ideal of $L$.

**Ideal:** Let $T \in I, S \in L$. Then clearly both $T \circ S$ and $S \circ T$ have finite rank. Thus
both are contained in \( I \) making \( I \) an ideal.

To see that this ideal is properly contained in \( L \) not that the identity map has infinite rank and is therefore not contained in \( I \) but is contained in \( L \).

\[ \square \]

Let \( R \) be a primitive ring with a faithful irreducible module \( M \) having commuting ring \( \Delta = C(M) \), a division ring. We regard \( M \) as a right vector space over \( \Delta \) via \( M \times \Delta \mapsto M \) by \((m, \alpha) = m\alpha \) (\( \alpha \) applied to \( m \)).

**Definition 48:** \( R \) is said to act **densely** on \( M \) over \( \Delta \) if for any \( v_1, \ldots, v_n \in M \), which are \( \Delta \)-independent, and any \( w_1, \ldots, w_n \in M \), there exists \( r \in R \) such that \( v_ir = w_i, \ i = 1, \ldots, n \).

**Theorem 49: Density Theorem:** Let \( R \) be a primitive ring and let \( M \) be a faithful irreducible \( R \)-module. If \( \Delta = C(M) \), then \( R \) is a dense ring of linear transformations on \( M \) over \( \Delta \).

**Proof:** Let \( V \) be a finite dimensional subspace of \( M \) over \( \Delta \). To prove the theorem it suffices to show that if \( m \in M, m \notin V \), then we can find an \( r \in R \) with \( Vr = (0) \) but \( mr \neq 0 \).

To see this, we will assume that such an \( r \in R \) does exist. Then in that case we have that \( mrR \neq (0) \) and by the irreducibility of \( M, mrR = M \). Thus, we can find an \( s \in R \) such that \( mrs \) is arbitrary and \( Vrs = (0) \). Let \( v_1, \ldots, v_n \in M \) be linearly independent over \( \Delta \) and \( w_1, \ldots, w_n \in M \). Let \( V_i \) be the linear span over \( \Delta \) of the \( v_j \) with \( i \neq j \). Since \( v_i \notin V_i \) we can find a \( t_i \in R \) with \( v_it_i = w_i, V_it_i = (0) \). If \( t = t_1 + \cdots + t_n \) we see that \( v_it = w_i, 1 \leq i \leq n \). Thus \( R \) is dense on \( M \).

Now we wish to prove that given \( V \subset M \) of finite dimension over \( \Delta \) and
We proceed by induction on the dimension of $V$ over $\Delta$. If $\dim V = 0$, then we are done.

Let $\dim V_0 = \dim V - 1$ and suppose $w \notin V_0$. Then $V = V_0 + w\Delta$. Let $A(V_0) = \{x \in R : V_0 x = (0)\}$. By our induction hypothesis, if $y \notin V_0$ then there is an $r \in A(V_0)$ such that $y r \neq 0$. This means, if $mA(V_0) = (0)$, then $m \in V_0$. Not that $A(V_0)$ is a right ideal of $R$. Since $w \notin V_0, wA(V_0) \neq (0)$, hence, as a submodule of $M, wA(V_0) = M$. Suppose for $m \in M, m \notin V$, if $Vr = (0)$ then $mr = 0$. We will show that this leads to a contradiction. Define a map $\tau : M \rightarrow M$ as follows: if $x \in M$ with $x = wa, a \in A(V_0)$, then $x \tau = ma$. To see that $\tau$ is well defined note that if $x = 0$ then $0 = x = wa$, hence, $a$ annihilates both $V_0$ and $w$ and therefore, it annihilates all of $V$. By assumption $ma = 0$. This gives us that $x = x \tau = ma = 0$ showing $\tau$ is well defined.

Clearly $\tau \in E(M)$. Also, if $x = wa$ with $a \in A(V_0)$ then for any $r \in R$, since $ar \in A(V_0)$ we get that

$$xr = (wa)r = w(ar)$$

hence

$$(xr)\tau = m(ar) = (ma)r = (wa)\tau r = x\tau r.$$  

This gives us that $\tau \in \Delta$. Hence for $a \in A(V_0)$

$$ma = (wa)\tau = (w\tau)a$$

That is, $(m - w\tau)a = 0$ for all $a \in A(V_0)$. By our induction hypothesis we get that $m - w\tau \in V_0$ and hence $m \in V_0 + w\Delta = V$, a contradiction. This contradiction gives us our desired result. □
Density is a generalization of the key property of linear transformations. In fact, we will sketch the proof that if $R$ acts densely on $M$ over $\Delta$ and $M$ is finite dimensional over $\Delta$, then $R \cong \text{Hom}_\Delta(M, M)$. To see this: for any $a \in R$ recall the map $T_a : M \rightarrow M$ defined by $mT_a = ma$. It is easy to see $T_a$ preserves addition. If $\alpha \in \Delta$, $(m\alpha)T_a = (m\alpha)a = (ma)\alpha$ (since $\alpha \in C(M)$) = $(mT_a)\alpha$. Thus, we have a map $T : R \rightarrow \text{End}_\Delta(M, M)$ by $T(a) = T_a$. $T$ is a ring homomorphism. If $a \in \ker T$, then $T_a = 0$ on $M$ which implies $Ma = 0$ and $M$ faithful implies $a = 0$. Thus $T$ is one-to-one. For onto, fix a $\Delta$-basis $\{v_1, ..., v_n\}$ for $M$. If $\phi \in \text{Hom}_\Delta(M, M)$ is arbitrary, set $w_i = v_i$. Then the density implies there is an $r \in R$ such that $v_ir = w_i$.

For any $m \in M$, let $m = \sum_{i=1}^{n} v_i \alpha_i$ and so

$$m\phi = \sum_{i=1}^{n} w_i \alpha_i = \sum_{i=1}^{n} v_i r \alpha_i = (\sum_{i=1}^{n} v_i \alpha_i)r = mr = mT_r,$$

which gives $\phi = T_r$ proving $T$ is onto. Thus $T$ is the desired isomorphism.

We remark that in this case $\text{Hom}_\Delta(M, M) \cong M_n(\Delta)$.

With the Density Theorem proven, we can now prove the Wedderburn-Artin Theorem, a fundamental result in the classification of noncommutative rings. The Wedderburn-Artin Theorem plays a key role in the theory of group representations.

**Theorem 50: Wedderburn-Artin:** $R$ is a simple Artinian ring if and only if $R \cong M_n(D)$, where $D$ is a division ring.

**Proof:** Suppose $R$ is a simple Artinian ring. We first show $R$ is primitive. $R$ simple implies $J(R) = (0)$ or $J(R) = R$. Then $R^2 = R$, together with the fact that $J(R)$ is nilpotent (since $R$ is Artinian) would yield $R = (0)$, a contradiction. Thus
\( J(R) = \langle 0 \rangle \) and so \( R \) is semisimple and simple, thus it must be primitive.

Let \( M \) be a faithful irreducible \( R \)-module with \( D = C(M) \). If \( \dim_D M = n \), then the density of \( R \) implies \( R \cong M_n(D) \) and we are done. Assume not and let \( \{v_1, v_2, \ldots\} \) be an infinite linearly independent set over \( D \). For each index \( m \), set \( V_m = \text{span}_D \{v_1, v_2, \ldots\} \) and let \( \rho_m = \{x \in R : V_m x = 0\} \). The \( \rho_m \)'s are right ideals of \( R \) and clearly \( \rho_1 \supset \rho_2 \supset \cdots \) is a descending chain of right ideals. By density, we can choose \( r \in R \) such that \( V_m r = \langle 0 \rangle \) and \( V_{m-1} r = \langle 0 \rangle \). This means that for all \( m \) we can find \( r \in \rho_m \) such that \( r \in \rho_{m+1} \). Hence this chain of right ideals is never stationary. Therefore, \( M \) must be finite dimensional over \( D \) and so \( R \cong M_n(D) \). We leave the converse for later in the thesis. \( \square \)

When we combine Wedderburn-Artin with Theorem 43 we get our final structure theorem for these simple/semisimple Artinian rings and we see why division rings are so important, as they are the building blocks for some very ubiquitous rings (that is, semisimple Artinian rings).

**Corollary 51:** If \( R \) is a semisimple Artinian ring then

\[ R \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k) \]

where the \( D_i \) are division rings.
Chapter 2: Division Rings

In this chapter we will give two methods for constructing division rings. The first method uses skew Laurent series. This construction just requires a field $F$ and an automorphism $\sigma$ of $F$. The second method is a generalization of the construction of the field of fractions of an integral domain. If $R$ is a noncommutative domain, Ore would hope that we could somehow embed $R$ in a division ring of fractions. There are noncommutative domains which don’t have division rings of fractions. However, if $R$ satisfies an additional condition, called the Ore condition, then $R$ will have a division ring.

Section 1: Power Series and Laurent Series

**Definition 1:** Let $R$ be a ring and let $x$ be an indeterminate over $R$. A **power series** over $R$ is an expression of the form $\sum_{i=0}^{\infty} r_i x^i$. A **Laurent series** over $R$ is an expression of the form $\sum_{i=n}^{\infty} r_i x^i$, where $n \in \mathbb{Z}$ (some Laurent series can involve negative powers of $x$).

One can turn the set of all power series, denoted $R[[x]]$ and the set of all Laurent series, denoted $R((x))$, into rings under the usual operations of addition and multiplication. If $D$ is a division ring then $D((x))$ will be a division ring. However, this does not give us a useful construction because we to start with a division ring.

Let $R$ be a ring and $\sigma : R \rightarrow R$ an automorphism of $R$. Let $R[[x; \sigma]]$ denote the set of all expressions of the form $\sum_{i=0}^{\infty} r_i x^i$ where $r_i \in R$. We make $R[[x; \sigma]]$ into a ring as
follows:

Addition:
\[
\sum_{i=0}^{\infty} r_i x^i + \sum_{i=0}^{\infty} s_i x^i = \sum_{i=0}^{\infty} (r_i + s_i) x^i,
\]
for all \( r_i, s_i \in R \)

Multiplication is defined assuming that for all \( r \in R, xr = \sigma(r)x \). Hence,
\[
\left( \sum_{i=0}^{\infty} r_i x^i \right) \left( \sum_{i=0}^{\infty} s_i x^i \right) = \sum_{i=0}^{\infty} t_i x^i
\]

Where \( t_i = \sum_{j=0}^{i} r_j \sigma(s_{i-j}) \).

Of course we have to check that \( R[[x; \sigma]] \) is in fact a ring under these operations.

Most of the ring axioms are straightforward to check. We remark that to check the
associative properties under multiplication, it suffices to check multiplication of
monomials is associative. We check this:
\[
((r_i x^i)(s_j x^j))(t_k x^k) = \\
(r_i \sigma^i(s_j) x^{i+j})(t_k x^k) = \\
r_i \sigma^i(s_j) \sigma^{i+j}(t_k) x^{i+j+k} = \\
r_i \sigma^i(s_j \sigma^j(t_k)) x^i x^j x^k = \\
(r_i x^i)(s_j \sigma^j(t_k) x^j x^k) = \\
(r_i x^i)((s_j x^j)(t_k x^k)),
\]
as desired.

Definition 2: \( R[[x; \sigma]] \) is called the ring of Skew power series over \( R \).

Proposition 3: The element \( p(x) = \sum_{i=0}^{\infty} r_i x^i \in R[[x; \sigma]] \) is invertible if and only if \( r_0 \)
is invertible in \( R \).
Proof: Suppose $r_0$ is an invertible element of $R$. Then we can write

\[ p(x) = r_0(1 + r_0^{-1}r_1x + r_0^{-1}r_2x^2 + \cdots) = r_0(1 + q(x)) \]

where $q(0) = 0$.

Consider the expression

\[ 1 - q(x) + q(x)^2 - \cdots. \]

$q(0) = 0$ implies that this expression is a skew power series ($q(0) = 0$ insures that none of the coefficients will involve infinite sums). Now

\[ (1 + q(x))(1 - q(x) + q(x)^2 - \cdots) = (1 - q(x) + q(x)^2 - \cdots) + (q(x) - q(x)^2 + q(x)^3 - \cdots) = 1, \]

implying $1 + q(x)$ is invertible. Since $r_0$ is invertible in $R$, it follows that $p(x) = r_0(1 + q(x))$ is the product of two invertible elements in $R[[x; \sigma]]$ and hence invertible.

Conversely, if $p$ is invertible then all the components of $p$ are invertible. Thus, $r_0$ must be invertible. \qed

We remark that the above construction would have worked if we had just assumed $\sigma : R \rightarrow R$ is an endomorphism. However, the existence of the negative exponents in Laurent series necessitates that we assume $\sigma$ is an automorphism.

In this case we have, for all $r \in R$, $r = x\sigma^{-1}(r)x^{-1}$ which gives $x^{-1}r = \sigma^{-1}(r)x^{-1}$ and so the skew multiplication is also valid for negative exponents. Let $R((x; \sigma))$ denote the set of all expressions of the form $\sum_{i=0}^{\infty} r_i x^i$, $n \in \mathbb{Z}$, where addition is as usual and multiplication is skewed. One can check the ring axioms and we note that the associative property of multiplication reduces to considering monomials (as above). We remark that $R((x; \sigma))$ can be obtained through localizing $R[[x; \sigma]]$ of the
Definition 4: $R((x; \sigma))$ is called the ring of Skew Laurent series over $R$.

Corollary 5: The element $p(x) = \sum_{i=n}^{\infty} r_i x^i \in R((x; \sigma))$ is invertible in $R((x; \sigma))$ if and only if $r_n$ is invertible in $R$.

Proof: If $r_n$ is invertible then

\[
p(x) = r_n x^n + r_{n+1} x^{n+1} + \cdots
= r_n x^n (1 + \sigma^{-n}(r_n r_{n+1}) x + \sigma^{-n}(r_n r_{n+2}) x^2 + \cdots)
= r_n x^n (1 + q(x)),
\]

where $q(x) \in R[[x; \sigma]]$ with $q(0) = 0$. By Proposition 3, $(1 + q(x))$ is invertible in $R[[x; \sigma]]$ hence in $R((x; \sigma))$. Since $r_n x^n$ is also invertible in $R((x; \sigma))$, the result follows. The converse is automatic.

The result immediately yields a method for constructing division rings and we have

Corollary 6: If $F$ is a field and $\sigma$ is an automorphism of $F$, then $F((x; \sigma))$ is a division ring.

Proof: By the preceding corollaries we know every nonzero element of $F((x; \sigma))$ is invertible. Hence $F((x; \sigma))$ is a division ring.

We remark that if $\sigma$ is not the identity automorphism then in general $xa = \sigma(a)x \neq ax$. Hence, we have a "real live" division ring.

The center of a division ring is a field and we can view the division ring as a vector space over its center. It is important to know the dimension of the division ring over
its center. We now turn to the problem of determining the center of a skew Laurent
series over a field and then determine its dimension over the center.

First we fix some notation. If \( \sigma \in \text{Aut}(F) \), let \( F_\sigma \) denote its fixed field. That is,
\( F_\sigma = \{ a \in F : \sigma(a) = a \} \). The center will depend on the period of \( \sigma \) (i.e. the order
of \( \sigma \) as an element in the group \( \text{Aut}(F) \)). Note that if \( a \in F_\sigma \), then for any \( n \in \mathbb{Z} \),
\( x^n a = \sigma^n(a)x^n = ax^n \) and thus \( F_\sigma \) is always contained in the center of \( F((x; \sigma)) \). If
\( \text{ord}(\sigma) = n < \infty \), then for any \( a \in F \), \( x^n a = \sigma^n(a)x^n = ax^n \), proving \( x^n \) is central.
Hence, if \( \text{ord}(\sigma) = n \), then \( F_\sigma((x^n)) \) is contained in the center of \( F((x; \sigma)) \) and
\( F_\sigma((x^n)) \) is the field of Laurent series in \( x^n \) with coefficients in \( F_\sigma \).

**Theorem 7:** The center of \( F((x; \sigma)) \) is

\[
C = \begin{cases} 
  F_\sigma, & \text{if } \sigma \text{ has infinite period} \\
  F_\sigma((x^n)), & \text{if } \sigma \text{ has finite period } n
\end{cases}
\]

**Proof:** If \( p(x) = \sum_{i=n}^{\infty} a_i x^i \in C \), then \( xp(x) = p(x)x \) gives
\[
\sum_{i=m}^{\infty} \sigma(a_i)x^{i+1} = \sum_{i=m}^{\infty} a_i x^{i+1}
\]
and thus \( \sigma(a_i) = a_i \) hence \( a_i \in F_\sigma \).

Now, for any \( b \in F \), we have \( bp(x) = p(x)b \) gives
\[
\sum_{i=m}^{\infty} ba_i x^{i+1} = \sum_{i=m}^{\infty} a_i \sigma^i(b)x^{i+1}
\]
and we obtain \( ba_i = a_i \sigma^i(b) \). If \( a_i \neq 0 \), then \( \sigma^i(b) = b \), that is \( \sigma^i \) is the identity
automorphism.

**Case 1:** \( \sigma \) has infinite order. Then \( \sigma^i = 1 \) implies \( i = 0 \) and thus \( p(x) = a_0 \in F_\sigma \),
proving $C \subset F_\sigma$. By the preceding discussion, we can conclude $C = F_\sigma$.

Case 2: $\sigma$ has finite order $n$. Then $\sigma^i = 1$ implies $n|i$, say $i = nq_i$, some $q_i \in \mathbb{Z}$. Thus

$$p(x) = \sum_{i=n}^{\infty} a_i x^i = \sum_{i=n}^{\infty} a_i x^{nq_i} \in F_\sigma((x^n)).$$

Hence, in this case, $C = F_\sigma((x^n))$. \hfill \qed

In order to determine the dimension of $F((x;\sigma))$ over its center we need a theorem from field theory. We state it here without proof.

**Artin’s Lemma:** Let $F$ be a field, let $F_\sigma$ be the fixed field of $\sigma$, and let $\sigma$ have period $n$. Then

$$[F : F_\sigma] = n.$$ 

**Theorem 8:** If $\sigma$ has finite order, say order $n$, then the dimension of $F((x;\sigma))$ over its center $F_\sigma((x^n))$ is $n^2$. Otherwise, the order of $\sigma$ is infinite and the dimension of $F((x;\sigma))$ over its center $F_\sigma$ is infinite.

**Proof:** If the period of $\sigma$ is infinite then it is clear that the dimension of $F((x;\sigma))$ is infinite over its center. Thus, assume that $\sigma^n = I$, some $n \in \mathbb{Z}$. First we show that $[F((x;\sigma)) : F((x^n))] = n$. Consider $S = \{1, x^2, \ldots, x^{n-1}\}$, we wish to show that $S$ is a basis for $F((x;\sigma))$ over $F((x^n))$. By the division algorithm $i = nt_i + r$, where $0 \leq r_i < n$ and $t_i \in \mathbb{Z}$. Thus $x^i = (x^n)^{t_i} x^{r_i}$ which implies $S$ spans $F((x;\sigma))$ over $F((x^n))$. Now assume $\sum_{j=0}^{n-1} h_j x^j = 0$, where $h_j = \sum_{i=0}^{\infty} a_{ij} x^{ni}$, then

$$0 = \sum_{j=0}^{n-1} \left( \sum_{i=0}^{\infty} a_{ij} x^{ni} \right) x^j = \sum_{j=0}^{n-1} \sum_{i=0}^{\infty} a_{ij} x^{ni+j}.$$

Since $j$ runs from 0 to $n - 1$ we get that the powers of $x$ must all be distinct which forces $a_{ij} = 0$. So that $S$ is a basis and indeed $[F((x;\sigma)) : F((x^n))] = n$.

Then by Artin’s Lemma we get that $[F : F_\sigma] = n$. If $\{c_1, \ldots, c_n\}$ is a basis for $F$ over $F_\sigma$ then this set must also be a basis for $F((x^n))$ over $F_\sigma((x^n))$. Then by the tower
We look at some examples:

(1) the first known example of a division ring is the ring of quaternions. The quaternions were discovered by Hamilton in 1831. The ring of quaternions, denoted $H$, is the four dimensional real vector space with $\mathbb{R}$ basis $\{1, i, j, k\}$ having relations

\[
i^2 = j^2 = k^2 = -1
\]

\[
ij = k, \ jk = i, \ ki = j
\]

\[
ji = -k, \ kj = -i, \ ik = -j
\]

(2) Let $\sigma : \mathbb{C} \rightarrow \mathbb{C}$ be complex conjugation. in $\mathbb{C}[x; \sigma]$ we have $xi = -ix$. So $\mathbb{C}[x; \sigma]$ is a non-commutative domain.

Suppose that $p = \sum_{i=0}^{n} a_i x^i$ is central in $\mathbb{C}[x; \sigma]$. Then $xp = px$ implies that $\sigma(a_i) = a_i$ and hence $a_i \in \mathbb{R}$, for all $i$. Also, for all $u \in \mathbb{C}$, $up = pu \Rightarrow ua_i = a_i \sigma^i(u)$. Hence, if $a_i$ is not zero then $\sigma^1 = \text{identity on } \mathbb{C}$ and since $\sigma$ has period two, we must have that $i$ is even. It follows that the center of $\mathbb{C}[x; \sigma] = \mathbb{R}[x^2]$.

**Section 2: Ring of Fractions and the Ore Domain**

In the previous section we were able to construct division rings from the Laurent and power series rings. Now we will show what conditions are required in order for a noncommutative ring to have a division ring of fractions. We will show that a necessary and sufficient condition for a division of fractions of $R$ to exist is that $R$ satisfy the Ore condition (or be an Ore domain). As this is relatively easy to check.
it becomes a great way of adding to our library of division rings.

A remark on notation: $R^\times$ will be the multiplicative group of a ring $R$.

**Definition 9a:** Given any domain $R$, a **Right Division Ring of Fractions of $R$** is a division ring $D$ with an embedding $\phi : R \rightarrow D$ such that $D$ is generated by $\phi(R)$.

Alternatively,

**Definition 9b:** We say that $R$ has a **Right Division Ring of Fractions** if there exists an embedding $\phi : R \rightarrow D$ such that for all $d \in D, d = \phi(r)\phi^{-1}(s), r, s \in R, s \neq 0$. We write $rs^{-1}$ for $\phi(r)\phi^{-1}(s)$

**Definition 10:** Let $R$ be a ring. $R$ is said to satisfy the **right Ore condition** if,

$$aR \cap bR \neq (0), \text{ for all } a, b \in R^\times.$$ 

That is, for all $a, b \in R^\times$ there exists $a', b' \in R$ such that $ab' = ba' \neq 0$

**Definition 11:** Any domain $R$ that satisfies the right ore condition will be know as a **right Ore domain**.

For the duration of this chapter we will only be dealing with right division ring of fractions and right ore domains. Therefore, we will refer to each without the “right”.

**Theorem 12:** Let $R$ be any ring. $R$ has a division ring of fractions $D$ if and only if $R$ is a Ore domain.

**Proof:** First suppose that $R$ has a division ring of fractions $D$ such that every element can be expressed as $rs^{-1}, r, s \in R^\times, s \neq 0$. Since this is a division ring (i.e.
non commutative) then there exists $r', s' \in R^\times$ such that $s^{-1}r = r's'^{-1} \Rightarrow rs' = sr'$.

Thus, for any $r, s \in R^\times$ we get that $rR \cap sR \neq (0)$

Conversely, since we want are elements of the form $rs^{-1}, r, s \in R$ then we naturally define a set $S = R \times R^\times$. We define an equivalence relation on $S$ by the rule

$(a, b) \sim (a', b') \Leftrightarrow au = a'u'$ and $bu = b'u' \neq 0$ for some $u, u' \in R^\times$. To show that $\sim$ is reflexive we let $u, u' = 1$ and we get that $au = au', bu = bu'$. Symmetry is trivial since if $au = a'u'$ then $a'u' = au$.

To show the transitive property holds suppose that $(a, b) \sim (a', b')$ and that $(a', b') \sim (a'', b'')$ Then there exists $u, u', v, v' \in R^\times$ such that

$$au = a'u'$$
$$a'v = a''v'$$
$$bu = b'u' \neq 0$$
$$b'v = b''v' \neq 0$$

Since $R$ is an Ore Domain then there exists $s \in R, s' \in R^\times$ such that $b'u's = b'vs'$.

Since $R$ is a domain and $s' \neq 0$ then we have $b'vs' \neq 0$ and thus $b'u's \neq 0$ and by cancellation (since $b, b' \neq 0$) we get $u's = vs'$. Since $aus = a'u's$ and $a'vs = a''v's$ we get

$$aus = a'u's = a'vs' = a''v's'$$

and similarly

$$bus = b'u's = b'vs' = b''v's' \neq 0$$

Thus, there exists $\alpha = us, \beta = v's \in R$ such that
\[a\alpha = a''\beta, b\alpha = b''\beta \neq 0 \Rightarrow (a, b) \sim (a'', b'')\]

The equivalence class of \((a, b)\) will be denoted \(a/b\). To define a ring structure on \(S/\sim\) we will define addition by

\[
a/b + c/d = (ar + cr')/m \text{ such that } br = dr' = m \neq 0, \text{ where } r, r' \in R^x
\]

and multiplication by

\[
(a/b)(c/d) = ax/gy, \text{ such that } n = bx = cy, \text{ where } x, y \in R^x
\]

Before we show that these operations are well-defined we will show that addition does not depend on the choice of \(m\) and it will follow that multiplication does not depend on our choice of \(n\).

Note that \(a/b = ar/br = ar/m\) and \(c/d = cr'/dr' = m\). If \(m' = bs = ds'\) where \(s, s' \in R^x\) and \(mu = m'u'\) (since we are in an Ore Domain such \(u, u' \in R^x\) do exist) then \(mu = m'u' \Rightarrow bru = bsu' \Rightarrow ru = su'\). Similarly we get \(r'u = s'u'\). Therefore,

\[
a/b + c/d = (ar + cr')/m
\]

\[
= (ar + cr')u/mu
\]

\[
= (aru + cr'u)/mu
\]

\[
= (asu' + cs'u')/m'u'
\]

\[
= (as + cs')/m'
\]

\[
= a/b + c/d
\]

To show that addition is well-defined let \(a/b = a'/b'\). Then there exists \(s, s', s''\) such...
that $m' = bs = ds = b's''$ and $as = a's''$. Then,

\[ \frac{a}{b} + \frac{c}{d} = \frac{(ar + cr')}{m} \]

\[ = \frac{(as + cs')}{m'}, \text{ by above} \]

\[ = \frac{(a's'' + cs')}{m'} \]

\[ = \frac{(a's'' + cs')}{b's''} \]

\[ = \frac{a's''/b's'' + cs'/ds'}{m'} \]

\[ = \frac{a'/b' + c/d}{m'} \]

To show that multiplication is well-defined we first note that $\frac{a}{b} = \frac{ax}{n}$ and $\frac{c}{d} = \frac{n}{dy}$. Again let $\frac{a}{b} = \frac{a'}{b'}$. Choose $u, v$ such that $bu = b'v$ and $au = a'v$. Let $x', y' \in R^\times$ such that $n' = bux' = cvy \Rightarrow n' = b'vx' = cvy$. We get that

\[ \left(\frac{a}{b}\right) \left(\frac{c}{d}\right) = \left(\frac{au}{bu}\right) \left(\frac{cv}{dv}\right) \]

\[ = aux/dvy \]

\[ = a'vx/dvy \]

\[ = a'x/dy \]

\[ = \left(\frac{a'/b'}{c/d}\right) \]

A similarly tedious argument will show that the ring laws do in fact hold. We will denote this ring by $D$. Now $\frac{a}{b} = 0$ is and only if $\frac{a}{b} = 0/b'$ for all $b' \in R^\times$ if and only if $au = 0u'$, $bu = b'u' \neq 0$ if and only if $a = 0$. Thus if $a \neq 0$ then $(a/b)^{-1} = b/a$. Hence every non-zero element in $D$ has an inverse in $D$ which gives us that $D$ is a division ring.

Clearly that the map $f : a \rightarrow a/1$ is an embedding of $R$ into $D$. \qed
Section 3: Noetherian Rings

In this section we prove that all Noetherian Rings satisfy the Ore Condition. This will allow us to more easily recognize Ore Domains and provide substantial examples of noncommutative rings with a ring of fractions. Before we get to this result we define some terms and prove a result concerning Noetherian Rings.

The ascending chain condition (ACC) on right ideals; Given any ascending chain of right ideals, \( A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \), there exists \( n \in \mathbb{N} \) such that \( A_n = A_j \) for all \( j \geq n \)

Maximum Condition on Right Ideals: Let \( S \neq \emptyset \) be a collection of right ideals, then \( S \) has a maximal element with respect to set inclusion.

Finitely Generated Right Ideal: Let \( A \) be a right ideal of \( R \), then there exist \( a_1, \ldots, a_n \in A \) such that \( A = a_1R + \cdots a_nR \)

Definition 13: A Noetherian Ring is a ring that satisfies any of the preceding three condition.

Theorem 14: The Following are all equivalent:

(a) The ACC on right ideals

(b) The Maximum condition on right ideals

(c) Finitely generated right ideals

Proof: “(a) \( \Rightarrow \) (b)” Let \( S \neq \emptyset \) be a collection of right ideals and suppose \( R \) satisfies the ACC on right ideals. Let \( A_1 \subset S \). If \( A_1 \) is not maximal then there is an \( A_2 \) such that \( A_1 \subsetneq A_2 \). If \( A_2 \) is not maximal then again there is an \( A_3 \) such that \( A_2 \subsetneq A_3 \).
We continue this way until we reach an \( n \in \mathbb{N} \) such that \( A_n = A_{n+1} = \cdots \) (because \( R \) satisfies ACC). Thus (b) is satisfied.

“(b) \( \Rightarrow \) (c)” Let \( R \) satisfy the maximum condition on right ideals and assume there exists a right ideal \( I \) of \( R \) that is not finitely generated. Let 
\[
S = \{ A \subset R : A \text{ is finitely generated and } A \subset I \}.
\]
Since \( (0) \subset I \Rightarrow S \neq \emptyset \Rightarrow S \) has a maximal element, say \( J \). Now \( J \subset S \Rightarrow J \subsetneq I \). Thus, there exists \( a \in S \) such that \( a \notin J \). This gives us \( J \subset J + aR \subset I \) which implies \( J + aR \subset S \) because \( aR \subset I \) and is finitely generated. But this contradicts the maximality of \( J \).

“(c) \( \Rightarrow \) (a)” Suppose that every right ideal is finitely generated. Given an ascending chain of right ideals \( A_1 \subset A_2 \subset \cdots \subset A_n \cdots \). It is clear that \( \bigcup_{i \geq 1} A_i \) is also a right ideal. This gives us that \( \bigcup_{i \geq 1} A_i \) is finitely generated and thus there exists \( b_1, \ldots, b_n \) such that \( \bigcup_{i \geq 1} A_i = b_1 R + \cdots + b_n R \). Since each \( b_i \in A_j \) for some \( j \) then there exists \( k \) such that \( b_i \in A_k \) for all \( i = 1, \ldots, n \). Thus \( \bigcup_{i \geq 1} A_i \subset A_k \) \( \square \)

**Theorem 15:** If \( R \) is a right Noetherian domain, then \( R \) is a right Ore domain.

**Proof:** Let \( a, b \neq 0 \in R \) and consider \( bR \subset bR + abR \subset bR + abR + a^2bR \subset \cdots \).

Since \( R \) is a right Noetherian ring it satisfies the ACC so that there exists \( n \) such that \( bR + abR + \cdots + a^n bR = bR + abR + \cdots + a^n bR + a^{n+1}bR \). Hence, 
\[
a^{n+1} b = \sum_{i=0}^{n} a^i bR.
\]
Let \( c_k, k \geq 0 \), be the first nonzero \( c_i \) of \( R \) such that 
\[
a^{n+1} b = a^k b c_k + \cdots + a^n b c_n \text{ where } k \leq n
\]
\[
= a^k (b c_k + \cdots + a^{n-k} b c_n)
\]
Since $R$ is a domain then cancellation holds. Thus

\[ a^{n-k+1}b = bc_k + \cdots + a^{n-k}bc_n \Rightarrow \]

\[ bc_k = a^{n+1-k}b - a^{n-k}bc_n - \cdots - abc_{k+1} \]

Now $a, b \in R \Rightarrow a^{n-k+1}b \in aR \cap bR \Rightarrow bc_k \in aR \cap bR$. Since $c_k \neq 0, b \neq 0 \Rightarrow aR \cap bR \neq (0)$

With this Theorem established we wish to expand our supply of division rings via Noetherian rings. We will prove that if $R$ is Noetherian then the set of all polynomials over $R$ with indeterminate $x$ and with skew multiplication $\sigma$ (denoted $R[x; \sigma]$) is as well. As a consequence we will get $R[x; \sigma]$ is an Ore Domain whenever $R$ is an Ore Domain (this is called the Hilbert Basis Theorem). However, before we can this Theorem we must state the Modules version of Theorem 13. We only state it because in the proof one needs only to replace the word ideal with the word module to get the desired result.

**The Ascending Chain Condition (ACC) on right Submodules:** Given any ascending chain of right submodules, $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$, there exists $n \in \mathbb{N}$ such that $A_n = A_j$, for all $j \geq n$

**Maximum Condition on Right Submodules:** Let $S \neq \emptyset$ be a collection of right submodules, then $S$ has a maximal element with respect to set inclusion.

**Finitely Generated Right Submodules:** Let $A$ be a right submodules of $R$, then there exist $a_1, \ldots, a_n \in A$ such that $A = a_1R + \cdots + a_nR$

**Theorem 16:** The Following are all equivalent:
Theorem 17: Hilbert Basis Theorem: Let \( R \) be a right Noetherian Ring and let \( \sigma \in Aut(R) \). Then \( R[x;\sigma] \) is right Noetherian.

Proof: Let \( I \neq 0 \) be a right ideal of \( R[x;\sigma] \). We will show that \( I \) is finitely generated.

Let \( J = \{ r \in R : rx^d + r_{d-1}x^{d-1} + \cdots + r_0 \in I, r_i \in R \} \cup \{0\} \). \( J \) is clearly an additive subgroup of \( R \). To see that it is also a right ideal of \( R \) let \( r \in J, a \in R \) and let \( p(x) = rx^d + r_{d-1}x^{d-1} + \cdots + r_0 \in I \). Then
\[
p(x)\sigma^{-d}(a) = rx^d\sigma^{-d}(a) + \cdots = rax^d + \cdots \in I \Rightarrow ra \in J.
\]

Since \( R \) is Noetherian then \( J \) must be finitely generated, say \( J = \langle r_1, \ldots, r_k \rangle \) with \( r_i \neq 0 \), for all \( i \). Now since \( r_i \in J \) for \( i = 1, \ldots, k \) \( \Rightarrow \) there are \( p_i(x) \in I \), for \( i = 1, \ldots, k \), such that \( r_i \) are the leading coefficients of the \( p_i(x) \) and where the \( \text{deg } p_i(x) = n_i \). Let
\[
n = \max\{n_1, \ldots, n_k\}.
\]

Note that \( p_i(x)x^{n-n_i} = (r_i x^{n_i} + \cdots) x^{n-n_i} = r_i x^n + \cdots \in I \). Thus, without loss of generality we may assume that \( p_i = r_i x^n \), for all \( i = 1, \ldots, k \).

Now let \( N = R + Rx + \cdots Rx^{n-1} \). That is, \( N \) contains all the elements of \( R[x;\sigma] \) that have \( \text{deg } < n \). Note that \( N = R + Rx + \cdots + x^{n-1}R \) since \( \sigma \) is an automorphism. Thus we get that \( N \) is a right submodule of \( R[x;\sigma] \). Since \( N \) is finitely generated then it must be Noetherian.

This gives us that \( I \cap N \) is finitely generated. Thus, we can let \( I \cap N = \langle q_1, \ldots, q_t \rangle \).

Let \( I_0 = \langle p_1, \ldots, p_k, q_1, \ldots, q_t \rangle \) be a right ideal of \( R[x;\sigma] \). Clearly \( I_0 \subset I \cap N \subset I \).
Now let $p(x) \in I$ such that $\deg p(x) < n$. This gives us that $p(x) \in I \cap N \Rightarrow$ there exist $a_1, \ldots, a_t \in R$ such that $p(x) = q_1a_1 + \cdots + q_ta_t \Rightarrow p(x) \in I_0$. Thus, $I = I_0$

Now we show that $I_0$ is finitely generated. Consider $p(x) \in I$ such that

\[ \deg p(x) = m \geq n \]

and suppose that for all $q(x) \in I$ with $\deg q(x) < m$ that $q(x) \in I_0$. Also, we let be the leading coefficient of $p(x)$. Then $p(x) = rx^m + \cdots$.

Since $p(x) \in I$ then $r \in J \Rightarrow$ there exist $a_1, \ldots, a_k \in R$ such that

\[ r = r_1a_1 + \cdots + r_ka_k. \]

We wish to construct an element in $I_0$ with degree exactly $m$ with leading coefficient $r$. Observe that for all $i, p_i\sigma^{-n}(a_i) = r_ia_i x^n + \cdots$. So if

\[ q = (p_1\sigma^{-n}(a_1) + \cdots + p_k\sigma^{-n}(a_k))x^{m-n} = [(r_1a_1 x^n + \cdots) + \cdots + (r_ka_k x^n + \cdots)]x^{m-n} = (r_1a_1 + \cdots + r_ka_k)x^m + \cdots \in I_0. \]

Now $p - q \in I$ with degree less than $m \Rightarrow p - q \in I_0 \Rightarrow p \in I_0$. \hfill \Box

**Theorem 18:** Let $R$ be a right Ore Domain and $\sigma \in Aut(R)$. Then $R[x; \sigma]$ is a right Ore Domain.

**Proof:** Please refer to Theorem 28 as that is the more generalized version. \hfill \Box

**Section 4: Skew Polynomial Rings:**

In this section we will define the skew polynomial ring $R[x; \sigma, \delta]$ and show that for any division ring $K$, $K[x; \sigma, \delta]$ has a division ring of fractions. We then widen the restrictions by allowing $R$ to be an ore domain and we will not only show that $R[x; \sigma, \delta]$ has a division ring of fractions but we will explicitly find it. To prove the latter we will develop a new concept called localization.

When dealing with polynomial rings it is important that the degree of the
polynomials are preserved via commutation and multiplication. Let \( R \) be a ring and let \( R < x > \) be the additive group freely generated over \( R \) by \( x \). The elements of \( R < x > \) are of the form \( r_0 + r_1 x + \cdots + r^n x^n \) with no pre-defined multiplication.

Earlier we defined \( \sigma \) to be the automorphism that skews multiplication, that is \( x r = x \sigma (r) \). Defining \( \sigma \) this way preserves the degree of the polynomial \( x r \) and thus giving us an integral domain (we earlier called this set \( R[x; \sigma] \)). We will further complicate matters by taking \( x r = \delta (r) + \sigma (r)x \). This is the more general case of skewing polynomials but it still preserves degrees.

**Definition 19:** Let \( \sigma, \delta : R \longrightarrow R \) such that \( x r = \delta (r) + \sigma (r)x \).

**Proposition 20:** (a) \( \sigma \) is an injective endomorphism. (b) \( \delta \) is an additive homomorphism such that for all \( r, s \in R \), \( \delta (rs) = \sigma (r)\delta (s) + \delta (r)s \). We call \( \delta \) a \( \sigma \)-derivation.

**Proof:** Since it is true that \( x (r + s) = xr + xs \) then \( x (r + s) = \delta (r + s) + \sigma (r + s)x \).

Also, \( xr + xs = \delta (r) + \sigma (r)x + \delta (s) + \sigma (s)x = \delta (r) + \delta (s) + [\sigma (r) + \sigma (s)]x \). Thus, \( \delta (r + s) = \delta (r) + \delta (s) \) and \( \sigma (r + s) = \sigma (r) + \sigma (s) \). So we get that both maps are additive homomorphisms. Now consider \( x (rs) \) and \( (xr)s \).

\[
x (rs) = \delta (rs) + \sigma (rs)x
\]

\[
(xr)s = \delta (r)s + \sigma (r)xs
\]

\[
= \delta (r)s + \sigma (r)[\delta (s) + \sigma (s)x]
\]

\[
= \delta (r)s + \sigma (r)\delta (s) + \sigma (r)\sigma (s)x
\]

Thus we get that \( \sigma (rs) = \sigma (r)\sigma (s) \) and \( \delta (rs) = \delta (r)s + \sigma (r)\delta (s) \). To show that \( \sigma \) is
injective suppose $r \neq r' \in R$. Then

$$xr \neq xr' \Rightarrow \delta(r) + \sigma(r)x \neq \delta(r') + \sigma(r')x \Rightarrow \sigma(r) \neq \sigma(r') \tag*{□}$$

**Definition 21:** Let $R$ be any ring and let $\sigma, \delta$ be as above. Then $R[x; \sigma, \delta]$ is called the **Skew Polynomial Ring**.

**Proposition 22:** The degree of $R[x; \sigma, \delta]$ satisfies the following three conditions for any domain $R$:

(i) $\deg f \in \mathbb{N}$, for all $f \neq 0$

(ii) $\deg(f - g) \leq \max\{\deg f, \deg g\}$

(iii) $\deg(fg) = \deg f + \deg g$

Moreover, $R[x; \sigma, \delta]$ is a domain

**Proof:** (i) and (ii) are trivial. To see that (iii) holds we note that in the previous proof $\sigma$ preserved degrees. Thus, if $\deg f = a, \deg g = b$ then $\deg(fg) = a + b$. That is, $(r_ax^a)(s_bx^b) = r_ax^as_bx^{a+b} = r_a\sigma^a(s_b)x^{a+b}$, where $r_a\sigma^a(s_b) \neq 0$ because $R$ is a domain. Thus, we also get that $R[x; \sigma, \delta]$ is a domain. $\tag*{□}$

In order to prove the next theorem it is important to note that every principal right ideal domain (P.R.I.D.) is a right noetherian ring. This is because in a P.R.I.D. every right ideal is finitely generated and thus satisfies the definition of noetherian.

**Theorem 23:** Let $K$ be a division ring with $\sigma, \delta$ as defined above. Then $K[x; \sigma, \delta]$ has a division ring of fractions.

**Proof:** We will show that a $K[x; \sigma, \delta]$ has a division ring of fractions by showing that it is P.R.I.D. which will give us that it is noetherian. This will prove the
theorem as earlier we proved that noetherian implies ore domain. To show that $K[x;\sigma,\delta]$ is a P.R.I.D. we use the division algorithm. Let $P = K[x;\sigma,\delta]$ and let $f, g \in P$. We claim that there exists $q, r \in P$ such that $f = qg + r$, $\deg r < \deg g$. If $\deg f < \deg g$ let $q = 0$, $r = f$ and we are done. Otherwise, $\deg f \geq \deg g$. In this case let $f = x^r a_r + \cdots + a_0$ and $g = x^s b_x + \cdots + b_0$. Then $r \geq s$. If $q_1 = b_s^{-1} x^{r-s} a_r$ then $gq_1$ has the same leading term as $f$ and so $\deg(f - gq_1) \leq \deg f$. If $\deg(f - gq_1) \geq \deg g$ we can continue this process and after at most $r - s$ steps we obtain an $r \in P$ such that $\deg r < \deg g$. Thus the claim is proven and we return to showing that $P$ is P.R.I.D.. Suppose $I \neq (0)$ let $h \in I$ be a non-zero polynomial of minimal degree. If $f \in I$ then there exists unique $q, r$ such that $f =hq + r$, where $\deg r < \deg h$. Since $f, h \in I \Rightarrow r = f - hq \in I$. By the minimality of $h$ we get that $r = 0$ and thus $f = hq \Rightarrow I$ is a right principal ideal. □

We will refer to the division ring of fractions of $K[x;\sigma,\delta]$ as the **Skew Function Field** denoted $K(x;\sigma,\delta)$

**Definition 24:** A regular right Ore set is a multiplicative set subset of a ring, say $S \subset R$, satisfying the following two conditions:

(i) Given any $s \in S, r \in R, sR \cap rS \neq (0)$

(ii) No element of $S$ is a zero-divisor.

The following theorem is a generalization of Theorem 12. The proofs are very similar and therefore we state the generalization without proof.

**Theorem 25:** Let $R$ be any ring and let $S$ be a regular right Ore set. Then $R$ can be embedded in a ring $K$ in which all the elements of $S$ have inverses and if
$f : R \rightarrow K$ is the embedding, then every element of $K$ has the form $f(a)f^{-1}(b)$, where $a \in R, b \in S$.

**Definition 26:** The ring $K$ constructed in the previous theorem is called the **localization** of $R$ at $S$ and will be denoted $R_S$. Note that when $R$ is an Ore domain we can take $S$ to be $R^\times$.

**Proposition 27:** Let $R$ be any ring, let $S$ be a right regular Ore set in $R$, and let $R_S$ be the localization. Then any set in $R_S$ can be brought to a common denominator.

**Proof:** Let $a/b, a'/b' \in R_S$. In Theorem 12 we showed that these two elements could be brought to a common denominator. In particular, we let $bu = b'u' = m$ for some $u, u' \in R^\times$ whereas in this case we will say that $u, u' \in S$. We got 

$$a/b = au/m, a'/b' = a'u'/m.$$ 

In the general case we use induction on the number of elements. So that if we are given $a_1/b_1, \ldots, a_n/b_n$ then we can bring $a_2/b_2, \ldots, a_n/b_n$ to a common denominator, say $a_2u_2/m, \ldots a nu_n/m$. By the Ore condition there exists $v \in R, v' \in S$ such that $b_1v = mv' \in S$. Hence $a_1/b_1 = a_1v/b_1v = a_1v/mv'$ and for any $i = 2, \ldots, n$, we have $a_i/b_i = a_iu_i/b_iu_i = a_iu_i/m = a_iu_i v'/mv'$. This concludes the proof. \hfill $\square$

**Theorem 28:** Let $R$ be a right Ore domain with an injective endomorphism $\sigma$ and an $\sigma$-derivation $\delta$. Then $R[x; \sigma, \delta]$ is a right Ore domain.

**Proof:** Proposition 22 gives us that $R[x; \sigma, \delta]$ is a domain and since we are assuming that $R$ is an Ore domain then it has a division ring of fractions, call it $K$. 

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We define $\sigma$ on $K$ by

$$\sigma(ab^{-1}) = \sigma(a)\sigma^{-1}(b), ab^{-1} \in K, a, b \in R$$

$\sigma$ is well-defined: Let $ab^{-1} = a' b'^{-1} \in K$, then there exists $u, u' \in R^\times \ni: au = a' u', bu = b' u' \neq 0$. We get that

$$\sigma(ab^{-1}) = \sigma(au(bu)^{-1})$$

$$= \sigma(au)\sigma^{-1}(bu)$$

$$= \sigma(a')\sigma^{-1}(b')$$

$$= \sigma(a' b'^{-1})$$

So $\sigma$ is well-defined. We now want to extend $\delta$ to an $\sigma$-derivation on $K$. To do this first note that $\delta(1) = \delta(1 \cdot 1) = \delta(1) + \sigma(1)\delta(1) \Rightarrow \sigma(1)\delta(1) = 0$. Since $\sigma$ is well-defined, injective and $\sigma(1) = \sigma(1)\sigma(1)$ then $\sigma(1) = 1 \Rightarrow \delta(1) = 0$. Now $\delta(bb^{-1}) = \delta(b)b^{-1} + \sigma(b)\delta(b^{-1}) = 0 \Rightarrow \delta(b^{-1}) = -\delta(b)b^{-1}\sigma^{-1}$. We can now define $\delta$ on $K$ by $\delta(ab^{-1}) = \delta(a)b^{-1} - \sigma(a)\delta(b)b^{-1}\sigma^{-1}$. By using the same argument to show that $\sigma$ is well-defined we can see that $\delta$ is well-defined. Thus we can now form the skew polynomial ring $K[x; \sigma, \delta]$. $K[x; \sigma, \delta]$ has a division ring of fractions $K(x; \sigma, \delta)$.

Every $h \in K(x; \sigma, \delta)$ has the form $f/g \ni: f, g \in K[x; \sigma, \delta]$. Let $f = a_n x^n + \cdots + a_0, g = b_m x^m + \cdots + b_0$. Then for all $i = 1, \ldots, n, j = 1, \ldots, m, a_i, b_j \in K \Rightarrow a_i = r_i/s_i, b_j = u_j/v_j \ni: r_i, s_i, u_j, v_j \in R^\times$. By Proposition 23 these elements can be brought to a common denominator. If the denominator is $w$ then $f, g$ have the forms $f = f_1/w =, g = g_1/w$ where
\( f_1, g_1 \in R[x; \sigma, \delta] \). Thus we get that \( h = f/g = \frac{f_1/w}{g_1/w} = f_1/g_1 \). Thus, every element of \( R[x; \sigma, \delta] \) can be written as a quotient and by Theorem 11 \( R[x; \sigma, \delta] \) is an Ore Domain. Note that we also showed that the division ring of fractions for \( R[x; \sigma, \delta] \) is in fact \( K(x; \sigma, \delta) \) \( \square \)

Section 5: Goldie Rings

**Definition 29:** A ring \( R \) is said to be **semiprime** if it has no non-zero nilpotent ideals.

**Definition 30:** The **left annihilators of** \( s \) is the set
\[ l(s) = \{x \in R : xs = 0, \forall s \in S\} \]

**Definition 31:** The **right annihilators of** \( s \) is the set
\[ r(s) = \{x \in R : sx = 0, \forall s \in S\} \]

**Definition 32:** \( R \) is said to be a (left) **Goldie ring** if:

(a) \( R \) satisfies the A.C.C. on left annihilators,

(b) \( R \) contains no infinite direct sums on left ideals.

**Definition 33:** A left ideal \( I \) of \( R \) is said to be **essential** if for all non-zero ideals \( J, I \) intersects \( J \) in a non-trivial fashion. That is, \( I \) is essential if for all ideals \( 0 \neq J, I \cap J = \{0\} \cup \{\text{other stuff}\} \).

**Lemma 34:** Let \( R \) be a semiprime ring satisfying the A.C.C on left annihilators. If \( A \supset B \) are left ideals of \( R \) and \( r(A) \neq r(B) \), then there is an \( a \in A \) such that \( Aa \neq (0) \) and \( Aa \cap B = (0) \).
Proof: It is easy to see that if \( A \supset B \Rightarrow r(A) \subsetneq r(B) \), since \( r(A) \neq r(B) \). Let \( U \) be a minimal right annihilator such that \( r(A) \subsetneq U \subsetneq r(B) \). Then

\[ AU \neq (0) \Rightarrow (AU)^2 \neq (0), \text{ since } R \text{ is a semiprime ring. Let } ua \in UA \text{ such that } AuaU \neq (0). \text{ Then } uaU \notin r(A). \text{ We claim } AuaB \cap B = (0). \text{ If not then there is an } x \in A \text{ such that } xua \neq 0 \text{ with } xua \in Aua \cap B. \text{ Since } x \in A \text{ then } r(x) \supset r(A). \]

Now consider \( r(x) \cap U \). This is clearly a right annihilator containing \( r(A) \) and itself is contained in \( U \). Since \( uaU \subset r(x) \) and \( uaU \notin r(A) \) then \( r(A) \) is properly contained in \( r(x) \cap U \). Thus, \( U \subset r(x) \). This gives us that \( xU = (0) \) which contradicts the earlier choice of \( ua \in UA \). That is, it contradicts \( xua \neq 0 \). \( \Box \)

We now proceed with two corollaries of the above lemma that are required to prove Goldies theorem.

**Corollary 35:** Let \( R \) be as in the lemma above and let \( Rx \) and \( Ry \) be essential left ideals. Then \( Rxy \) is essential.

Proof: Suppose that \( A \) is a non-zero left ideal of \( R \). Let \( \overline{A} = \{ r \in R : ry \in A \} \). \( Ry \) is essential implies that there is a \( y_0 \neq 0 \in Ry \) such that \( y_0 \in Ry \cap A \). Thus \( y_0 \in \overline{A} \) which implies that \( \overline{A} \neq (0) \) and \( \overline{A}y = Ry \cap A \neq (0) \).

Now if \( w \in l(y) \) then \( wy = 0 \). Since \( 0 \in A \) then \( wy \in A \). Thus, \( w \in \overline{A} \). This shows that \( l(y) \subset \overline{A} \). Also note that \( l(y)y = 0 \).

By the lemma above we get that there is a non-zero left ideal \( T \subset \overline{A} \) such that \( T \cap l(y) = (0) \).

Now let \( \overline{T} = \{ r \in R : rx \in T \} \). \( Rx \) being essential gives us that \( \overline{T}x = Rx \cap T \neq (0) \).

This implies that \( \overline{T}xy \neq (0) \). Notice that since \( T \subset \overline{A} \) then for all \( t \in T, ty \in A \).
Now $T_{xy} = \{rxy : r \in R, rx \in T\}$. $rx \in T$ implies that $rxy \in A$ which gives us that $T_{xy} \subset A$.

Thus, $Rxy \cap A \neq (0)$ since $T_{xy} \subset Rxy \cap A$. □

**Corollary 36:** Let $R$ be as in the lemma above. If $Ra$ is essential in $R$ then $a$ is regular.

**Proof:** Since $Ra$ is essential then $Ra \cap R \neq (0)$. Coupled with the lemma we get that $r(A) = 0$. That is, in the special case when $A = R, B = Ra$.

We now wish to show that $l(a) = 0$. Combined, $l(a) = 0$ and $r(a) = 0$ will give us that $a$ is regular.

By that A.C.C. on left annihilators there is an integer $n$ such that $l(a^n) = l(a^{n+1})$. If $x \in Ra^n \cap l(a)$ then $xa = 0$ and for some $y \in R, xa = ya^n$.

Thus, we get that $ya^{n+1} = 0$ which implies that $y \in l(a^{n+1}) = l(a^n)$. Hence, $x = ya^n = 0$. Since $x$ was chosen arbitrarily then $Ra^n \cap l(a) = (0)$. But $Ra^n$ is essential (by previous corollary applied inductively n-times). Thus we must have that $l(a) = (0)$. □

For the remainder of the section we will assume that $R$ is a semiprime Goldie Ring.

We will prove four more lemmas before we prove Goldie’s theorem.

**Lemma 37:** $R$ satisfies the D.C.C. on left annihilators.

**Proof:** Let $L_1 \supset L_2 \supset \cdots \supset L_n \supset \cdots$ be a properly descending chain of left annihilators. Thus, $r(L_i) \neq r(L_{i+1})$. Applying the above lemma we get that there exist non-zero left ideals $C_n$ of $R$ such that $C_n \subset L_n$ and $C_n \cap L_{n+1} = (0)$. The $C_n$
form a direct sum of left ideals. Since $R$ is Goldie this sum is finite and hence the chain of annihilators must terminate.

**Lemma 38:** If $l(c) = 0$ then $Rc$ is essential and so $c$ is regular.

**Proof:** Let $A$ be a non-zero left ideal of $R$ such that $A \cap Rc = (0)$. Since $l(c) = 0$ we claim that $ac^n$ form a direct sum:

If $a_0 + a_1 c + \cdots + a_n c^n = 0, a_i \in A$, then $a_0 \in A \cap Rc \Rightarrow a = 0$ because $a_0 + a_1 c + \cdots + a_n c^n = 0 \in A \cap Rc$ and by closure for ideals $a_0 \in A \cap Rc$. Thus we get $a_1 c + \cdots + a_n c^n = 0 = (a_1 + \cdots + a_n c^{n-1})c = 0$. Since $l(c) = 0$ then $a_1 + \cdots + a_n c^{n-1} = 0$. Now repeat the first argument to get that $a_i = 0$, for all $i = 1, n$.

Since $R$ is goldie then we do not have any infinite direct sums and thus $A \cap Rc \neq (0)$. Hence $Rc$ is essential and $c$ is regular.

**Definition 39:** A two-sided ideal $S$ if $R$ is said to be an **annihilator ideal** is $S$ is a left annihilator of some ideal $T$.

Note that if $S$ is an annihilator ideal for some ideal $T$ then $ST = (0)$. Thus, $(TS)^2 = TSTS = (0)$. Thus, since $R$ is semiprime then $TS = (0)$.

**Lemma 40:** A non-zero minimal annihilator ideal of $R$ is a prime Goldie ring; moreover there is a finite direct sum of such ideals which is an essential left ideal of $R$.

**Proof:** Let $S$ be a non-zero minimal annihilator ideal. Let $T \neq (0)$ is a left ideal of $S$. Since $R$ is semiprime, $ST \neq (0)$ but $ST \subset T$ is a left ideal of $R$. This gives us
that $S$ has no infinite direct sums of left ideals. Since $R$ satisfies the A.C.C. on left annihilators then so does $S$, which makes $S$ a Goldie ring. Now suppose that $A, B$ are ideals of $S$ such that $AB = (0)$. From $SB \subset B$ we have that $ASB = (0)$ which implies that $A \subset l(SB) \cap S$. Now $l(SB) \cap S$ is an annihilator ideal and if $A \neq (0)$ then by the minimality of $S$ we get that $S \subset l(SB)$. This implies that $(SB)^2 = (0) \Rightarrow B = (0)$. Thus, $S$ is a prime Goldie Ring.

Let $A = \bigoplus_{i=1}^{n} S_i$ be a maximal direct sum of minimal annihilator ideals. We claim that $A$ is essential.

If $A \cap K = (0)$ for some non-zero left ideal $K$ of $R$ then $AK \subset A \cap K = (0) \Rightarrow K \subset r(A)$. Since $R$ is semiprime, $A \cap r(A) = 0$ and in $r(A)$ we can find a non-zero minimal annihilator ideal which fails to intersect $A$. In this case we can lengthen the direct sum if $A$ to include this ideal. However, this new lengthened ideal contradicts the maximality of $A$. Thus $A$ is essential. \hfill \square

**Lemma 41:** If $I$ is an essential left ideal of $R$ then is contains a regular element.

**Proof:** Case 1: $R$ is a prime ring. Let $a \in I$ such that $l(a)$ is a minimal ideal. Such a choice can be made by lemma 44. If $a$ is regular we are done. Thus, suppose $a$ is not regular. By corollary 43 $Ra \cap J = (0)$ for some non-zero left ideal $J$ of $R$. Since $I$ is essential, $I \cap J \neq (0)$. Thus, we may suppose that $J \subset I$. If $x \in J$ and $b \in (a + x)$ then $b(a + x) = 0$. This gives us that $ba = -bx \in Ra \cap J = (0) \Rightarrow b \in l(a) \cap l(x)$. By the minimality of $l(a)$ we get that $l(x) \supset l(a) \cap l(x) \supset l(a) \Rightarrow l(a)x = (0)$, for all $x \in J$.

Now in a prime ring $l(a)J = (0)$ coupled with $J \neq (0)$ forces us to have that $l(a) = (0)$. By lemma 45 we get that $a$ is regular.
Case 2: $R$ is a semiprime ring. Let $A = \bigoplus_{i=1}^{n} S_i$ be a maximal direct sum of minimal annihilator ideals. Since $S_i$ are prime and since $I \cap S_i$ are essential in $S_i$ then by the argument above we get that $I \cap S_i$ contains a regular elements $r_i \in S_i$.

We claim that $r = r_1 + \cdots + r_n$ is regular in $R$. If $l(r) \neq (0)$ then by the essentiality of $A$ we get that $l(r) \cap A \neq (0)$. Thus there is a non-zero element $t = t_1 + \cdots + t_n \in A$ with $tr = 0$. Notice that $tr = t_1 r_1 + \cdots + t_n r_n$. Since $A = \bigoplus_{i=1}^{n} S_i \Rightarrow t_i r_i = 0$, for all $i = 1, n$. But $r_i$ are all regular which force all $t_i = 0$ and hence $t = 0$. But this is a contradiction. \hfill \Box

Now we are finally ready to prove Goldie’s theorems.

**Theorem 42:** Let $R$ be a semiprime left Goldie ring. Then $R$ has a left quotient ring $Q = Q(R)$.

**Proof:** Let $a, b \in R$ such that $a$ is regular. This gives us that $l(a) = r(a) = (0)$, which also gives us that $Ra$ is essential. Let $M = \{ r \in R : rb \in Ra \}$. We will show that $M$ is essential. Let $S \neq (0)$ be a left ideal. Let $x \in M \cap S$. Since $x \in M$ then $xb \in Ra$. Thus, since $Ra$ is essential we must have that $M$ is essential. Hence, there is a regular element $c \in M$ such that $cb = da$.

We have now shown that $R$ satisfies the Ore condition. \hfill \Box

We state a few facts about $Q$ without proof:

(i) If $I$ is a left ideal of $Q$ then $I = Q(I \cap R)$.

(ii) If $A_1 \oplus \cdots \oplus A_n$ is a direct sum if left ideals in $R$ then $QA_1 \oplus \cdots \oplus QA_n$ is a direct sum of left ideals in $Q$. 60
Thus far we have shown that these semiprime Goldie rings have a quotient ring.

The second of Goldie's theorems explicitly defines the structure of these rings. Since the following theorem is not necessary in this section we state it without proof.

**Theorem 43: Goldie's theorem** For a semisimple Goldie ring $R, Q(R)$ is a semisimple Artinian ring.
Chapter 3: Simple Rings

The Wedderburn-Artin Theorem has shown us why simple rings are so important and so we will shift our focus to these rings and, amongst other interesting facts, show that if $R$ is simple then $R((x; \sigma))$ must also be simple.

Section 1: Construction of Simple Rings

In this section we pick up where we left off at the end of chapter 2 section 1. We will give conditions that give us simple rings from $R((x; \sigma))$.

**Definition 1:** A Ring $R$ is simple if $R \neq 0$ and if the only two sided ideals of $R$ are $(0)$ and $R$.

**Lemma 2:** Suppose $R$ is a simple ring. Let $a \in R$ be non-zero, then

$$\sum_{i=1}^{n} r_ias_i = 1, r_i, s_i \in R.$$ Also, if $aR = Ra$, then $a$ is invertible in $R$.

**Proof:** If $a$ is non-zero then the set $RaR = \{ \sum_{i=1}^{n} r_ias_i : r_i, s_i \in R \}$ is a non-zero two sided ideal or $R$. By the simplicity of $R$ we get that $R = RaR$. Thus, there exists $r_i, s_i \in R$ such that $\sum_{i=1}^{n} r_ias_i = 1$. Now suppose that for all $r \in R, ar = sa$, some $s \in R$. We show that $aR$ is a two sided ideal (it will follow that $Ra$ is a two sided ideal). Let $x \in R, xar = xsa = x'a = ar' \in aR$ because $r' \in R, arx \in aR$ because $rx \in R$. Thus, since $aR, Ra$ are both non-zero two sided ideals then we must have by the simplicity of $R$ that $Ra = aR = R$ $\Rightarrow$ there exists $a_1, a_2 \in R$ such that $a_1a = aa_2 = 1$. So $a$ is invertible. □

**Theorem 3:** If $R$ is a simple ring and $\sigma \in Aut(R)$, then $R((x; \sigma))$ is simple.
Proof: Suppose $I$ is a nonzero two-sided ideal of $R((x;\sigma))$. For any nonzero $p(x) \in I$, there exists $n \geq 0$ such that $h(x) = p(x)x^n \in R[[x;\sigma]]$. In fact, $h(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ where $a_0 \neq 0$. By the above lemma, there exist $r_i, s_i \in R$, $i = 1, \ldots, m$, such that $\sum_{i=1}^{m} r_i a_0 s_i = 1$. Hence

$$g(x) = \sum_{i=1}^{m} r_i h(x)s_i = 1$$

$$= 1 + b_1 x + b_2 x^2 + \cdots$$

for some $b_j \in R$. Thus $g(x)$ is an invertible element in $I$ which implies $I = R((x;\sigma))$.

We now try to understand the center of the skew-Laurent series ring with coefficients from a non-commutative ring, as we did before for Division Rings. However, before we can try and formalize any properties of the center of these rings we must first deal with inner automorphisms, as they will be integral in our comprehension.

Definition 4: Let $R$ be a ring. $\phi \in Aut(R)$ is called an inner automorphism determined by $u$ if $\phi(r) = u^{-1}ru$ for some unit $u \in R$. Note that $inn(u) = inn(u)(r)$

Remark: If $u$ and $v$ are invertible in $R$, then $inn(u) \circ inn(v) = inn(vu)$. To see why note that $inn(u) \circ inn(v)(r) = inn(u)(v^{-1}rv) = u^{-1}(v^{-1}rv)u = (u^{-1}v^{-1})r(vu) = (vu)^{-1}r(vu) = inn(vu)(r)$

Lemma 5: Let $\sigma \in Aut(R)$ such that $\sigma^k = inn(u)$ for some $k \in \mathbb{N}$. Then there exists $l \in \mathbb{N}$ such that $\sigma^l = inn(v)$, where $\sigma(v) = v$
**Proof:** For any \( k \geq 1 \) and for any \( i \geq 0 \) we get that

\[
\sigma^k = (\sigma^i \circ \sigma^k \circ \sigma^{-i})(r) \\
= \sigma^i(\text{inn}(u)(\sigma^{-i}(r))) \\
= \sigma^i(u^{-1}\sigma^{-i}(r)u) \\
= \sigma^i(u^{-1}r\sigma^i(u) \\
= \sigma^{-i}(u)\sigma^i(u) \\
= \text{inn}(\sigma^1(u))
\]

Thus,

\[
\sigma^k = \text{inn}(\sigma^{k-1}(u))\text{inn}(\sigma^{k-2}(u))\cdots\text{inn}(\sigma(u))\text{inn}(u) \\
= \text{inn}(u\sigma(u)\cdots\sigma^{k-1}(u))
\]

Where the last equality comes from repeating the remark above \( k \) times. Let \( a = u\sigma(u)\cdots\sigma^{k-1}(u) \). Since \( \sigma^k(u) = u \), we have

\[
\sigma(u\sigma(u)\cdots\sigma^{k-1}(u)) = \sigma(u)\sigma^2(u)\cdots\sigma^{k-1}(u)\sigma^k(u) \\
= \sigma(u)\sigma^2(u)\cdots\sigma^{k-1}(u)u \\
= u^{-1}[\sigma(u)\sigma^2(u)\cdots\sigma^{k-1}(u)]u \\
= u^{-1}au \\
= \sigma^k(u) = u
\]

Thus, the result holds. \( \square \)

**Theorem 6:** Let \( R \) be a simple ring with center \( F \) and let \( C \) denote the center of \( R((x; \sigma)) \). Then \( C = \)
(1) $F_\sigma$, if $\sigma^k$ is not inner for all $k \in \mathbb{Z}$

(2) $F_\sigma((ux^n))$, where $n$ is the least positive integer for which $\sigma^n$ is an inner auto determined by a $\sigma$ invariant element $u$

**Proof:** Let $p = \sum_{i=m}^{\infty} s_i x^i \in C$. As in the proof of Theorem 6 from Chapter 2, if $xp = px$ we get $\sigma(s_i) = s_i$ for all $i \geq m$. Thus $s_i \in F_\sigma$ for all $i \geq m$. Now let $b \in R$. If $bp = pb$, then (again in proof of Theorem 6 from Chapter 2) $bs_i = s_i \sigma^i(b)$. Since $\sigma \in Aut(R)$ then $Rs_i = s_i R$. Thus if $s_i \neq 0$ then $s_i$ is invertible. Then from $bs_i = s_i \sigma^i(b)$ we get $\sigma^i(b) = s_i^{-1} bs_i$ which implies $\sigma^i = inn(s_i)$

**Case 1:** Assume $\sigma^k$ is not inner for all $k \in \mathbb{Z}$

Clearly $F_\sigma \subset C$. To see that $C \subset F_\sigma$ note that since $\sigma^k$ is not inner for all $k \in \mathbb{Z}$ and $\sigma^i = inn(s_i)$ then $i = 0 \Rightarrow p = s_0 \in F_\sigma$

**Case 2:** Choose $n$ to be the smallest positive integer such that $\sigma^n = inn(u)$ for some $\sigma$-invariant $u$ (we can do this by previous lemma). Since $\sigma^i = inn(s_i)$ and $\sigma(s_i) = s_i$, then by the division algorithm we have that $i = nt_i, t_i \in \mathbb{Z}$. So that $inn(s_i) = \sigma^i = \sigma^{nt_i} = [inn(u)]^{t_i} = inn(u^{t_i})$. So that

$s_i^{-1} rs_i = u^{-t_i} ru^{t_i} \Rightarrow s_i u^{-t_i} r = rs_i u^{-t_i} \Rightarrow s_i u^{-t_i} \in F$. Hence, for some $\lambda_i \in F_\sigma, s_i = \lambda_i u^{-t_i}$. Since $u$ is $\sigma$-invariant then it must commute with $x$ an so $p = \sum \lambda_i u^{-t_i} x^{nt_i} = \sum \lambda_i (u^{-1} x^n)^{t_i} \in F_\sigma((ux^n))$. From here it is clear that $F_\sigma((ux^n)) \subset C$

We now give another skew construction that produces simple rings using automorphisms. We will invert the indeterminate $x$, but the elements in the ring will not be infinite series as they were above.
**Definition 7:** Let $R$ be a ring and let $\sigma \in Aut(R)$. A **Laurent polynomial** is a polynomial expression of the form $\sum_{i=k}^{n} r_i x^i$, where $k, n \in \mathbb{Z}$.

Equality and addition of there polynomials is defined in chapter 2. Multiplication will be according to the commutation rule

$$xr = \sigma(r)x$$

As in chapter 2, we need $\sigma$ to be an automorphism in order to extend the commutation rule to the negative powers of the indeterminate. The set of all that Laurent polynomials under the usual addition and this skew multiplication will be denoted $R[x, x^{-1}; \sigma]$ and will be referred to as the **Ring of Skew-Laurent Polynomials**. Note that we may call $R[x, x^{-1}; \sigma]$ a ring as we have proved that a more general structure is a ring in chapter 2.

**Theorem 8:** Let $R$ be a simple ring, Let $C$ be the center of $R[x, x^{-1}; \sigma]$ and $F$ the center of $R$. Then $C =$

1. $F_\sigma$, if $\sigma^k$ is not inner for all $k \in \mathbb{Z}$
2. $F_\sigma[ux^n, ux^{-n}]$, where $n$ is the least positive integer such that $\sigma^n$ is an inner automorphism determined by a $\sigma$-variant element $u$. $F_\sigma[ux^n, ux^{-n}]$ is the set of all commutative Laurent polynomials.

The proof is similar to the proof of Theorem 6, so we leave it out.

**Theorem 9:** Let $R$ be a simple ring and $I$ a non-zero ideal of $R[x, x^{-1}; \sigma]$. Then $I$ contains a non-zero central element of $R[x, x^{-1}; \sigma]$

**Proof:** Let $R[x; \sigma]$ be set of all polynomials over $R$ with the skew multiplication
defined above. Let $I$ be a non-zero ideal of $R[x, x^{-1}; \sigma]$ and let $I_x = I \cap R[x; \sigma]$.

Clearly $I_x \neq (0)$ is an ideal of $R[x; \sigma]$. By Lemma 2 $I_x$ contains some monic polynomial, say $g = \sum_{i=k}^{m} a_i x^i$, where $a_m = 1$. Suppose $g$ is of minimal degree such that $m - k$ is as small as possible. Since $g$ is monic, then for any $r \in R$,

$$rg - g\sigma^{-m}(r) = \sum_{i=k}^{m} (ra_i - a_i\sigma^{i-m}(r))x^i$$

must have degree less than $g$. To see this note that $rg = rx^m + \cdots$ and $g\sigma^{-m}(r) = x^m \sigma^{-m}(r) + \cdots = rx^m + \cdots$. By the minimality of $g$ we get that $rg - g\sigma^{-m}(r) = 0$. If $a_i \neq 0 \Rightarrow ra_i = a_i\sigma^{i-m}(r) \Rightarrow Ra_i = a_iR = R$. Thus $a_i$ invertible in $R$. This gives us that $g = a_i^{-1}ra_i \Rightarrow \sigma^{i-m}(r) = inn(a_i)$.

If no positive power of $\sigma$ is an inner-automorphism then $i = m$ and $g = x^m \in I_x \Rightarrow x^m \in I$. Since $x^m$ is invertible in $R[x, x^{-1}; \sigma]$, then $I = R[x, x^{-1}; \sigma]$.

Now suppose there is some $n \in \mathbb{N}$ such that $\sigma$ is an inner-automorphism. By the minimality of $g$ we have that $xg - gx = 0 \Rightarrow \sigma(a_i) = 0$, for all $i$. By Lemma 5, we can choose $n$ to be the least positive integer for which $\sigma^n = inn(u)$ is an inner-automorphism determined by a $\sigma$-invariant element $u$. Then

$$\sigma^{i-m} = inn(a_i) \Rightarrow i - m = ns_i, \quad s_i \in \mathbb{Z}.$$ Thus, $m = i - ns_i = k - ns_k$, which gives

$$i = k + n(s_i - s_k) = k + nt_i, \quad t_i > 0.$$ We also get that

$$inn(a_i) = \sigma^{i-m} = \sigma^{ns_i} = inn(u^{s_i}) \Rightarrow a_i = \lambda_i u^{s_i},$$ for some $\lambda_i \in F_{\sigma}$. Since $u$ commutes with $x^n$ then $a_i x^i = \lambda_i u^{s_k + t_i} (x^n)^{t_i} x^k = u^{s_k} \lambda_i (ux^n)^{t_i} x^k$. Thus,

$$g = u^{s_k} \left( \sum_{i=k}^{m} \lambda_i (ux^n)^{t_i} \right) x^k = vx^kh,$$

where $v$ is a unit and $h$ is in the center of $R[x, x^{-1}; \sigma]$. Since $vx^k$ is invertible in $R[x, x^{-1}; \sigma]$, it follows that $h \in I$. \qed
Corollary 10: If $R$ is a simple ring and $\sigma \in Aut(R)$ such that no positive power of $\sigma$ is an inner-automorphism, then $R[x, x^{-1}; \sigma]$ is a simple ring.

Proof: In the above proof we showed that if $R$ is a simple ring and $\sigma \in Aut(R)$ such that no positive power of $\sigma$ is an inner-automorphism, then $I = R[x, x^{-1}; \sigma]$ □

Section 2: Central Localization

We now give a construction of a simple ring not unlike our construction of a division ring given that the ring was an Ore Domain. It turns out we don’t need any strong requirement like an Ore domain but instead we will require that the center of our ring not be trivial. Now let $R^\times$ be all the non-zero elements of $R$ and let $S$ be the center of all the non-zero elements of $R$, denoted $S = Z(R^\times)$. We define a relation on $\sim$ by $(r_1, s_1) \sim (r_2, s_2)$ if and only if $r_1s_2 = r_2s_1$.

Proposition 11: The relation $\sim$ defined above is an equivalence relation.

Proof: Let $r, r', r'' \in R$, and $s, s', s'' \in S$.

Reflexivity: $(r, s) \sim (r, s)$, since $rs = rs$.

Symmetry: $(r, s) \sim (r', s') \Rightarrow rs' = r's \Rightarrow r's = rs' \Rightarrow (r', s') \sim (r, s)$.

Transitivity: Let $(r, s) \sim (r', s')$ and $(r', s') \sim (r'', s'') \Rightarrow rs' = r's$ and $r's'' = r''s'$.  


Then

\[ rs' = r's \iff rs'' = r's's'' \iff \]
\[ rs's' = r's's' \iff \]
\[ rs's' = r's's' \iff \]
\[ rs'' = r''s \]

Thus, \((r, s) \sim (r'', s'')\). \qed

We will denote the equivalence class of \((r, s)\) by \(rs^{-1}\) and we call the set of all these equivalence classes \(RS^{-1}\). In order to get a ring structure on \(RS^{-1}\) we define addition by,

\[ rs^{-1} + r's'^{-1} = (rs' + r's')(ss')^{-1} \]

and multiplication by,

\[ (rs^{-1})(r's'^{-1}) = (rr')(ss')^{-1} \]

We note that the additive identity is given by \(0 = 0 \cdot 1^{-1}\) and the multiplicative identity is \(1 = 1 \cdot 1^{-1}\). The proof of \(RS^{-1}\) being a ring is extremely uninspiring, thus we will leave it out. We do however show that these operations are well-defined. To see this let \(rs^{-1} = r's'^{-1}\). We will show that for any \(ab^{-1} \in RS^{-1}\),

\[ rs^{-1} + ab^{-1} = r's'^{-1} + ab^{-1} \text{ and that } (rs^{-1})(ab^{-1}) = (r's'^{-1})(ab^{-1}). \]
First, \( rs^{-1} = r's'^{-1} \Rightarrow rs' = r's \), thus

\[
rs^{-1} + ab^{-1} = (rb + as)(sb)^{-1}
\]
\[
= (rbs' + ass')(sbs')^{-1}
\]
\[
= (rs'b + as's)(s'bs)^{-1}
\]
\[
= (r'sb + as's)(s'bs)^{-1}
\]
\[
= (r'bs + as's)(s'bs)^{-1}
\]
\[
= (r'b + as')(s'b)^{-1}
\]
\[
= r's'^{-1} + ab^{-1}
\]

And similarly we get

\[
(rs^{-1})(ab^{-1}) = (ra)(sb)^{-1}
\]
\[
= (ras')(sbs')^{-1}
\]
\[
= (rs'a)(s'bs)^{-1}
\]
\[
= (r'sa)(s'bs)^{-1}
\]
\[
= (r'as)(s'bs)^{-1}
\]
\[
= (r'a)(s'b)^{-1}
\]
\[
= (r's'^{-1})(ab^{-1})
\]

This procedure for forming \( RS^{-1} \) from \( R \) and a multiplicative subset of the center of \( R \) is called the **Central Localization** of \( R \).

**Theorem 12:** Let \( R \) and \( S \) be as described above. Then

1. There is a natural embedding of \( R \) into \( RS^{-1} \).
2. In this embedding every \( s \in S \) has a multiplicative inverse.
(3) Every ideal of $RS^{-1}$ is of the form $IS^{-1}$, where $I$ is an ideal of $R$.

(4) If every non-zero ideal $I$ of $R$ intersects $S$ in a non-trivial way, then $RS^{-1}$ is a simple ring.

**Proof:** Let $f : R \rightarrow RS^{-1}$ defined by $r \mapsto r \cdot 1^{-1}$. This is clearly an injective homomorphism. Thus (1) is proven. For the second statement let $s \in S$ and consider $(1, s) \cdot (s, 1) = (1 \cdot s^{-1})(s \cdot 1^{-1}) = (1 \cdot s)(s \cdot 1)^{-1} = 1$. Now let $J$ be an ideal of $RS^{-1}$. Then $I = f^{-1}(J)$ is an ideal of $R$ and clearly $J = f(I) = IS^{-1}$. Thus, every ideal of $RS^{-1}$ can be written as $IS^{-1}$ where $I$ is an ideal of $R$. The fourth statement is trivial given (3) and (4).

Section 3: Examples of Simple Rings

**Theorem 13:** A matrix ring over a field is simple.

**Proof:** Let $e_{ij}$ be a matrix with a 1 in the $(i, j)$ entry and 0 everywhere else and let $F$ be a field. It is clear that $\{e_{ij} : 1 \leq i, j \leq n\}$ is a basis for $M_n(F)$. Notice that,

$$e_{ij}e_{rs} = \begin{cases} e_{is}, & \text{if } j = s \\ 0, & \text{otherwise} \end{cases}$$

Let $I \trianglelefteq M_n(F)$ be non-zero. Let $A$ be a non-zero matrix in $I$. Then,

$$e_{1i}Ae_{j1} = \begin{bmatrix} a_{ij} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Where $a_{ij}$ is the $ij$-th element of $A$. 71
Since $F$ is a field and $a_{ij} \in F$ then there must exist $(a_{ij})^{-1} \in F$ such that

$$a_{ij}(a_{ij})^{-1} = (a_{ij})^{-1}a_{ij} = 1.$$  

We have that $I$ is an ideal, $A \in I$ and $\{e_{ij} : 1 \leq i, j \leq n\} \in M_n(F)$, thus $e_{1i}Ae_{j1} \in I$.

Let $A'$ be the matrix equal to $A$ everywhere except we let the $ij$-th entry be $(a_{ij})^{-1}$.

Since $e_{1i}A'e_{j1} \in M_n(F)$ then the product of $e_{1i}Ae_{j1}$ and $e_{1i}A'e_{j1} \in M_n(F)$ is an element of the ideal $I$.

But the product is precisely,

$$e_{11} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0
\end{bmatrix}$$

To see that $e_{22}, \ldots, e_{nn}$ are all elements of $I$ note that $e_{kk}$ is the product of $e_{ki}Ae_{jk}$ and $e_{ki}A'e_{jk}$ for all $1 \leq k \leq n$. Where $e_{ki}Ae_{jk} \in I$ for all $1 \leq k \leq n$ and $e_{ki}A'e_{jk} \in M_n(F)$ for all $1 \leq k \leq n$.

Thus, we have $\sum_{i=1}^{n} e_{ii} \in I$. But this sum is the $n \times n$ identity matrix. □

We will now prove the converse of Wedderburn-Artin. Notice that the above proof gives us that $M_n(D)$ is simple, for any division ring $D$. We must just show that it is also Artinian. To do this we prove a few lemmas. Note that all the proofs are in terms of either right ideals or left ideals but the proof of the opposite side is exactly the same.

**Lemma 13:** Let $R = M_n(D)$, for any division ring $D$. For each $i = 1, 2, \ldots, n$, $e_{ii}R$ is a minimal right ideal.
proof: Clearly \( e_i R \) is a right ideal. Let \( A \) be a non-zero right ideal of \( R \) such that \( A \subset e_i R \). Let \( 0 \neq x = x_1 e_{i1} + \cdots + x_n e_{in} \in A \). If \( x_{ij} \neq 0 \) for some \( j \) then

\[
x e_{ij} x_{ij}^{-1} e_{ji} = e_i \in A
\]

This implies that \( A = e_i R \). \( \square \)

Lemma 14: Suppose \( R \) is a ring which can be written as the finite direct sum of minimal right ideals. Then every right ideal of \( R \) is of the form \( e R \), for some idempotent \( e \in R \).

Proof: Assume \( R = \bigoplus_{i=1}^{k} M_i \), where each \( M_i \) is a minimal right ideal. Let \( A \) be a right ideal of \( R \). If \( A = (0) \) or \( A = R \) then the result is true. Thus suppose that \( A \) is a non-zero right ideal properly contained in \( R \). Now, \( A \cap M_i \subset M_i \) as right ideals and by the minimality of each \( M_i \) we have that \( A \cap M_i = (0) \) or \( A \cap M_i = M_i \). Since \( A \neq R \) it must be the case that some \( M_i \) is not contained in \( A \). Hence \( A \cap M_i = (0) \) and we can thus form the direct sum \( A_1 = A \oplus M_1 \). If \( A_1 \neq R \) then, as above we can assume that \( M_2 \) is not contained in \( A_1 \). This gives us that \( A_1 \cap M_2 = (0) \). And we can form the direct sum

\[
A_2 = A_1 \oplus M_2 = A \oplus M_1 \oplus M_2
\]

Since \( R = M_1 \oplus \cdots \oplus M_k \) then at some point we reach an index \( l \) such that \( A_l = R \Rightarrow R = A \oplus B \), for some right ideal \( B \) of \( R \). Thus we have that

\[
1 = e + f, e \in A, f \in B \text{ and so } e = e^2 + fe, \text{ which implies that}
\]

\[
e - e^2 = fe \in A \cap B = (0). \text{ Thus, } e^2 = e \text{ and } fe = 0. \text{ So, for all}
\]

\[
a \in A, a = ea + fa \Rightarrow a - ea = fa \in A \cap B = (0). \text{ Therefore, } a = ea \Rightarrow A \subset e R \text{ and}
\]

since \( e \in A \), \( A = e R \). \( \square \)
Thus, any ring which has the property that it can be written as the finite direct sum of minimal right ideals must be ring Noetherian (by the finitely generated property of Noetherian rings).

**Lemma 15:** Every right ideal of $R = M_n(D)$ is of the form $eR$, for some idempotent $e \in R$.

**proof:** $R = e_{11}R \oplus \cdots \oplus e_{nn}R$ is a finite direct sum of minimal right ideals of $R$. hence, the result from lemma 14. Similarly for left ideals. In particular $M_n(D)$ is left and right Noetherian \[ \square \]

Recall that for the left annihilator of a ring $R, l(R)$ if $S \subset T$ then $l(S) \supset l(T)$ and similarly for the right annihilators.

**Lemma 16:** If $R$ is any ring and $e \in R$ is idempotent, then $r(Re) = (1 - e)R$ and $l[(1 - e)R] = Re$.

**Proof:** Clearly $1 - e \in r(Re) \Rightarrow (1 - e)R \subset r(Re)$. Conversely, suppose $x \in r(Re)$, then in particular $ex = 0$. Now, $1 = (1 - e) + e \Rightarrow$

\[ x = (1 - e)x + ex = (1 - e)x \Rightarrow \]

$x \in (1 - e)R$. Thus $r(Re) = (1 - e)R$.

The equality of $l[(1 - e)R] = Re$ is done similarly. \[ \square \]

**Theorem 17:** If $D$ is a division ring then $D_n = M_n(D)$ is both left and right Artinian.
Proof: By lemma 15 a random descending chain of left ideals of $R$ is of the form:

$$Re_1 \supset Re_2 \supset \cdots \supset R2_n \supset \cdots$$

Then

$$r(Re_1) \subset r(Re_2) \subset \cdots \subset r(Re_n) \subset \cdots$$

is an ascending chain of right ideals of $R$. But since $R$ is Noetherian then there is some index $m$ such that $r(Re_m) = r(Re_k)$, for all $k \geq m$. By the above lemma we have that $Re_m = l(r(Re_m)) = l(r(Re_{m+1})) = Re_{m+1} = \cdots$. A similar proof works for right Artinian. □

Definition 18: Let $R$ be any ring. The smallest integer $n$ such that $n1 = 0$ is called the characteristic of $R$, denoted Char($R$). If not such $n$ exists then Char($R$) = 0.

Definition 19: Let $K$ be a polynomial ring. A map $\delta : K \to K$ defined by:

$$\delta(ab) = a\delta(b) + \delta(a)b$$

$$\delta(a + b) = \delta(a) + \delta(b)$$

for all $a, b \in K$ is called a derivation on $K$. So $xa = ax + \delta(a)$, where $x$ is indeterminate over $K$ and not commutative.

If $R$ is the ring of all $C^\infty$ real valued functions on the real line, then the classical derivative, $\frac{d}{dx}$, is a derivation on $R$.

Definition 20: $K[x; \delta]$ is known as the differential polynomial ring.

Definition 21: $\delta$ is an inner derivation if there is a $c \in K$ such that $\delta(a) = ca - ac$, for all $a \in K$. We will refer to these as $\delta$-inn.
**Definition 22:** An ideal $I \subset R$ is called a $\delta$-ideal if $\delta(I) \subset I$

**Definition 23:** The ring $R$ is called $\delta$-simple if the only $\delta$-ideals of $R$ are $(0)$ and $R$.

To see that $\delta$-inn is a derivation notice that

$$(x - c)a = xa - ca = ax + \delta(a) - ca = ax - ac = a(x - c)$$

So if $t = x - c$ we can show that $K[x; \delta] \cong K[t]$, where $K[t]$ is the usual polynomial ring, since $(x - c)$ commutes with $K$.

**Theorem 24:** Let $K$ be a ring with $\text{Char}(K) = 0$ and let $\delta$ be a derivation. Then $R = K[x; \delta]$ is simple if and only if $K$ is $\delta$-simple and $\delta \neq \delta$-inn on $K$.

**Proof:** First we assume that $\delta$ is inner. Then, for some $c \in K$, $\delta(b) = cb - bc$, for all $b \in K$. Thus, for $t = x - c$, $R = K[t] \nRightarrow R$ is not simple.

Now assume $U \neq (0)$ is a $\delta$-ideal of $K$. Let $I = \{ \sum a_i x^i \in R : a_i \in U \} = UR$. This is clearly an ideal of $R$.

Conversely, assume $K$ is $\delta$-simple but that $R$ is not simple. Let $I \neq (0)$ be an ideal of $R$. We will show that $\delta$ must be inner.

Let $n$ be the minimum degree for the non-zero polynomials in $I$ and let

$$U = \{ a \in K : ax^n + a_{i-1}x^{i+1} + \cdots \in I \} \cup \{0\}$$

Let $f \in I$ such that the deg $f = n$. Now

$$fx = ax^{n+1} + a_{i-1}x^i + \cdots$$

and

$$xf = x(ax^n + a_{i-1}x^{i+1} + \cdots) = ax^{n+1} + a_{i-1}x^i + \cdots + \delta(a)x^n + \delta(a_{i-1})x^{i+1} + \cdots$$
so that

\[ xf - fx = \delta(a)x^n + \delta(a_{i-1})x^{i+1} + \cdots \]

This gives us that \( \delta(a) \in U \Rightarrow U \) is a \( \delta \)-ideal of \( K \). Since

\[ g = x^n + dx^{n-1} + \cdots \in I \Rightarrow 1 \in U. \]

Note that since \( xb = bx + \delta(b) \) then

\[ x^2b = x(xb) \]
\[ = x(bx + \delta(b)) \]
\[ = xbx + x\delta(b) \]
\[ = (bx + \delta(b))x + x\delta(b) \]
\[ = bx^2 + 2\delta(b)x + \delta(b) \]

Thus, inductively, we get

\[ x^n b = bx^n + n\delta(b)x^{n-1} + \cdots \]

Now for some \( b \in K \) we get that

\[ bg = bx^n + bdx^{n-1} + \cdots \]

and

\[ gb = x^n b + dx^{n-1}b + \cdots \]
\[ = bx^n + n\delta(b)x^{n-1} + dbx^{n-1} + \cdots \]
\[ = bx^n + (n\delta(b) + db)x^{n-1} + \cdots \]

so that

\[ bg - gb = (bd - db - n\delta(b))x^{n-1} + \cdots \]
since $bg - gb \in I$.

Since the minimum degree of any polynomial in $I$ is $n$ then

$$bg - gb = (bd - db - n\delta(b))x^{n-1} + \cdots = 0$$

$$\Rightarrow bd - db = n\delta(b) \Rightarrow$$

$$\delta(b) = \frac{b - d}{n}$$

Thus, $\delta$ is inner. \hfill \Box$

Let $K$ be any ring and $\{x_i\}$ be non-commuting indeterminates over $K$.

Now let $R = K < x_i >$ and let $F = \{f_j\} \subset R$. $(F)$ is the ideal generated by $F$ in $R$.

Thus we may talk about $\overline{R} = R/(F)$, the ring generated over $K$ by $\{x_i\}$ with relations $F$.

**Definition 25:** If $R = K < x, y >$ and $F = \{xy - yx - 1\}$ then $\overline{R}$ is known as the

**First Weyl Algebra** over $K$ and is denoted

$$A_1(K) = K < x, y > / (xy - yx - 1)$$

The $n$-th Weyl algebra, $A_n(K)$ is generated by $\{x_i, y_i\}$, each commuting with elements of $K$ and having relations

$$\begin{align*}
(1) & \quad x_i y_i - y_i x_i = 1; 1 \leq i \leq n \\
(2) & \quad x_i y_j - y_j x_i = 0; i \neq j \\
(3) & \quad x_i x_j - x_j x_i = 0; i \neq j \\
(4) & \quad y_i y_j - y_j y_i = 0; i \neq j
\end{align*}$$

We make two notes:
Lemma 26: The center of a simple ring is a field.

Proof: Let $R$ be a simple ring with center $C$. Now suppose that $C$ is not a field. Then for some non-zero element $a \in C$ there does not exist a non-zero element $b$ such that $ab = 1$. This gives us that the ideal generated by $a, (a)$, does not contain the identity. This means that $R$ strictly contains non-zero ideals and is therefore not simple.

Lemma 27: Let $K$ be a simple ring of characteristic 0. Then for any non-inner derivation $\delta$ on $K$, $R = K[x, \delta]$ is a non-artinian simple ring.

Proof: That $R$ is simple follows from theorem 19. Since the center of $K$ is a field and $\text{Char}(K) = 0$ then $\mathbb{Q} \subset C$, where $C$ is the center of $K$. The descending chain

$$Rx \supseteq Rx^2 \supseteq \cdots$$

shows that $R$ is not artinian.

Theorem 28: Let $K$ be any simple ring of characteristic zero. Then $A_n(K), n \geq 1$ are all nonartinian simple rings.

Proof: Since $A_n = A_1(A_{n-1})$ it suffices to prove the theorem for $A_1$. It is clear by the lemma that $A_1(K)$ is nonartinian. To show that $A_1$ is simple we use the identification $A_1(K) = K[y][x; \delta]$ where $\delta = d/dy$ on $K[y]$. Since $y$ is in the center of $K[y]$ but $\delta(y) = \frac{d}{dy}(y) = 1$, we see that $\delta$ is not an inner derivation on $K$. Let $U$ be a non-zero $\delta$-ideal of $K$. If $f = ay^n + \cdots$ has minimal degree $n$ among the non-zero
polynomials in $U$, then

$$\delta(f) = nay^{n-1} + \cdots$$

implies that $na = 0$. Since $a \neq 0$ and $\mathbb{Q} \subset K$ we must have that $n = 0$. Thus $f = a \in U \cap K$ and the fact that $K$ is simple implies that $1 \in U \cap K$. Thus $U = K[y]$. □
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