Fermi Coordinates and Relative Motion in Inflationary Power Law Cosmologies

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by

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Dedications

For Bob and Roberta, Cyril and Esther, & Ed and Arlene
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Abstract

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A power law cosmology is a Robertson-Walker spacetime in which the scale factor, $a(t)$, is of the form $a(t) = t^\alpha$. This work has two primary goals: to give exact Fermi coordinates for geodesic observers, and to give formulas for the Fermi relative velocity, both in the context of power law cosmologies. To those ends, the author gives the necessary mathematical and physical background, beginning at the level that a first-year graduate student of mathematics should find comfortable. In Chapter 2, we develop the theory of manifolds, focusing on Lorentzian manifolds. In Chapter 3, the necessary components of Einstein's general theory of relativity are presented. A cosmology is said to be inflationary if it expands at an accelerated rate; in Chapter 4, the author shows that formulas which were known to hold for non-inflationary cosmologies also hold for inflationary power law cosmologies (see [1], [2]). This is of particular interest because there are multiple recent papers which suggest modeling the universe as an inflationary power law cosmology.
Chapter 1

Introduction

General relativity models the universe as a four-dimensional, Lorentzian manifold. Because of this, relatively simple physical questions can have correspondingly difficult answers. For example, the relative velocity of a non local object is not a well-defined concept: there are four competing, geometrically-defined versions of relative velocity. Since the manifold we work in is non-Euclidean, it takes substantial mathematical machinery to make sense out of the relative velocity of one body compared to another at two distinct points on the manifold, due to the non-equivalence of the tangent spaces at different points. Furthermore, to compare the velocity of a particle relative to an observer, we must decide what events are simultaneous to that observer. Generally, cosmologists work in “normal coordinates;” simultaneity is taken to mean a fixed “normal” time coordinate (also known as “cosmological time”), and velocity is taken to be the instantaneous rate of change of the Hubble distance with respect to the observer’s time. There are problems with this approach, however. The foliations of spacetime created by fixing the normal time coordinate generate submanifolds which are not natural, in the sense that a geodesic in the submanifold is not a geodesic in the full manifold. Distances measured on this submanifold, then, are not geometrically meaningful, even when they are proper distances.

In this paper, the author will introduce the reader to Fermi Coordinates, and the corresponding notions of simultaneity and velocity that go along with them. Chapter 2 will deal with mathematical preliminaries, mainly results from differential geometry and the theory of differentiable manifolds. The treatment of manifolds will follow [3], which the author recommends highly. Chapter 3 will introduce the reader to General Relativity – primarily how it is applied to modern cosmology. The following chapter will be dedicated to the construction of Fermi Coordinates, and the author will provide a new result regarding their maximal extension under certain conditions. The fifth chapter will deal with formulae for relative velocities. Fermi Coordinates provide a more natural setting to explore the topic of relative velocity because their coordinate slices are physically meaningful; in a sense they contain all geodesics which are perpendicular to an observer’s motion through time. Chapter 6 contains some concluding remarks.

Much of this work is based on an upcoming publication with Drs. David Klein and Vicente Bolós entitled Relative Velocities, Geometry, and Expansion of Space, which is to appear in the book by NOVA Publishers entitled Recent Advances in Cosmolgy.
This thesis frequently makes use of the Einstein summation convention, which uses a super-script to replace the summation symbol in the following manner:

\[ x^i y_i := \sum_{i=1}^{n} x_i y_i. \]

This is done when the limits of the summation are clear from the context, or have or will be explained.

Unless specified otherwise, the units of all physical formula are such that the speed of light, \( c \), is equal to 1.
Chapter 2

Mathematical Preliminaries

As any journey begins with preparation, so must yours, reader; in this context, that preparation is an introduction to certain mathematical concepts. Differential geometry is often called the language of general relativity, and with good reason. Selections from this robust field will be given, along with needed parts of manifold theory. The chapter ends with some useful results from analysis.

Note: all topological spaces considered are Hausdorff spaces with countable bases. The interested reader can consult any topology text for a formal definition.

2.1 Manifolds

Mathematicians, physicists and engineers are, for the most part, comfortable working in $\mathbb{R}^n$, the set of all $n$-tuples of real numbers, $(x^1, x^2, ..., x^n)$. And at first, it seems unreasonable to leave that comfortable space. However, it turns out there are reasons to do mathematics on objects that are not $\mathbb{R}^n$; the most important in the context of this thesis being that not all $n$-dimensional objects can be embedded in $\mathbb{R}^n$. The universe, for example, is all that exists - it should not need to be embedded in some larger ambient space.

The mathematical objects mentioned above are “manifolds” (or “differentiable manifolds”). Manifolds are objects that can be thought of as patches of $\mathbb{R}^n$ sewn together, for a constant $n$. It is frequently said that manifolds locally “look like” $\mathbb{R}^n$. Before that notion is explained, following [3], some examples of manifolds are:

**Example 1 ($\mathbb{R}^n$)** $\mathbb{R}^n$ is itself a manifold, which should not be surprising - $\mathbb{R}^n$ locally looks quite a bit like $\mathbb{R}^n$.

**Example 2 (The $n$-Sphere)** The $n$-sphere, $S^n$, which is the collection of points that are unit distance from the origin in $\mathbb{R}^{n+1}$.

There are two important points to make here: thinking of the sphere as a surface in a larger space is convenient, but unnecessary. Furthermore, the reader familiar with cartography will recall that the 2-sphere has the interesting property that a single map cannot be made of it that is both complete and accurate. In mathematical terms, it can be shown that there is no isometry from the sphere to the plane.

**Example 3 (Torus)** The surface of a doughnut, known as a torus, which results from identifying opposite sides on a square.
Since calculus is only defined in $\mathbb{R}^n$, if we wish to do analysis on some other object, we must find a way to relate that object to $\mathbb{R}^n$. What follows will allow us to do just that, but it requires that we revisit some basic concepts. A map, $\phi$, between $\mathbb{R}^n$ and $\mathbb{R}^m$ sends an $n$-tuple $(x^1, x^2, \ldots, x^n)$ to an $m$-tuple $(y^1, y^2, \ldots, y^m)$, and so can be thought of as $m$ distinct functions of $n$ variables in the following manner:

\[
\begin{align*}
y^1 &= \phi^1(x^1, x^2, \ldots, x^n) \\
y^2 &= \phi^2(x^1, x^2, \ldots, x^n) \\
\vdots \\
y^m &= \phi^m(x^1, x^2, \ldots, x^n).
\end{align*}
\]

Each $\phi^i$ is said to be $C^p$ if its $p^{th}$ derivative exists and is continuous; $\phi$ is $C^p$ if all $\phi^i$ are at least $C^p$. $C^0$ is used to denote a continuous function, and a function is said to be $C^\infty$ if it is infinitely continuously differentiable; this is also referred to as being smooth.

**Definition 1** Let $U$ be an open subset of $\mathcal{M}$, a topological space. A chart, or coordinate system, is the set $U$ coupled with a continuous, one-to-one map $\phi : U \to \mathbb{R}^n$, such that $\phi(U)$ is an open subset of $\mathbb{R}^n$.

**Definition 2** Given a collection of charts, $\{(U_\alpha, \phi_\alpha)\}$, the transition maps of that collection are the functions $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$, defined:

\[
\phi_{\alpha\beta} = (\phi_\beta \circ \phi_{\alpha}^{-1})|_{\phi_\alpha(U_\alpha \cap U_\beta)}.
\]

We note that $\phi_{\alpha\beta}^{-1} = \phi_{\beta\alpha}$.

**Definition 3** A $C^p$ atlas, $0 \leq p \leq \infty$, is a collection of charts, $\{(U_\alpha, \phi_\alpha)\}$, such that:

1. $\bigcup_\alpha U_\alpha = \mathcal{M}$.
2. All transition maps, $\phi_{\alpha\beta}$, are $C^p$ and surjective (where defined).

**Definition 4** Two atlases $\{U_\alpha, \phi_\alpha\}$, $\{V_i, \psi_i\}$ are $C^p$ compatible if their union is an atlas.

This definition means that the transition functions, $\psi_i \phi_{\alpha}^{-1}$, are $C^p$ and surjective. As noted in [4], compatibility is an equivalence relation, and so gives rise to equivalence classes.

**Definition 5** A maximal $C^p$ atlas or $C^p$ structure on $\mathcal{M}$ is an equivalence class of $C^p$ atlases.

**Definition 6** A $C^p$ manifold $\mathcal{M}$ is a connected topological space together with a maximal $C^p$ atlas.
Since the space is connected, it follows that the range of each \( \phi_\alpha \) is a subset of \( \mathbb{R}^n \) for some fixed \( n \); this unique value is called the \textbf{dimension} of the manifold. All manifolds discussed herein are smooth, that is, they are \( C^\infty \) manifolds. For more on smooth manifolds, the reader is encouraged to consult [5], [6].

We will also consider certain subsets of \( \mathcal{M} \) called \textit{submanifolds}, in particular we will consider submanifolds known as \textbf{coordinate slices}.

\textbf{Definition 7} Let \( \mathcal{M} \) be an \( n \)-dimensional smooth manifold, \( S \subset \mathcal{M} \), and let \( m < n \). We say \( S \) is a coordinate slice of \( \mathcal{M} \) if for all \( q \in S \), there exists a chart \((U, \phi) = (U, x^1, \ldots, x^n)\) on \( \mathcal{M} \) such that \( U \) contains \( q \) and

\[
U \cap S = \{ p \in \mathcal{M} \mid x^{m+1}(p) = x^{m+2}(p) = \ldots = x^n(p) = 0 \}.
\]

\((U', \phi') = (U \cap S, x^1, \ldots, x^m)\) is then a chart on \( S \), and the collection of such charts comprises a smooth atlas on \( S \).

(Adapted from [7]).

\subsection*{2.2 Vectors, Paths, and Tangent Spaces}

The material in the preceding section, upon first reading, most likely seems unnecessarily technical. It was necessary though, among other reasons, to define manifolds without reference to any ambient space in which they may be embedded. Again, we are developing mathematical machinery that will allow us to describe the universe, which is, by definition, not embedded in an ambient space. Another technical issue that must now be resolved, though, is the location of tangent vectors. In manifolds embedded in \( \mathbb{R}^n \), vectors which are tangent to the manifold live in \( \mathbb{R}^n \) as well; this turns out not to be the case in general. As before, formalism comes to the rescue; however, the results, while technical, are intuitive as well: if we want to visualize the tangent space at a point \( p \) on a manifold, \( \mathcal{M} \), we consider all parametrized curves through \( p \), find the tangent vectors of these curves, and verify that indeed this is the desired space. However, we must make those notions precise.

In what follows, let \( \mu, \nu \) run from 0 to 3 (it is common for Greek summation indices to cover this range, while Latin ones run from 1 to 3).

\textbf{Definition 8} \( f : \mathcal{M} \to \mathbb{R} \) is called a \textbf{smooth function} on \( \mathcal{M} \) if for all charts on \( \mathcal{M} \), \((U, \phi), f \circ \phi^{-1}\) is a smooth function on \( \phi(U) \).

\textbf{Definition 9} \( C^\infty(\mathcal{M}) := \{ f : \mathcal{M} \to \mathbb{R} \mid f \text{ is smooth} \} \).

Given a point \( p \in \mathcal{M} \), each curve through \( p \) defines an operator on \( C^\infty(\mathcal{M}) \) – the directional derivative. To understand this, we must discuss paths.
Definition 10 A smooth path $\gamma$, parametrized by the parameter $\rho$ on $\mathcal{M}$ is a function $\gamma : I \subseteq \mathbb{R} \rightarrow \mathcal{M}$ such that for all charts $(U, \phi) = (U, x^\mu)$ with $\gamma(I) \subseteq U$, $(\phi \circ \gamma) : I \rightarrow \mathbb{R}^4$ is a smooth path.

Definition 11 The partial derivative with respect to $x^\mu$,
\[
\partial_\mu := \frac{\partial}{\partial x^\mu},
\] (2.1)
is the directional derivative (see implicit definition below) along the curve defined by $x^\nu = \text{constant for } \nu \neq \mu$, parametrized by $x^\nu$.

Definition 12 Under the same conditions as Definition 10, the velocity “vector,” or “tangent vector to $\gamma$” is the “vector field”
\[
\gamma' = \frac{d\gamma}{d\rho} := \frac{d(\phi \circ \gamma)^\mu}{d\rho} \partial_\mu.
\] (2.2)
This additionally defines what it means to take a derivative of a function on $\mathcal{M}$; one uses the chart to send the function to $\mathbb{R}^n$, where calculus is already defined. However, we will often suppress the explicit reference to the chart; if coordinates are used, they are assumed to be from a chart. That means we will write (2.2) as
\[
\gamma' = \frac{dx^\mu(\rho)}{d\rho} \partial_\mu.
\] (2.3)
To justify our nomenclature in the above definition, we present the following claim.

Claim 1 Given $p \in \mathcal{M}$, each path (written in coordinates as $x^\mu(\rho)$) through $p$ defines a directional derivative, $\frac{d}{d\rho} : C^\infty(\mathcal{M}) \rightarrow \mathbb{R}$. These directional derivatives form a vector space.

Proof. Let $\frac{d}{d\lambda}$ and $\frac{d}{d\eta}$ represent derivatives along two curves through $p$: $x^\mu(\lambda)$ and $x^\mu(\eta)$. For scalars $a$ and $b$, it is clear that
\[
\left( a \frac{d}{d\lambda} + b \frac{d}{d\eta} \right) (f) = a \frac{df}{d\lambda} + b \frac{df}{d\eta},
\] (2.4)
however, we must also show that the space is closed under addition and scalar multiplication, that is, that $a(\frac{d}{d\lambda}) + b(\frac{d}{d\eta})$ is a derivative; we do this by verifying that it satisfies the Leibniz product rule.
\[
\left( a \frac{d}{d\lambda} + b \frac{d}{d\eta} \right) (fg) = af \frac{dg}{d\lambda} + ag \frac{df}{d\lambda} + bf \frac{dg}{d\eta} + bg \frac{df}{d\eta}
\] (2.5)
\[
= \left( a \frac{df}{d\lambda} + b \frac{df}{d\eta} \right) (g) + \left( a \frac{dg}{d\lambda} + b \frac{dg}{d\eta} \right) (f).
\]
We call this vector space \( T_p(M) \), the tangent space to \( M \) at \( p \). To see that this name is justified, we find a basis for it. Let \( (U, x^\mu) \) be a coordinate chart on an \( n \)-dimensional manifold \( M \).

**Claim 2** The partial derivative operators at \( p \), \( \{\partial_\mu\}_{p}\), form a basis for the tangent space \( T_p(M) \).

**Proof.** Following [3], let \( M \) be an \( n \)-dimensional manifold, \( (U, \phi) = (U, x^\mu) \) a chart on \( M \) containing \( p \), \( \gamma : \mathbb{R} \to M \) a curve, and \( f : M \to \mathbb{R} \) a smooth function, with both the image of \( \gamma \) and domain of \( f \) containing \( p \). Let \( \lambda \) be the parameter along \( \gamma \). We must express \( \frac{d}{d\lambda} \) in terms of \( \partial_\mu \).

\[
\frac{d}{d\lambda}f = \frac{d}{d\lambda}(f \circ \gamma)
\]

\[
= \frac{d}{d\lambda}[(f \circ \phi^{-1}) \circ (\phi \circ \gamma)]
\]

\[
= \frac{d}{d\lambda}(\phi \circ \gamma)^\mu \frac{\partial (f \circ \phi^{-1})}{\partial x^\mu}
\]

\[
= \frac{dx^\mu}{d\lambda} \partial_\mu f.
\]

(2.6)

**Remark 2.2.1** The above computation takes place at the point \( p \in M \), though the notation does not indicate it – this is fairly common when the formula holds at all \( p \) under consideration. The motivation for this is not just laziness (see vector fields, below).

The first line of (2.6) formalizes what is meant by the left-hand side of the equation. The second line sends us to \( \mathbb{R}^n \); the third line is the chain rule. The fourth line simplifies the notation. Since \( f \) was arbitrary, we have

\[
\left. \frac{d}{d\lambda} \right|_p = \frac{dx^\mu}{d\lambda} \partial_\mu |_p,
\]

(2.7)

proving the claim.

Hence an arbitrary tangent vector \( v \in T_p(M) \) can be written \( v = v(x^\mu) \partial_\mu |_p \). Setting \( v = \partial_\mu \) yields a change of basis formula which is just the standard multi-dimensional chain rule, as stated below.
**Proposition 1**  Given two charts on \( M, (U, \phi) = (U, x^\mu) \) and \( (V, \psi) = (V, x^{\mu'}) \) with \( p \in (U \cap V) \), each of the coordinate vectors of \( \psi \) can be expressed in terms of the coordinate vectors of \( \phi \) using the following formula:

\[
\partial_{\mu'}|_p = \frac{\partial x^{\mu'}}{\partial x^{\mu}}(p) \partial_{\mu}|_p.
\]

(2.8)

An arbitrary vector, though it may appear different in various coordinate systems, is a geometrically defined object, and hence must actually remain unchanged by a change of basis. That is, for \( p \in M \) in the domain of two charts, \( x^\mu \) and \( x'^\mu \), and \( v \in T_p M \):

\[
v = v^\mu \partial^\mu|_p = v'^\mu \partial'^\mu|_p,
\]

(2.9)

hence

\[
v'^\mu = \frac{\partial x'^\mu}{\partial x^\mu}(p) v^\mu.
\]

(2.10)

As noted in [2] and elsewhere, there is no a priori method for adding tangent vectors based at different points, say \( p \) and \( q \), on \( M \); this is because they are elements of different spaces, \( T_p M \) and \( T_q M \). Briefly considering one’s intuitive understanding of velocity gives one an idea of the problems one encounters when trying to add relative velocities in general.

We now turn our attention to vector fields. A vector field, \( V \), assigns a distinct vector \( V_p \), to each point \( p \) in the manifold. We say that the vector field is smooth provided

\[
(Vf)(p) := V_p(f)
\]

(2.11)

is a smooth function on the manifold for all smooth functions \( f \). All vector fields that follow will be assumed to be smooth unless stated otherwise. Vector fields lie in what is known as the tangent bundle, \( TM = \bigsqcup_{p \in M} T_p M \).

**Example 4**  Given a chart, \( (U, \phi) = (U, x^\mu) \), the coordinate vector fields are the collection \( \{\partial_\mu\} \). At each point in \( U \), they assign the vector \( \partial_\mu|_p \).

**Definition 13**  Given two vector fields, \( X \) and \( Y \), on a manifold \( M \), their commutator, \( [X,Y] \), is defined by its action on a smooth function \( f \) in the following manner:

\[
[X,Y](f) := X(Y(f)) - Y(X(f)).
\]

(2.12)

Since mixed partials of a smooth function commute, the commutator of the vector field \( \{\partial_i\} \) will always vanish.

The commutator is a special case of the Lie derivative; as such it is sometimes called the Lie bracket. The Lie derivative is used to study a type of symmetry of vector
fields known as Killing symmetry. Though interesting, it is beyond the scope of this paper. The interested reader is advised to consult [3] for an introduction.

2.3 One-Forms and Tensors

All vector spaces have a dual space, and \( T_p \mathcal{M} \) is no exception.

**Definition 14** The dual space of \( T_p \mathcal{M}, T_p \mathcal{M}^* \), is the collection of linear maps \( \omega : T_p \mathcal{M} \to \mathbb{R} \).

Elements of the dual space are not lacking in names: one-forms, dual vectors, and covectors, to name a few. Given a function \( f \), its differential, \( df \), is the most common example of a one-form. Its action on a vector, \( \frac{df}{dt} \), is taking the directional derivative of the function in the direction of the vector:

\[
\frac{df}{dt} \left( \frac{d}{dt} \right) = \frac{df}{dt}.
\]

In terms of a vector field, \( V \),

\[
df(V) := V(f).
\]

As with vectors, we now consider objects defined throughout the manifold, which smoothly assign a one-form to each point on the manifold. Sadly, these objects are also known as one-forms; luckily, the meaning is usually clear from context.

It is a fact from linear algebra that given a basis, \( \{v_1, \ldots, v_n\} \), of a vector space \( V \), there is a corresponding dual basis, \( \{\omega^1, \ldots, \omega^n\} \), of the dual space \( V^* \) such that \( \omega^i(v_j) = \delta^i_j \). It is then natural to ask what the dual basis of the coordinate vectors, \( \{\partial_1, \ldots, \partial_n\} \), is; the answer is known as the one-form dual basis, and is shown in [5].

**Proposition 2** Given a manifold \( \mathcal{M} \) containing point \( p \) in the domain of chart \( (U, x^1, \ldots, x^n) \), the one form dual basis for \( T_p \mathcal{M}^* \) is \( \{dx^1, \ldots, dx^n\} \). That is:

\[
dx^i(\partial_j) = \delta^i_j. \tag{2.13}
\]

Furthermore, any \( \omega \in T_p \mathcal{M}^* \) can be written in components as \( \omega = \omega_i dx^i \).

Given a vector space \( V \) and its dual, \( V^* \), we may define a \((k, l)\) tensor, \( T \).

**Definition 15** A \((k, l)\) **tensor**, \( T \), is a multilinear map from \( k \) copies of \( V^* \) and \( l \) copies of \( V \) into the real numbers, that is

\[
T : V^* \times \cdots \times V^* \times V \times \cdots \times V \to \mathbb{R}.
\]

The tensor of primary concern is the metric tensor. It has the following properties.
Definition 16 A **symmetric bilinear form** on a vector space \( V \) is a \((0,2)\) tensor, \( g \), with the property that for all \( u, v \in V \), \( g(u, v) = g(v, u) \).

Definition 17 A symmetric bilinear form, \( g \), is called:

- **non-degenerate** if \( g(u, v) = 0 \) for all \( v \in V \) implies \( u = 0 \);
- **positive definite** if \( g(v, v) > 0 \) for all non-zero \( v \in V \);
- **negative definite** if \( g(v, v) < 0 \) for all non-zero \( v \in V \).

Given a symmetric bilinear form, \( b \), and a vector space \( V \), let \( W \) and \( W' \) be the largest subspaces of \( V \) such that for all \( v \in W \), \( b(v, v) > 0 \) and for all \( v \in W' \), \( b(v, v) < 0 \). We may now define the signature of the symmetric bilinear form.

Definition 18 The **signature** of the symmetric bilinear form discussed above is \((p, q)\), where \( \dim(W') = p \) and \( \dim(W) = q \).

We conclude with a convenient new operation.

Definition 19 Let \( \omega^{(1)}, \ldots, \omega^{(k)}, \ldots, \omega^{(k+m)} \) be one-forms, and let \( V^{(1)}, \ldots, V^{(l)}, \ldots, V^{(l+n)} \) be vectors. Given a \((k, l)\) tensor \( T \) and a \((m, n)\) tensor \( S \), the **tensor product**, denoted \( \otimes \), is defined by

\[
T \otimes S(\omega^{(1)}, \ldots, \omega^{(k)}, \ldots, \omega^{(k+m)}, V^{(1)}, \ldots, V^{(l)}, \ldots, V^{(l+n)}) = T(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)})S(\omega^{(k+1)}, \ldots, \omega^{(k+m)}, V^{(l+1)}, \ldots, V^{(l+n)})
\]

(2.14)

2.4 The Metric Tensor

The metric tensor is a type of function which generalizes the dot product to more general manifolds. It is incredibly important, among other things, it: allows for the computation of proper distance and time, determines the shortest path between points, replaces the Newtonian gravitational field, and determines causality.

Definition 20 Let \( M \) be a manifold. The **metric tensor**, \( g \), is a non-degenerate, symmetric bilinear form of constant signature on \( M \) such that for all \( X, Y \in TM \), \( g(X, Y) \) is a smooth function on \( M \).

Just as a linear transformation is often written in matrix form (with respect to some basis), the metric tensor is often written “in coordinates;” that is, for a chart \((U, x^1, \ldots, x^n)\), we write

\[
g_{ij} := g(\partial_i, \partial_j)
\]

(2.15)
on \( U \).

Proposition 3 Let \((U, x^1, \ldots, x^n)\) be a chart on \( M \), and let \( V, W \in TM \), and at each \( p \in M \), \( v = V^i_p \partial_i|_p \), \( w = W^j_p \partial_j|_p \). Then

\[
g(v, w) = g_{ij} dx^i \otimes dx^j(v, w).
\]

(2.16)
Proof.

\[ g(v, w) = g(V^i_p \partial_i|_p, W^j_p \partial_j|_p) \]
\[ = V^i_p W^j_p g(\partial_i|_p, \partial_j|_p) \]
\[ = V^i_p W^j_p g_{ij} \]
\[ = g_{ij} V^i_p W^j_p \]
\[ = g_{ij} dx^i(v) dx^j(w) \]
\[ = g_{ij} dx^i \otimes dx^j(v, w) \]

by linearity, (2.15), commutativity, (2.13), and (2.14).

It is common to write \( ds^2 \) for \( g(\cdot, \cdot) \), that is:

\[ ds^2 := g(\cdot, \cdot) = g_{ij} dx^i dx^j, \]

where the tensor product sign in the last term has been dropped, as is also common.

This notation is partly due to the fact that the symbol \( g \) is commonly reserved for the determinant of the \( n \times n \) matrix \( \{g_{ij}\} \). It follows from the symmetry and non-degeneracy of \( g \) that \( \{g_{ij}\} \) is a symmetric, invertible matrix. Its inverse is denoted \( \{g^{ij}\} \).

The reader is cautioned: there is no \( s \) of which \( ds^2 \) is the square of the differential; it is merely a convenient shorthand. As stated at the beginning of the chapter, the metric tensor is a generalization of the dot product. The notation is a reminder that we take the square root to get a physically meaningful quality, as we will see later.

2.5 Spacetimes

We now focus our attention on specific types of manifolds. A **Lorentzian manifold**, \( \mathcal{M} \), is a manifold of dimension \( n \) equipped with a metric tensor \( g \) with signature \((1,n-1)\). As noted in sources such as [2] and [8], this is effectively a requirement that one dimension be different than the others; this dimension is time. Again, the metric is a generalization of the dot product, which gives the length of a vector. Since our metric may give the length as being positive, negative, or zero, this gives three possible classes of vectors.

**Definition 21** Let \( \mathcal{M} \) be a Lorentzian manifold, whose metric tensor is \( g \), and \( V \) a
vector in $T_p\mathcal{M}$. We say $V$ is:

\begin{align*}
\text{lightlike if } & g(V, V) = 0, \\
\text{timelike if } & g(V, V) < 0, \\
\text{spacelike if } & g(V, V) > 0.
\end{align*}

(2.17)

Now, an especially important definition.

**Definition 22** A **spacetime** is 4-dimensional Lorentzian manifold that is time-orientable. The points in spacetimes are referred to as “events.”

The spacetime may sometimes be referred to by making explicit reference to the manifold and its metric, written as the ordered pair $(\mathcal{M}, g)$.

The term time-orientable intuitively refers to being able to identify a past and future. Timelike vectors in the tangent space are divided into two equivalence classes; one is denoted “future-directed,” the other “past-directed.” It will not be rigorously defined here; the interested reader is advised to consult [5] and [9].

The reader familiar with special relativity will recognize the following example.

**Example 5 (Minkowski Space)** Minkowski space, with its global coordinates $(t, x, y, z)$ has the metric

\[ \eta := \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \]

(2.18)

Following (2.17), we label a path as light-, space-, or timelike provided $\gamma'$ is a vector of that type at each point along the path. This is called the causal structure of the path. Lightlike paths are also called null paths. Though there is nothing to prevent a path from having the property that its tangent vector is at some points spacelike, and at other points timelike (for example); these paths are not considered, hence remain nameless.

**Definition 23** Let $\gamma : I \to \mathcal{M}$ be a spacelike path. The parameter $\rho$ that gives $g(\gamma'(\rho), \gamma'(\rho)) = 1$ is called the proper distance along $\gamma$.

**Definition 24** Let $\gamma : I \to \mathcal{M}$ be a timelike path. The parameter $\tau$ that gives $g(\gamma'(\tau), \gamma'(\tau)) = -1$ is called the proper time along $\gamma$.

Proper distance (time) will not exists if the path fails to be spacelike (timelike).

### 2.6 The Covariant Derivative

There is one more technical issue to resolve before we can move on - that of connections. A connection on a manifold connects the tangent spaces of nearby
points, allowing vector fields to be differentiated. More precisely, if we denote the space of vector fields on $\mathcal{M}$ by $C^{\infty}(TM)$, we can define it as follows.

**Definition 25** A connection on $\mathcal{M}$, $\nabla$ is a bilinear map $\nabla : C^{\infty}(TM) \times C^{\infty}(TM) \to C^{\infty}(TM)$ which for all $f \in C^{\infty}(\mathcal{M})$ and vector fields, $X, Y, W$ on $\mathcal{M}$, satisfy

\[
\nabla_{fX+gY}W = f\nabla_XW + g\nabla_YW \tag{2.19}
\]

and

\[
\nabla_XfY = X(f)Y + f\nabla_XY \tag{2.20}
\]

$\nabla_XY$ is called the **covariant derivative** of $Y$ in the direction on $X$.

There is only one connection we will be concerned with; that is the Levi-Civita connection; it has two more properties.

**Definition 26** The Levi-Civita connection on a Lorentzian manifold $\mathcal{M}$ is the unique torsion-free connection that satisfies

\[
[V,W] = \nabla_VW - \nabla_WV \tag{2.21}
\]

and

\[
X(g(V,W)) = g(\nabla_XV,W) + g(V,\nabla_XW) \tag{2.22}
\]

for vector fields $X$, $V$, and $W$.

Proof of its uniqueness, as well as the properties stated, can be found in [5], where the following is also shown. For $(U,x^\mu)$ a chart on $\mathcal{M}$,

\[
\nabla_{\partial_i}(W^j\partial_j) = \left\{ \frac{\partial W^k}{\partial x^i} + \Gamma^k_{ij}W^j \right\}\partial_k \tag{2.23}
\]

where

\[
\Gamma^k_{ij} = \frac{1}{2}g^{km}\left( \frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right) \tag{2.24}
\]

are called the **Christoffel symbols**, or connection coefficients. As noted in [5], “the Christoffel symbols of a coordinate system measure the failure of its coordinate vector fields to be parallel.” The term “torsion-free” in the above definition means that the torsion tensor, defined

\[
T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 2\Gamma^\lambda_{[\mu\nu]} \tag{2.25}
\]

is zero; that is, that the Christoffel symbols associated with the connection are symmetric in their lower indices.

All manifolds from here on will assumed to be equipped with a Levi-Civita connection $\nabla$. 

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Finally, we note that in general, for $W = W^\beta \partial_\beta$, $V = V^\alpha \partial_\alpha$,

$$\nabla_W V = \left\{ W^\beta \left( \frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\delta} V^\delta \right) \right\} \partial_\alpha,$$

which, at a point, is simply an ordinary differential equation (upon being set equal to a value).

### 2.7 Parallel Transport and Geodesics

Gravity ceases to be viewed as a force in general relativity. In classical mechanics, an object moving under the influence of no external force moves in a straight line path. In general relativity, this generalizes to: an object under the influence of no external, non-gravitation force moves in a geodesic path. Geodesics are defined in terms of parallel transport: moving a vector along a path while keeping it constant. Unlike in flat space, the result of parallel transport in curved space will depend on the path taken. A vector field is said to be parallel along a path if its covariant derivative is zero at all points on the path. This is stated as a definition.

**Definition 27** Let $M$ be a Lorentzian manifold, and $\gamma : I \to M$ be a smooth path. A vector field $V$ is said to be parallel along $\gamma$ if at all points on $\gamma(I)$,

$$\nabla_\gamma' V = 0.$$  

**Theorem 3** Given a vector $v \in T_p M$ and a path $\gamma$ whose image contains $p$, there exists a unique vector field $V$ that is parallel along $\gamma$, and has $V(p) = v$. $V$ is called the parallel transport of the vector $v$ along $\gamma$.

Uniqueness follows from the note after (2.26) and the theory of ordinary differential equations.

One would hope that a generalized notion of the angle between to vectors would remain constant throughout parallel translation. That is the case.

**Proposition 4** Let $\gamma$ be a smooth path on $M$, Lorentzian, with metric $g$. If $X$ and $Y$ are two vector fields parallel along $\gamma$, then $g(X, Y)$ is constant along $\gamma$.

The proof of this follows from (2.22) and the definition of $X, Y$ parallel along $\gamma$,

$$\gamma'(g(X, Y)) = g(\nabla_\gamma' X, Y) + g(X, \nabla_\gamma' Y) = g(0, Y) + g(X, 0) = 0.$$

A primary goal of this paper is constructing Fermi coordinates, which are defined in terms of orthogonal vectors at an event, and then parallel translated along a path. This proposition guarantees that they remain orthogonal.

We may now define precisely what we mean by geodesics.
Definition 28 A geodesic on a manifold $\mathcal{M}$ is a path $\gamma(t)$ such that parallel transport along the curve preserves the tangent vector to the curve.

So, at all points along a geodesic

$$\nabla_{\gamma'} \gamma' = 0. \quad (2.28)$$

Following (2.26), in coordinates, (2.28) looks like

$$\frac{d^2 x^\alpha}{dt^2} + \Gamma_{\mu\nu}^\lambda \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0, \quad (2.29)$$

which is known as the geodesic equation. Again, it is an ordinary differential equation, and from the theory, if we are given an initial point and vector, (2.29) has a unique solution in a neighborhood of the initial point.

Since $g(\gamma', \gamma')$ is a constant for a geodesic, we note that the causal structure of a geodesic is incapable of change: that is, each geodesic must be null, timelike or spacelike.

### 2.8 The Exponential Map

In this section, we exploit the relation between a manifold and its tangent space inherent in the definition of a geodesic to define a map called the exponential map, which we later use to construct Fermi coordinates. The construction relies on the theorem on existence and uniqueness for solutions of ordinary differential equations – this means our map, like those solutions, are only defined in a neighborhood. A proof of the following propositions are given in [5], beginning on page 71.

**Proposition 5** For a point $p$ on a manifold $\mathcal{M}$, and $v$ an element of $T_p \mathcal{M}$, there exists an open set $U_{p,v} \subseteq \mathcal{M}$ and unique geodesic $\gamma_v$ which satisfies $\gamma_v(0) = p$ and $\gamma'_v(0) = v$, with $\text{Image}(\gamma_v) \cap U_{p,v} \neq \emptyset$.

**Definition 29** Using the notation of the above proposition, the exponential map at $p$, $\exp_p(v) : T_p \mathcal{M} \rightarrow \mathcal{M}$ is given by

$$\exp_p(v) = \gamma_v(1) \quad (2.30)$$

provided the right side exists.

**Proposition 6** For each $q \in \mathcal{M}$ there exists an open set $\widehat{U}_{q,v} \subseteq T_q \mathcal{M}$ containing 0 such that the exponential map defined above is a diffeomorphism onto $U_{p,v}$. The inverse function, $\exp_p^{-1} q$, sends a point on the geodesic $\gamma_v$ joining $p = \gamma_v(0)$ and $q = \gamma_v(1)$ to the unique vector $v \in T_p \mathcal{M}$ with $\exp_p(v) = \gamma_v(1) = q$.

Briefly, the exponential map takes lines through the origin of $T_q \mathcal{M}$ to geodesics through $q$. 

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2.9 Lebesgue Dominated Convergence Theorem and Leibniz Rule

We end the chapter with a section of a different flavor. Herein, we present the Lebesgue Dominated Convergence Theorem (the “DCT”) and Leibniz Rule, two important results from the field of real analysis. Though worthy of study intrinsically, they are also made use of later in the thesis.

**Theorem 4 (General Lebesgue Dominated Convergence Theorem)** Let \( \{f_n\} \) be a sequence of measurable functions on a set \( E \) that converges pointwise almost everywhere on \( E \) to \( f \). Suppose there is a sequence \( \{g_n\} \) of nonnegative measurable functions on \( E \) that converges pointwise almost everywhere on \( E \) to \( g \) and dominates \( \{f_n\} \) in the sense that, for all \( n \), \(|f_n| \leq g_n\) on \( E \). If

\[
\lim_{n \to \infty} \int_E g_n = \int_E g < \infty, \tag{2.31}
\]

then

\[
\lim_{n \to \infty} \int_E f_n = \int_E f. \tag{2.32}
\]

In the case that each \( g_n \) is identical and does not depend on \( n \), that is, \( g_n = g \), we refer to the theorem as simply the Dominated Convergence Theorem. A proof of this is found in [10].

**Theorem 5 (Leibniz Rule)** For \( \tau \) in some interval \( I \), and a continuously differentiable (with respect to \( \tau \)) \( f \) and differentiable \( u, v \),

\[
\frac{d}{d\tau} \int_{u(\tau)}^{v(\tau)} f(\sigma, \tau) d\sigma = \int_{u(\tau)}^{v(\tau)} \frac{\partial}{\partial \tau} f(\sigma, \tau) d\sigma + \frac{dv(\tau)}{d\tau} f(v(\tau), \tau) - \frac{du(\tau)}{d\tau} f(u(\tau), \tau). \tag{2.33}
\]

We are concerned with applying this formula when \( u \) is a constant, \( c \), and \( f(c, t) \) is undefined.

**Lemma 1** Using the notation of (2.33) with \( \tau \in [t, \infty) \), \( u(\tau) = 1 \), \( v(\tau) = \sigma(\tau) \), and

\[
f(\sigma, \tau) = \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sqrt{\sigma} \sqrt{\sigma - 1}}, \tag{2.34}
\]

Leibniz rule gives

\[
\frac{d}{d\tau} \int_1^{\sigma(\tau)} \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sqrt{\sigma} \sqrt{\sigma - 1}} d\sigma = \int_1^{\sigma(\tau)} \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma \sqrt{\sigma - 1}} d\sigma + \frac{d\sigma(\tau)}{d\tau} \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma(\tau)}} \right)}{\sqrt{\sigma(\tau)} \sqrt{\sigma(\tau) - 1}}. \tag{2.35}
\]
Proof. Choose \(d \in (1, \sigma(\tau))\), so

\[
\frac{d}{d\tau} \int_1^{\sigma(\tau)} \frac{b\left(\frac{a(\tau)}{\sqrt{\sigma}}\right)}{\sqrt{\sigma} \sqrt{\sigma - 1}} d\sigma = \frac{d}{d\tau} \left[ \int_d^{\sigma(\tau)} \frac{b\left(\frac{a(\tau)}{\sqrt{\sigma}}\right)}{\sqrt{\sigma} \sqrt{\sigma - 1}} d\sigma + \int_1^d \frac{b\left(\frac{a(\tau)}{\sqrt{\sigma}}\right)}{\sqrt{\sigma} \sqrt{\sigma - 1}} d\sigma \right]
\]

\(2.36\)

The first term is well behaved at its lower limit, hence Leibniz rule applies, giving:

\[
\frac{d}{d\tau} \int_1^{\sigma(\tau)} \frac{b\left(\frac{a(\tau)}{\sqrt{\sigma}}\right)}{\sqrt{\sigma} \sqrt{\sigma - 1}} d\sigma = \int_1^{\sigma(\tau)} \frac{b\left(\frac{a(\tau)}{\sqrt{\sigma}}\right) \dot{a}(\tau)}{\sigma \sqrt{\sigma - 1}} d\sigma + \frac{d\sigma(\tau)}{d\tau} \frac{b\left(\frac{a(\tau)}{\sqrt{\sigma(\tau)}}\right)}{\sqrt{\sigma(\tau)} \sqrt{\sigma(\tau) - 1}}.
\]

\(2.37\)

So to show (2.35), it remains to show that the second term is

\[
\frac{d}{d\tau} \int_1^d \frac{b\left(\frac{a(\tau)}{\sqrt{\sigma}}\right)}{\sqrt{\sigma} \sqrt{\sigma - 1}} d\sigma = \int_1^d \frac{b\left(\frac{a(\tau)}{\sqrt{\sigma}}\right) \dot{a}(\tau)}{\sigma \sqrt{\sigma - 1}} d\sigma,
\]

\(2.38\)

or, using the notation from the statement,

\[
\frac{d}{d\tau} \int_1^d f(\sigma, \tau) d\sigma = \int_1^d \frac{\partial f}{\partial \tau}(\sigma, \tau) d\sigma.
\]

\(2.39\)

The derivative on the left exists provided that for any non-zero sequence \((\tau_n)\) with \(\lim_{n \to \infty} \tau_n = 0\),

\[
\lim_{n \to \infty} \int_1^d \frac{f(\sigma, \tau + \tau_n) - f(\sigma, \tau)}{\tau_n} d\sigma
\]

\(2.40\)

exists and the limit is the same regardless of the sequence chosen (this follows quickly from contradiction). So we have, first by definition, then by the above observation,

\[
\frac{d}{d\tau} \int_1^d f(\sigma, \tau) d\sigma = \lim_{h \to 0} \int_1^d \frac{f(\sigma, \tau + h) - f(\sigma, \tau)}{h} d\sigma
\]

\(2.41\)

for any non-zero sequence \((\tau_n)\) with \(\lim_{n \to \infty} \tau_n = 0\), so pick one (that is, now \((\tau_n)\) refers to a specific sequence). We would like to bring the limit inside the integral using the DCT, but to do this, we must find a bound on the integrand. By the Mean Value Theorem for derivatives,

\[
\frac{f(\sigma, \tau + \tau_n) - f(\sigma, \tau)}{\tau_n} = \frac{\partial f}{\partial \tau}(\sigma, \hat{\tau}_n)
\]

\(2.42\)
for some \( \hat{\tau}_n \in (\tau, \tau + \tau_n) \). So a bound on \( \frac{\partial f}{\partial \tau}(\sigma, \tau) \) will suffice. Since
\[
\frac{\partial f}{\partial \tau}(\sigma, \tau) = \frac{\ddot{b}\left(\frac{a(\tau)}{\sqrt{\sigma}}\right) \dot{a}(\tau)}{\sigma \sqrt{\sigma - 1}},
\]
which, for reasons that will be obvious in Chapter 4 (when this Lemma is used), is
\[
\frac{\ddot{b}\left(\frac{a(\tau)}{\sqrt{\sigma}}\right) \dot{a}(\tau)}{\sigma \sqrt{\sigma - 1}} = \frac{1 - \alpha}{\alpha \tau^\alpha} \frac{1}{\sigma^{1/2\alpha} \sqrt{\sigma - 1}}.
\]
As a function of \( \tau \), this has maximum value on its domain equal to
\[
\frac{1 - \alpha}{\alpha \tau^\alpha} \frac{1}{\sigma^{1/2\alpha} \sqrt{\sigma - 1}} := g(\sigma),
\]
and
\[
\int_1^d g(\sigma) d\sigma < \infty
\]
(the closed form of the integral is very messy and not instructive), so since we have found an \( L^1 \) dominating function, by the DCT,
\[
\frac{d}{d\tau} \int_1^d f(\sigma, \tau) d\sigma = \lim_{h \to 0} \int_1^d \frac{f(\sigma, \tau + h) - f(\sigma, \tau)}{h} d\sigma
\]
\[
= \lim_{n \to \infty} \int_1^d \frac{f(\sigma, \tau + \tau_n) - f(\sigma, \tau)}{\tau_n} d\sigma
\]
\[
= \int_1^d \lim_{n \to \infty} \frac{f(\sigma, \tau + \tau_n) - f(\sigma, \tau)}{\tau_n} d\sigma
\]
\[
= \int_1^d \frac{\partial f}{\partial \tau}(\sigma, \tau) d\sigma
\]
\[
= \int_1^d \ddot{b}\left(\frac{a(\tau)}{\sqrt{\sigma}}\right) \dot{a}(\tau) \frac{1}{\sigma \sqrt{\sigma - 1}} d\sigma,
\]
completing the proof.

\[
\]
Chapter 3

Cosmology

In this chapter, the reader will be introduced to the general theory of relativity and its applications to cosmology.

3.1 General Relativity

“...conflicts with one’s scientific understanding to conceive of a thing which acts but cannot be acted upon;” this was Einstein’s critique of absolute space – the preferred reference frame of Newtonian mechanics [11]. To resolve this problem, Einstein had to posit three axioms which are initially counter-intuitive, and the end result is the general theory of relativity. He began by assuming:

1. The laws of physics are the same in all inertial frames.

2. Light travels at a constant speed, \( c \), with respect to all inertial frames.

These two assumptions for the basis of what became known as special relativity. One consequence of these is that the metric of special relativity is that of \( \mathbb{R}^4 \) with the metric \( \eta \), as mentioned in (2.18). Einstein’s third assumption can be stated then, as:

3. The Equivalence Principle - The laws of physics reduce to those of special relativity in a sufficiently small region of spacetime.

The mathematical framework for these ideas is exactly the differential geometry explained in the preceding chapter. The universe is posited to be a spacetime whose points are called events (justifying the names). The metric of this spacetime is determined by the Einstein Field Equations, a system of partial differential equations that relate the matter-energy distribution within the spacetime to the curvature of the spacetime. (For the sake of completeness, the Einstein Field Equations are discussed further in the appendix). As mentioned prior, in the absence of forces, observers move along timelike geodesics - this is a generalization of Newton’s first law of motion, the law of inertia. Let us be clear about what this means.

The path of objects can, in certain scenarios, be observed to deviate from straight lines. Prior to Einstein’s theory, if there were no known forces acting on the object (a “free” body), this deviation was explained by a new force called “gravity.” In that sense, gravity no longer exists in general relativity. The deviation from the expected path comes not from a force, but from having the wrong expectations. If the assumption is made that space is flat, then a free body would move in a straight line - but they sometimes do not; space is curved! This is the main idea. The field equations answer the question “what is the curvature of space?”
Then, to rephrase the Equivalence Principle, at each point on a geodesic observer’s worldline, there is a system of coordinates in which the metric is that of special relativity. These coordinates are known as Fermi coordinates, and they are the focus of this thesis. However, to construct them, we must first establish a notion of simultaneity.

### 3.2 Fermi Space Slices

Given a manifold $\mathcal{M}$ with metric $g$ that is a spacetime, suppose that $p \in \mathcal{M}$ is in the domain of $n$ charts, $(U_1, \phi_1), \ldots, (U_n, \phi_n)$. That is, for each $1 \leq i \leq n$, $\phi_i(p) = (t_i(p), x^1_i(p), x^2_i(p), x^3_i(p))$. What does it mean for an event to be simultaneous to $p$? It seems that each submanifold $\mathcal{M}_i = \{q \in \mathcal{M} | t_i(q) = t_i(p)\}$ could be argued to contain points simultaneous to $p$, and indeed, one could argue just that. However, there is a preferred submanifold, called a Fermi space slice (or a Landau submanifold); preferred in that it has Fermi coordinates on it.

To find a Fermi space slice, consider the worldline, $\gamma$, of a geodesic observer (and called a Fermi observer) in the spacetime $(\mathcal{M}, g)$. A Fermi space slice at the time $t = \tau$, $\mathcal{M}_\tau$, is precisely all space-like geodesics orthogonal to $\gamma'(\tau)$. That is, let

$$f(q) = g(\gamma'(\tau), exp_{\gamma(\tau)}^{-1}(q)).$$

**Definition 30** The Fermi space slice of $\tau$-simultaneous events relative to the geodesic observer $\gamma$ at proper time $\tau$, $\mathcal{M}_\tau$, is defined

$$\mathcal{M}_\tau := f^{-1}(0).$$

To say that an event $q$ is “simultaneous” to $\gamma(\tau)$ will mean that $q \in \mathcal{M}_\tau$.

For a further discussion of this definition of simultaneity, see [12] and [13]. We will see later that each $\mathcal{M}_\tau$ is a submanifold of the type discussed above, that is, $\mathcal{M}_\tau = \{p| \phi_i(p) = (\tau, x^1_i(p), x^2_i(p), x^3_i(p))\}$, where $\phi_i$ is the Fermi coordinate system. It is for this reason that events on $\mathcal{M}_\tau$ are called simultaneous. By their construction, Fermi space slices have the important property that geodesic paths in $\mathcal{M}_\tau$ originating at the observer’s worldline are still geodesics when thought of as lying in the ambient spacetime. This is important because physical measurements are done in coordinates, and conclusions are then drawn about all of spacetime.

The reader may be concerned that we have flippantly referred to geodesics on $\mathcal{M}_\tau$ without concerning ourselves whether it is even a manifold. However, that concern would be misplaced; each $\mathcal{M}_\tau$ is a coordinate slice, as defined in Chapter 2, and the restriction of the metric $g$ to $\mathcal{M}_\tau$ produces $(\mathcal{M}_\tau, g_\tau)$, a Riemannian manifold. We say that the collection $\{\mathcal{M}_\tau\}$ foliates an open subset of $\mathcal{M}$ around the observer’s path.
One last important feature of Fermi space slices is the way in which distance is measured on them.

**Definition 31** Let $\gamma$ be a geodesic observer. Let $p \in \mathcal{M}_\tau$ be simultaneous to $\gamma(\tau)$. The Fermi distance, $\rho_{\mathrm{Fermi}}(\gamma(\tau), p)$ with respect to $\gamma(\tau)$ is

$$\rho_{\mathrm{Fermi}}(\gamma(\tau), p) = \sqrt{g(p, \exp^{-1}_\gamma p, \exp^{-1}_\gamma p)}$$  \hspace{1cm} (3.3)

This definition of distance can be shown to be the proper distance along the spacelike geodesic from $\gamma$ to $p$.

### 3.3 Fermi Coordinates

In this section, we will use Fermi space slices to put coordinates on a neighborhood around an observer’s worldline. These coordinates, Fermi coordinates, were mentioned in the preceding section as a direct consequence of the Equivalence Principle. Though the underlying structure of a spacetime is, of course, coordinate-independent, viewing spacetime through Fermi coordinates allows one to see the similarities the spacetime has to Minkowski space. In fact, near the path of a geodesic observer in Fermi coordinates, the metric is Minkowskian to the first order [3].

Furthermore, as we will see later, certain definitions of relative velocity are defined on Fermi space slices, and so are natural to study in terms of Fermi coordinates.

Following [1], [2], [12], [14], [15], [16], [17], [18], and [19], Fermi coordinates are associated with the slice $\mathcal{M}_\tau$ in the following way. Each event is given time coordinate $\tau$, and the spatial coordinates are defined relative to an orthonormal reference frame which is parallel transported along the observer’s worldline. Specifically, using the notation of (3.2), a Fermi coordinate system along $\gamma$ is determined by an orthonormal frame, $e_0(\tau), e_1(\tau), e_2(\tau), e_3(\tau)$ parallel (see (2.27)) along $\gamma$, where $e_0(\tau) = \gamma'(\tau)$, the four-velocity of the observer.

A Fermi coordinate chart, $(U, x^0, x^1, x^2, x^3)$, relative to this tetrad are defined by

$$x^0 \left( \exp_{\gamma(\tau)}(\lambda^j e_j(\tau)) \right) = \tau$$

$$x^k \left( \exp_{\gamma(\tau)}(\lambda^j e_j(\tau)) \right) = \lambda^k,$$  \hspace{1cm} (3.4)

where Latin indices run from 1 to 3, while Greek indices run from 0 to 3. The open set on which this map is a diffeomorphism is guaranteed to exist in some neighborhood $U$ around $\gamma(\tau)$; see [5], page 200. Determining the largest possible $U$ in certain classes of spacetimes is a goal of Chapter 4. In Chapter 5, these coordinates are used to find Fermi relative velocities.
3.4 Friedmann-Lematre-Robertson-Walker Cosmology

On large scales, the spatial universe appears homogeneous and isotropic. Starting with this as an assumption, one can construct a spacetime that satisfies the field equations [20]. This model is known as the Friedmann-Lematre-Robertson-Walker (FLRW) model (more general models which do not satisfy the field equations are known as Robertson-Walker, or RW models). Informally, one begins with a collection of maximally symmetric 3-dimensional Riemannian manifolds \( \{ \Sigma_t \} \). Maximal symmetry, in this case, being constant curvature - a condition forced by the assumptions of spatial homogeneity and isotropy. There are three possible 3-dimensional Riemannian manifolds with constant curvature: Euclidean 3-space, the 3-sphere, or hyperbolic 3-space. These three possibilities are denoted \( k = 0 \), \( k = 1 \) or \( k = -1 \), respectively. The value of \( k \) is determined by experiment, though currently unknown; however, observations put the curvature of the universe very near 0 [21].

The metric for this manifold can be written in spherical coordinates

\[
ds^2_{\Sigma} = a^2 \left( d\chi^2 + S_k^2(\chi)d\Omega^2 \right),
\]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) and \( a \) is a scale factor; in the case that \( k = 1 \), \( a \) is the radius of the sphere, and

\[
S_k(\chi) = \begin{cases} 
\sin(\chi) & \text{if } k = 1 \\
\chi & \text{if } k = 0 \\
\sinh(\chi) & \text{if } k = -1
\end{cases}
\]

Introduce the time coordinate \( t \). It turns out that the collection \( \{ \Sigma_t \} \), indexed by \( t \), called public time, foliates the entire spacetime. The coordinates \( \{ t, \chi, \theta, \phi \} \) are therefore called global. If the size of the universe changes, (as has been assumed since the observations of Edwin Hubble in 1929 [22]), the term \( a \) becomes a function of time, \( a(t) \), called, appropriately, the scale factor of the universe. The Robertson-Walker metric is then

\[
ds^2 = -dt^2 + a(t)^2 \left[ d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]

or, using the abreviation above,

\[
ds^2 = -dt^2 + a(t)^2 \left[ d\chi^2 + S_k^2(\chi) d\Omega^2 \right]
\]

There is a coordinate singularity at \( \chi = 0 \), but it does not affect us. The radial coordinate \( \chi \) takes all nonnegative values for \( k = 0 \) or \( -1 \), but is restricted to \( [0, \pi) \) when \( k = 1 \). For clarity, we assume hereon that \( k = 0 \) or \( -1 \) so that the range of \( \chi \) is unbounded.
The expansion of space is sometimes called the Hubble flow, and an observer that is stationary relative to this expansion (that is, an observer with constant $\chi$ coordinate), is called a *co-moving observer*; co-moving with the Hubble flow.

### 3.5 Inflationary Spacetimes

Since Hubble, it has been universally assumed (based on observation) that $a(t)$ is an increasing function of time. In 2011, the Nobel Prize in Physics was awarded to Saul Perlmutter, Brian Schmidt, and Adam Riess for research that indicated that not only is $a(t)$ increasing, it is accelerating, that is $\ddot{a}(t) > 0$, where the overdot represents differentiation with respect to $t$. Spacetimes which include a period in which $a(t) > 0$ are called *inflationary spacetimes*.

A specific class of inflationary spacetimes are *power law cosmologies*, ones in which the metric is given by (3.7), with

$$a(t) = t^\alpha, \quad \alpha > 1.$$  

(3.9)

This is an attractive area of study, for multiple reasons. Noninflationary power law cosmologies, i.e. those with $\alpha < 1$, are well-studied. They include the Milne universe, the matter-dominated universe, and the radiation-dominated universe ($\alpha = 1, \alpha = \frac{2}{3}$, and $\alpha = \frac{1}{2}$, respectively). Additionally, [21], [23], [24], and [25] provide arguments for modeling the universe as an inflationary power law cosmology. Possible values of the curvature of the universe and of $\alpha$ are given in [21], with the values constrained by measurements of the X-ray gas mass fraction of galaxy clusters. At least one paper argues against the viability of the power law model [26]. However, the authors argue against a constant value of $\alpha$, which neither we nor the papers arguing in favor of power law cosmology suggest. We argue that certain epochs of the universe may be modeled by power law scale factors with fixed $\alpha$.

### 3.6 The Cosmological Event Horizon

In certain inflationary cosmologies, including inflationary power law cosmologies, there exists a boundary (relative to an observer) known as the cosmological event horizon [27], past which the observer can never receive light signals.

**Definition 32** *At time $t_s$, a co-moving observer at $\chi = 0$ observes a cosmological event horizon at $\chi$ value*

$$\chi_{\text{horiz}}(t_s) := \int_{t_s}^{\infty} \frac{1}{a(t)} dt.$$  

(3.10)

*If the integral is infinite, all events are within the observer’s horizon.*

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In the case of inflationary power law cosmologies, i.e., for $\alpha > 1$, this takes the form

$$\chi_{\text{horiz}}(t_s) = \int_{t_s}^{\infty} \frac{dt}{t^\alpha} = \frac{t_s^{1-\alpha}}{\alpha - 1},$$

(3.11)

which is finite.

Light signals travel along null geodesics, so this is equivalent to saying that there exist events which cannot be joined to an observer by a null geodesic. To convince the reader, we assume spherically-symmetry and manipulate (3.7):

$$ds^2 = -dt^2 + a(t)^2 \left(d\chi^2\right)$$

(3.12)

Therefore, since $ds = 0$ along a null path, $dt^2 = a(t)^2 d\chi^2$, and so

$$d\chi = \frac{dt}{a(t)} \implies \int_{\chi_{\text{horiz}}}^{0} d\chi = -\int_{t_s}^{\infty} \frac{1}{a(t)} dt,$$

(3.13)

where $t_s$ is the current cosmological time for the observer at $\chi = 0$, giving (3.10).
Chapter 4

Fermi Coordinates

As stated in the previous chapter, Fermi coordinates are an important tool for the study of cosmology. In this chapter, previously found formulas are given, and a new result is proved: that in the case of inflationary power law cosmologies, Fermi coordinates extend \textit{qu a patet orbis}, that is, to the cosmological event horizon.

4.1 Formulas for Fermi Coordinates

\textit{Note: this section follows the treatments of [1], [2], [16], and [19].}

We begin by considering the submanifold $M_{\theta_0,\varphi_0} = M_{\theta_0,\varphi_0,k}$ determined by constant angular coordinates $\theta = \theta_0$ and $\varphi = \varphi_0$. The restriction of the RW metric (3.7) to $M_{\theta_0,\varphi_0}$ is

$$ds^2 = -dt^2 + a^2(t)d\chi^2,$$

(4.1)

for which there is no longer a coordinate singularity. We restrict our attention to events with nonnegative $\chi$ value. This is justified by symmetry.

Consider the observer with geodesic path $\gamma(t) = (t, 0)$ in $M_{\theta_0,\varphi_0}$. Let $\rho$ be the proper distance along a spacelike geodesic orthogonal to $\gamma$. Then, following [2] and [19], the vector field

$$X = \partial_\rho = \frac{dt}{d\rho} \partial_t + \frac{d\chi}{d\rho} \partial_\chi = -\sqrt{\left(\frac{a(\tau)}{a(t)}\right)^2 - 1} \partial_t + \frac{a(\tau)}{a(t)} \partial_\chi$$

(4.2)

is geodesic, spacelike, unit, and orthogonal to $\gamma'$ at each point $p = (\tau, 0)$ along $\gamma$, that is, it is tangent to $M_\tau$.

As noted in [16] and [19], since $dt/d\rho < 0$, coordinate time decreases with increasing proper distance along the spacelike geodesic originating at $\gamma(\tau)$, which we denote $\psi$.

Using that $a(t)$ is increasing and $a(0) = 0$, we choose a nonaffine parameter for this geodesic

$$\sigma = \left(\frac{a(\tau)}{a(t)}\right)^2 \quad \text{with} \quad \sigma \in [1, \infty),$$

(4.3)

so $\psi = \psi(\sigma)$. The following theorem was proved in [1] and [2].

\textbf{Theorem 6} For smooth, increasing scale factors $a(t)$ with inverse $b(t)$, the spacelike geodesic orthogonal to $\gamma(t)$ at $t = \tau$ and parametrized by the parameter $\sigma$ defined in (4.3) is given by $\psi_\tau(\sigma) = (t(\tau, \sigma), \chi(\tau, \sigma))$, where
\[ t(\tau, \sigma) = b \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \]  \quad (4.4)

\[ \chi(\tau, \sigma) = \frac{1}{2} \int_1^\sigma b \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \frac{1}{\sqrt{\sigma} \sqrt{\sigma} - 1} \, d\sigma. \]  \quad (4.5)

Furthermore, at a fixed \( \tau \), proper distance along \( \psi_\tau \) is given by

\[ \rho = \rho_\tau(\sigma) = \frac{a(\tau)}{2} \int_1^\sigma \frac{b \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\frac{1}{\sigma^{3/2} \sqrt{\sigma} - 1}} \, d\sigma. \]  \quad (4.6)

Since the integrand in (4.6) is positive, it follows that for a fixed value of \( \tau \), \( \rho \) is a smooth, increasing function of \( \sigma \) with smooth inverse denoted

\[ \sigma_\tau(\rho) = \sigma(\rho). \]  \quad (4.7)

The change of variables \( \tilde{\sigma} = \left( \frac{a(\tau)}{a(t)} \right)^2 \), for fixed \( \tau \), changes (4.6) to the form

\[ \rho = \int_{t_s}^\tau \frac{a(t)}{\sqrt{a^2(\tau) - a^2(t)}} \, dt \]  \quad (4.8)

It follows from Theorem 3 and (4.7) that we may parametrize \( \psi_\tau \) by proper length, as stated in the following corollary.

**Corollary 7** With the hypotheses and notation of Theorem 3,

\[ \psi_\tau(\rho) = (t(\tau, \sigma(\rho)), \chi(\tau, \sigma(\rho))). \]  \quad (4.9)

**Remark 4.1.1** If we return our attention to the entire manifold \( \mathcal{M} \), it follow from symmetry that the unique spacelike geodesic orthogonal to the observer \( \gamma(t) = (t, 0, 0, 0) \) at \( t = \tau \) with fixed angular coordinates \( \theta_0, \varphi_0 \) is given by

\[ \Psi_\tau(\rho) = (t(\tau, \sigma(\rho)), \chi(\tau, \sigma(\rho)), \theta_0, \varphi_0) \]  \quad (4.10)

Let \( t_s > 0 \) be arbitrary but fixed. Define

\[ \sigma(\tau) := \left( \frac{a(\tau)}{a(t_s)} \right)^2. \]  \quad (4.11)

By (4.4), \( \sigma(\tau) = \sigma \) when \( t(\tau, \sigma) = t_s \). With this, (4.11), and (4.5) in mind, define

\[ \chi_\sigma(\tau) := \chi(\tau, \sigma(\tau)) = \frac{1}{2} \int_1^{\sigma(\tau)} b \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \frac{1}{\sqrt{\sigma} \sqrt{\sigma} - 1} \, d\sigma. \]  \quad (4.12)

In the case of interest, namely \( a(t) = t^\alpha \), we get that (4.8) becomes

\[ \rho = \int_{t_s}^\tau \frac{t^\alpha}{\sqrt{\tau^{2\alpha} - t^{2\alpha}}} \, dt, \]  \quad (4.13)
(4.11) becomes
\[
\sigma(\tau) = \frac{\tau^{2\alpha}}{t_{s}^{2\alpha}},
\] (4.14)
and most importantly, (4.12) becomes
\[
\chi_s(\tau) = \frac{1}{2\alpha\tau^{\alpha-1}} \int_{1}^{\sigma(\tau)} \frac{1}{\tilde{\sigma}^{1/2\alpha}\sqrt{\tilde{\sigma} - 1}} d\tilde{\sigma}.
\] (4.15)

4.2 Maximal Extension of Fermi Coordinates in Inflationary Cosmologies

It was shown in [1] that the Fermi coordinate chart for a comoving observer covers the entire Robertson-Walker spacetime, provided the scale factor \(a(t)\) is smooth, increasing, unbounded, and for all \(t > 0\), \(\ddot{a}(t) \leq 0\), i.e., for non inflationary cosmologies (with or without a big bang).

Cosmologies with scale factors of the form \(a(t) = t^\alpha\), with \(0 < \alpha \leq 1\) fall within this category and have global Fermi coordinates for comoving observers. Included are the radiation dominated universe \((\alpha = 1/2)\) and the matter dominated universe \((\alpha = 2/3)\).

In this section, we show that Fermi coordinates exist on a physically meaningful open set around the observer when the scale factor is of the form \(a(t) = t^\alpha\) with \(\alpha > 0\). Recall (3.10) and let
\[
V := \{(t, \chi) : t > 0 \text{ and } 0 < \chi < \chi_{\text{horiz}}(t)\}.
\]
Intuitively, \(V\) is the set of all eventually observable events. For \(0 < \alpha < 1\), \(V = (0, +\infty) \times [0, +\infty)\) represents all spacetime points.

We will show that the set of spacetime points with coordinates in \(V\) is a maximal chart for Fermi coordinates for a central observer in an inflationary power law cosmology. This involves first showing that \((\tau, \sigma)\) are coordinates on that chart. For that purpose, we establish a series of lemmas.

**Lemma 2** For \(a(t) = t^\alpha\) with \(\alpha > 1\), and any \(\tau > t_s > 0\),
\[
\chi_s(\tau) < \chi_{\text{horiz}}(t_s).
\]

**Proof.** Substituting \(x^{2\alpha} = \tilde{\sigma}\) into (4.15) gives,
\[
\chi_s(\tau) = \frac{1}{\tau^{\alpha-1}} \int_{1}^{\tau/t_s} \frac{x^{2\alpha-1}}{x \sqrt{x^{2\alpha} - 1}} \, dx.
\] (4.16)
Integration by parts in (4.16) yields,
\[
\chi_s(\tau) = \frac{1}{\alpha \tau^{\alpha-1}} \left[ \frac{\sqrt{(\tau/t_s)^{2\alpha} - 1}}{\tau/t_s} + \int_{1}^{\tau/t_s} \frac{\sqrt{x^{2\alpha} - 1}}{x^2} \, dx \right]
\]
\[
< \frac{1}{\alpha \tau^{\alpha-1}} \left[ \left( \frac{\tau}{t_s} \right)^{\alpha-1} + \int_{1}^{\tau/t_s} x^{\alpha-2} \, dx \right]
\]
\[
< \frac{1}{\alpha \tau^{\alpha-1}} \left[ \left( \frac{\tau}{t_s} \right)^{\alpha-1} + \frac{1}{\alpha - 1} \left( \frac{\tau}{t_s} \right)^{\alpha-1} \right]
\]
\[
= \frac{t_s^{1-\alpha}}{\alpha - 1} = \chi_{\text{horiz}}(t_s).
\]

**Lemma 3** For \(\alpha(t) = t^\alpha\) with \(\alpha > 1\), and any \(t_s > 0\),
\[
\lim_{\tau \to +\infty} \chi_s(\tau) = \chi_{\text{horiz}}(t_s).
\]

**Proof.** From L'Hospital’s and Leibniz’s rules applied to (4.15), we have,
\[
\lim_{\tau \to +\infty} \chi_s(\tau) = \lim_{\tau \to +\infty} \left( \frac{t_s^{1-\alpha}}{\alpha - 1} \right) \left( \frac{\tau^\alpha}{\sqrt{\tau^{2\alpha} - t_s^{2\alpha}}} \right) = \frac{t_s^{1-\alpha}}{\alpha - 1} = \chi_{\text{horiz}}(t_s).
\]

**Lemma 4** For \(\alpha(t) = t^\alpha\) with \(\alpha > 1\), and any \(\tau > t_s > 0\),
\[
\frac{d\chi_s}{d\tau}(\tau) > 0.
\]

**Proof.** For convenience, rewrite (4.15) as \(\chi_s(\tau) = \frac{f(\tau)}{g(\tau)}\) where,
\[
f(\tau) := \int_{1}^{\sigma(\tau)} \frac{1}{\sigma^{1/2\alpha} \sqrt{\sigma - 1}} \, d\sigma, \quad \text{and} \quad g(\tau) := 2\alpha \tau^{\alpha-1}.
\]

By the quotient rule, \(\frac{d\chi_s}{d\tau} > 0\) if and only if
\[
\chi_s(\tau) = \frac{f(\tau)}{g(\tau)} < \frac{f'(\tau)}{g'(\tau)},
\]
\[ (4.18) \]
By Leibniz’ rule, the quotient on the right in (4.18) is,
\[
f' (\tau) = \left( \frac{t_s^{1-\alpha}}{\alpha - 1} \right) \left( \frac{\tau^\alpha}{\sqrt{\tau^{2\alpha} - t_s^{2\alpha}}} \right).
\] (4.19)

Since the last term on the right in (4.19) is always greater than 1, we have,
\[
\frac{f' (\tau)}{g' (\tau)} > \frac{t_s^{1-\alpha}}{\alpha - 1} = \chi_{\text{horiz}} (t_s),
\]
Therefore \( \frac{d\chi}{d\tau} > 0 \) for any \( \tau \) such that \( \chi (\tau) < \chi_{\text{horiz}} (t_s) \). The result now follows from Lemma 2.


Lemma 5 For \( a(t) = t^\alpha \), the map \( F : (0, \infty) \times (1, \infty) \rightarrow V \) given by
\[
F (\tau, \sigma) := (t(\tau, \sigma), \chi (\tau, \sigma)),
\]
where the functions \( t \) and \( \chi \) are defined by (4.4) and (4.5), respectively, is a diffeomorphism.

Proof. Let \( (t_s, \chi_s) \in V \) be arbitrary but fixed. To prove that \( F \) is a bijection, we must show there exists a unique pair \( (\tau_0, \sigma_0) \in (0, +\infty) \times [1, +\infty) \) such that \( F(\tau_0, \sigma_0) = (t_s, \chi_s) \). From (4.4), it follows that \( \sigma_0 \) is uniquely determined by \( \tau_0 \) and,
\[
\sigma_0 = \left( \frac{a(\tau_0)}{a(t_s)} \right)^2.
\]
So it remains only to find \( \tau_0 \). To that end, let \( \sigma (\tau) \) be given by (4.11). It then follows from (4.12) and (4.5) that
\[
\chi (\tau, \sigma (\tau)) = \chi_s (\tau).
\]
Since by assumption, \( \chi_s < \chi_{\text{horiz}} (t_s) \), it follows from Lemmas 3 and 4 that there is a unique \( \tau_0 > t_s \) such that \( \chi (\tau_0, \sigma (\tau_0)) = \chi ((\tau_0, \sigma_0) = \chi_s \). Thus, \( F \) is a bijection.

The Jacobian determinant \( J(\tau, \sigma) \) for the transformation \( F \) was calculated in \([?]\) for a general class of scale factors including the power law scale factors considered here, and is given by,
\[
J(\tau, \sigma) = \frac{\dot{a}(\tau)}{2\sigma} \left( a(\tau) \right) \left( b \left( \frac{a(\tau)}{\sqrt{\sigma}} \right) \frac{\sigma}{\sqrt{\sigma} - 1} + \frac{a(\tau)}{2\sqrt{\sigma}} \int_1^\sigma \frac{b \left( \frac{a(\tau)}{\sqrt{\tilde{\sigma}}} \right)}{\tilde{\sigma} \sqrt{\tilde{\sigma} - 1}} d\tilde{\sigma} \right). \] (4.20)
Thus, from (4.20),

\[ J(\tau, \sigma(\tau)) = \frac{a(\tau) \dot{a}(\tau)}{2\sigma(\tau)^{3/2}} \left( \sqrt{\sigma(\tau)} \right) \left[ \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma(\tau)}} \right) \sqrt{\sigma(\tau)}}{a(\tau) \sqrt{\sigma(\tau)} - 1} + \frac{1}{2} \int_{1}^{\sigma(\tau)} \frac{\ddot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma} \right. \right] \left. \frac{d\sigma}{d\tau} \right] . \]

(4.21)

The first term in the square brackets in (4.21) may be rewritten:

\[ \left( \sqrt{\sigma(\tau)} \right) \left[ \frac{\dot{b} \left( \frac{a(\tau)}{\sqrt{\sigma(\tau)}} \right) \sqrt{\sigma(\tau)}}{a(\tau) \sqrt{\sigma(\tau)} - 1} = \frac{a(\tau) \dot{b}(a(t_s))}{a(t_s)^2 \sqrt{\sigma(\tau)} \sqrt{\sigma(\tau)} - 1} \right. \]

(4.22)

Now, applying Leibniz’ rule (see Chapter 2, Section 9) to (4.12) and (4.11) yields,

\[ \frac{d\chi(\tau)}{d\tau} = \dot{a}(\tau) \left[ \frac{a(\tau) \dot{b}(a(t_s))}{a(t_s)^2 \sqrt{\sigma(\tau)} \sqrt{\sigma(\tau)} - 1} + \frac{1}{2} \int_{1}^{\sigma} \frac{\ddot{b} \left( \frac{a(\tau)}{\sqrt{\sigma}} \right)}{\sigma} \right. \right] \left. \frac{d\sigma}{d\tau} \right] . \]

(4.23)

Combining (4.21), (4.22) and (4.23) gives,

\[ J(\tau, \sigma(\tau)) = \frac{a(\tau) \dot{b}(a(t_s)) d\chi(\tau)}{2\sigma(\tau)^{3/2} d\tau} . \]

(4.24)

Thus, applying Lemma 4 in (4.24), \( J(\tau, \sigma(\tau)) > 0 \) for all \( \tau \). Now given any \( \tau > 0 \) and \( \sigma > 1 \) there exists a positive \( t_s < \tau \) such that \( \sigma(\tau) = (\tau/t_s)^{2\alpha} = \sigma \). Therefore \( J(\tau, \sigma) > 0 \) for all \((\tau, \sigma)\), and by the inverse function theorem, \( F \) is a diffeomorphism.

To finish the construction of Fermi coordinates, we return to (4.8) in the form,

\[ \rho = \int_{t_s}^{\tau} \frac{a(t)}{\sqrt{a^2(\tau) - a^2(t)}} dt, \]

(4.25)

which gives the proper distance along the spacelike geodesic \( \psi_{\tau} \) orthogonal to \( \beta \), from the initial point \((\tau, 0) \in \beta \) to the unique point whose \( t \)-coordinate is \( t_s \). The radius \( \rho_{M_{\tau}} \) of the Fermi slice, \( M_{\tau} \), at proper time \( \tau \) of the central observer is obtained by replacing \( t_s \) by zero in (4.25). This can be understood by considering the last line of (4.17); the largest \( \chi_s \) values correspond to the smallest \( t_s \) values. The result for \( a(t) = t^\alpha \) is,

\[ \rho_{M_{\tau}} = \rho_{M_{\tau}}(\alpha) = \frac{\sqrt{\pi} \Gamma \left( \frac{1+\alpha}{2\alpha} \right)}{\Gamma \left( \frac{1}{2\alpha} \right)} \tau, \]

(4.26)

which holds for all \( \alpha > 0 \) (see [1]). As the authors noted, the spacelike geodesic joining any two events on a Fermi space slice is bounded above by (4.26), and as such, the quantity may be thought of, in some senses, as the radius of the universe.
Recall (4.6) and define

\[ U_{\text{Fermi}} := \{ (\tau, \rho) : \tau > 0 \text{ and } 0 < \rho < \rho_{M^*} \}, \]

and let \( G(\tau, \sigma) := (\tau, \rho(\sigma)) \). Then \( G \) is a diffeomorphism with inverse \( G^{-1}(\tau, \rho) = (\tau, \sigma(\rho)) \). Define \( H : V \to U_{\text{Fermi}} \) by

\[ H(t, \chi) := G \circ F^{-1}(t, \chi). \]

Then \( H \) is a diffeomorphism. Taking into consideration that \( \chi \) and \( \rho \) are both positive, radial coordinates, we can now state the main result of this section:

**Theorem 8** For a comoving observer in a Robertson-Walker cosmology whose scale factor is \( a(t) = t^\alpha \) with \( \alpha > 0 \), the maximal domain of Fermi coordinates is the set of all spacetime points whose curvature coordinates take values in \( V \), i.e., Fermi coordinates extend to the cosmological event horizon. The range of Fermi coordinates is \( U_{\text{Fermi}} \).

In the case that \( 0 < \alpha \leq 1 \), \( V = M \), this is due to [1].

### 4.3 The Metric

Following the notation of the preceding section, define the open set \( \mathcal{V} \subset M \) by

\[ \mathcal{V} = \{ p \in M \mid t(p) > 0 \text{ and } 0 < \chi(p) < \chi_{\text{horiz}}(t(p)) \}. \tag{4.27} \]

**Theorem 9** \((\mathcal{V}, \tau, \rho, \theta, \varphi)\) is a chart on \( M \) with line element given by

\[ ds^2 = g_{\tau\tau} d\tau^2 + d\rho^2 + \frac{a^2(\tau)}{\sigma(\rho)} S^2_{k}(\chi(\tau, \sigma(\rho))) d\Omega^2, \tag{4.28} \]

where

\[ g_{\tau\tau} = -\left( \dot{a}(\tau) \right)^2 \left( b \left( \frac{a(\tau)}{\sqrt{\sigma(\rho)}} \right) + a(\tau) \frac{\sqrt{\sigma(\rho)} - 1}{2 \sqrt{\sigma(\rho)}} \int_{1}^{\sigma(\rho)} \frac{\dot{b}}{\sigma \sqrt{\sigma - 1}} d\sigma \right)^2, \tag{4.29} \]

where \( \sigma(\rho) \) is given by (4.7) and \( \chi(\tau, \sigma(\rho)) \) is obtained from substituting (4.7) into (4.5).

That this is the case follows from direct calculation, detailed in [1]. In the case of power law cosmologies, this takes the form:

\[ g_{\tau\tau} = -\left( \frac{1}{\sqrt{\alpha \sigma(1-\alpha)/2\alpha}} + \frac{(1 - \alpha)\sqrt{\sigma(\rho) - 1}}{2\alpha^3/2 \sqrt{\sigma(\rho)}} \int_{1}^{\sigma(\rho)} \frac{1}{\tilde{\sigma}^{1/2\alpha} \sqrt{\tilde{\sigma} - 1}} d\tilde{\sigma} \right)^2 \tag{4.30} \]
In the following chapter, we will consider radial relative motion, and so for convenience revert to the 2-dimensional case. The metric, then, is

\[ ds^2 = g_{rr} d\tau^2 + d\rho^2, \]  

(4.31)

where \( g_{rr} \) is given above.

### 4.4 Examples

In this section we apply the above results to two particular cosmologies: the Milne Universe, and the model suggested by [21], in which \( k = -1 \) and \( \alpha = \frac{8}{7} \), which we refer to as the Zhu-Hu-Liu-Alcaniz (ZHuLA) Universe, after the authors.

Though not inflationary, we begin by examining the Milne Universe - a pedagogically interesting example of Fermi coordinates.

**Example 6 (The Milne Universe)** The Milne Universe is a prototype universe with no matter or energy, negative spatial curvature, and \( a(t) = t \).

To find Fermi coordinates for a comoving observer, we integrate (4.6) and obtain

\[ \rho = \tau \sqrt{\frac{\sigma - 1}{\sigma}}, \]  

(4.32)

Inverting this gives

\[ \sigma = \frac{1}{1 - \left(\frac{\rho}{\tau}\right)^2}. \]  

(4.33)

Substituting into (4.4) and (4.5) gives

\[ t = \frac{\tau}{\sqrt{\sigma}} = \sqrt{\tau^2 - \rho^2} \]  

(4.34)

and

\[ \chi = \ln(\sqrt{\sigma} + \sqrt{\sigma - 1}) = \tanh^{-1} \sqrt{\frac{\sigma - 1}{\sigma}} = \tanh^{-1} \left(\frac{\rho}{\tau}\right). \]  

(4.35)

(4.34) and (4.35) are inverted to give Fermi coordinates, \((\tau, \rho)\) in terms of \((t, \chi)\):

\[ \tau = t \cosh \chi \]

\[ \rho = t \sinh \chi. \]  

(4.36)

On what domain is this coordinate transformation valid? One way to answer this, in light of section 4.2, is to consider the cosmological event horizon, (3.10).

\[ \chi_{\text{horiz}}(t_0) = \int_{t_0}^{\infty} \frac{1}{t} dt = \infty \]  

(4.37)
for all $t_0$; that is, all spacetime points are always within an observer’s horizon. Theorem 5, then, says that this coordinate change is valid for all $(t, \chi)$. The metric, (4.31), reduces to simply

$$ds^2 = -d\tau^2 + d\rho^2.$$  

(4.38)

We have recovered the metric of special relativity; this well-known result shows that The Milne Universe can be thought of as the inside of a forward-pointing light cone of an event in Minkowski space. We note that in this case, $|g_{\tau\tau}| = 1$, the speed of light. This is not a coincidence, as we shall see in the following chapter.

**Remark 4.4.1** This shows that it is possible to have a universe in which $a(t)$ is increasing, but observers working in local coordinates view as static. If we wish to argue for an expanding universe, we may not do so solely on the basis of the sign of $\dot{a}(t)$.

Using (4.26), we see that

$$\rho_{M_r}(1) = \frac{\sqrt{\pi}\Gamma(1)}{2\Gamma(\frac{3}{2})}\tau = \tau.$$  

(4.39)

Although not new, this model has interesting features; most notably the ambiguity mentioned above. Would an observer in the Milne universe view space as expanding? Since we have identified the Milne universe with Minkowski space, most would say no - Minkowski space is unequivocally static. In fact, the authors of [28] say that “no one would seriously suggest that [the Milne universe] is expanding.” However, that is exactly what we propose: that the Milne universe *does* expand, on the basis that $\rho_{M_r}$, in some sense the radius of the universe, is an increasing function of $\tau$. As a proper distance, it has the advantage of being a measurement that is free of coordinates.

**Example 7 (ZHuLA Universe)** The ZHuLA Universe is an empirically derived, inflationary power law cosmological model of the universe, with $k = -1$ and $a(t) = t^\alpha$, with $1.09 \leq \alpha \leq 1.19$. For definitiveness, and for comparison with other standard models, we use the value $\alpha = \frac{8}{7}$.

We follow a similar procedure, and again using (4.26), we see that in this case

$$\rho_{M_r}\left(\frac{8}{7}\right) = \frac{\sqrt{\pi}\Gamma\left(\frac{15}{16}\right)}{2\Gamma\left(\frac{7}{16}\right)}\tau \approx 0.911\tau < \tau = \rho_{M_r}(1).$$  

(4.40)

This motivates taking the limit as $\alpha \to \infty$ in (4.26); the result is shocking:

$$\lim_{\alpha \to \infty} \rho_{M_r}(\alpha) = \lim_{\alpha \to \infty} \frac{\sqrt{\pi}\Gamma\left(\frac{1+\alpha}{2\alpha}\right)}{\Gamma\left(\frac{1}{2\alpha}\right)}\tau = 0.$$  

(4.41)

The *faster* the universe expands, the *less* available space there is.
Chapter 5

Relative Velocity

Most students of the subject are surprised to learn that general relativity provides no *a priori* definition of non-local relative velocities in curved space. An observer can only compare the velocity of a particle relative to her own at a single point. It is only at that point that we can say the particle’s velocity is bounded by \( c \); away from the observer a particle’s relative velocity may exceed \( c \) (in certain scenarios). Concern regarding the physical interpretation of superluminal velocities, and their existence or lack thereof, motivated a series of papers, including [17] and references 3, 4, and 5 therein.

We consider only radial motion, and as such, restrict ourselves to two dimensions. An observer, \( \gamma \), will travel along the path \( \gamma(t) = (t, 0) \). The vector \( \gamma' \) is called the observer’s *four-velocity* - though we work in two dimensions, we keep in mind that we are actually exploiting spherical symmetry in the 4-dimensional case. \( \gamma' = (1, 0) \) clearly has the property that \(|\gamma'| = g(\gamma', \gamma') = -1\); it is a consequence of the speed of light being constant in all local inertial frames that an arbitrary timelike path describing the motion of an observer, \( \beta \), has \(|\beta'| = -1\).

5.1 Remarks on the Competing Definitions of Relative Velocity

There are four geometrically defined, nonequivalent definition of relative velocity: Fermi, kinematic, astrometric, and spectroscopic. They rely on two separate notions of simultaneity: spacelike simultaneity (or Fermi simultaneity), which is notion of simultaneity Fermi coordinates were based on, and lightlike simultaneity. Lightlike simultaneity will not be discussed in detail, but we proceed with an informal discussion. A collection of lightlike surfaces of simultaneity \( \{N_\tau\} \) which foliate a portion of spacetime are constructed analogously to Fermi space slices, except the geodesics which define them are null.

We focus our attention on the Fermi velocity, which is based on spacelike simultaneity.

5.2 Fermi Relative Velocity

As stated at the end of section 4.3, we restrict to fixed angular coordinates in this portion of the thesis.

Fermi relative velocity is defined in terms of Fermi coordinates. At the point \( p = (\tau, 0) \), the observer \( \gamma \) measures the Fermi relative velocity of a radial moving test
particle, \( \beta \) at \( q = (\tau, \rho) \) (hence \( q \in \mathcal{M}_\tau \)) to be

\[
\psi_{\text{Fermi}} = \frac{d\rho}{d\tau} \big|_p \partial_p, \tag{5.1}
\]

where \( \partial_p \big|_p \in T_p \mathcal{M} \).

An interesting connection can be made here (see [17] and [16]). Return to the polar Fermi metric, (4.31) and consider the velocity of a photon at the event \( (\tau, \rho) \) by setting

\[
0 = ds^2 = g_{\tau\tau} d\tau^2 + d\rho^2, \tag{5.2}
\]

gives, for that photon,

\[
\left| \frac{d\rho}{d\tau} \right| = \sqrt{-g_{\tau\tau}(\tau, \rho)}. \tag{5.3}
\]

This gives the following definition and proposition.

**Definition 33** In Fermi coordinates, the non-local speed of light is defined

\[
c(\tau, \rho) = \sqrt{-g_{\tau\tau}(\tau, \rho)}. \tag{5.4}
\]

**Proposition 7** Given an observer, \( \gamma \) at the event \( (\tau, 0) \) the magnitude of the Fermi velocity of a radially moving test particle at \( (\tau, \rho) \) is bounded by the non-local speed of light at \( (\tau, \rho) \), namely \( c(\tau, \rho) \). This is a consequence of the postulates of special relativity, and can be written

\[
\|\psi_{\text{Fermi}}\| < \sqrt{-g_{\tau\tau}(\tau, \rho)}. \tag{5.5}
\]

### 5.3 Fermi Velocity in Power Law Cosmologies

The non-local speed of light, \( c(\tau, \rho) \), is not necessarily equal to the local speed of light, \( c \) (which is equal to one in this paper). This means that relative to an observer, a comoving test particle can move faster than the local speed of light. In fact, in [1], it was shown that in the radiation-dominated universe, \( \alpha = 1/2 \), \( c(\tau, \rho) \) approaches \( \frac{\pi}{2} \) for increasingly distant comoving observers. More generally, see the following remark ([1], [19]).

**Remark 5.3.1** For power law cosmologies, \( a(t) = t^\alpha \) with \( 0 < \alpha < 1 \),

\[
\lim_{\rho \to \rho_M} \|\psi_{\text{Fermi}}\| = \frac{\rho_M}{\tau} = \frac{\sqrt{\pi} \Gamma\left(\frac{1+\alpha}{2\alpha}\right)}{\Gamma\left(\frac{1}{2\alpha}\right)} > 1, \tag{5.6}
\]

as proved in [1].

However, in inflationary power law cosmologies, this is not the case – the situation is more complicated. To analyze it, we follow [16] and define the parameter \( v \) to be
\( v = \alpha \tau^{\alpha-1} \chi \), which, for \( \alpha > 1 \) has no upper bound, and

\[
C_\alpha := \frac{\sqrt{\pi} \Gamma\left(\frac{1+\alpha}{2\alpha}\right)}{\Gamma\left(\frac{1}{2\alpha}\right)} = \rho M_\tau. \tag{5.7}
\]

By (4.12),

\[
\left(\frac{t_s}{\tau}\right)^{1-\alpha} \left\{ 2 F_1 \left( \frac{1}{2}, \frac{1-\alpha}{2\alpha}; \frac{1+\alpha}{2\alpha}; \left(\frac{t_s}{\tau}\right)^{2\alpha} \right) \right\} = C_\alpha + \frac{\alpha - 1}{\alpha} v, \tag{5.8}
\]

where \( 2 F_1 (\cdot, \cdot; \cdot; \cdot) \) is the Gauss hypergeometric function. Define the bijective function

\[
F_\alpha(z) := z^{\frac{1-\alpha}{2}} \left\{ 2 F_1 \left( \frac{1}{2}, \frac{1-\alpha}{2\alpha}; \frac{1+\alpha}{2\alpha}; z^2 \right) \right\}, \tag{5.9}
\]

where \( 0 < z < 1 \) and \( \alpha > 0 \). Then it has an inverse, \( F^{-1}_\alpha \), and we define

\[
G_\alpha(v) := (F^{-1}_\alpha(C_\alpha + \frac{\alpha - 1}{\alpha} v))^{1/\alpha}, \tag{5.10}
\]

and finally, the following remark.

**Remark 5.3.2** For inflationary power law cosmologies, \( a(t) = t^\alpha \) with \( \alpha > 1 \), we have

\[
\|v_{\text{Fermi}}\| = G^{-1}_\alpha(v)\sqrt{1 - G^{2\alpha}_\alpha(v)} - \frac{\alpha - 1}{\alpha} (1 - G^{2\alpha}_\alpha(v)) v < 1 \tag{5.11}
\]

[16]. It follows that \( \|v_{\text{Fermi}}\| < 1 \).

For completeness, we remind the reader that the case that \( \alpha = 1 \) was discussed at the end of Chapter 4. In it,

\[
\lim_{\rho \to \rho M_\tau} \|v_{\text{Fermi}}\| = \frac{\rho M_\tau}{\tau} = 1. \tag{5.12}
\]

That is, in power law cosmologies, the magnitude of the Fermi velocity of a test particle is necessarily less than the local speed of light only in the case of inflation. In noninflationary power law cosmologies, a test particle may recede at velocities that equal or exceed the local speed of light. This may be taken as another argument for the consideration of the ZHuLA Universe as a viable model for study.
Extending the work of Randles & Klein in [1], we have shown that Fermi coordinates, though not necessarily global, cover all observable spacetime events in all power law cosmologies. When the cosmology is inflationary, this is a proper subset of the entire spacetime, with a boundary we referred to as the cosmological event horizon. This boundary is physically meaningful, which suggests the result is more than a mathematical curiosity. These coordinates were then used to discuss a candidate for the radius of the universe, and to develop the Fermi relative velocity. It was shown that the Fermi relative velocity of a comoving test particle does not exceed the local speed of light.

It is common to divide the universe chronologically into three epochs: radiation-dominated, matter-dominated, and dark-energy-dominated. The first two epochs are characterized by a scale factor of the form $a(t) \propto t^\alpha$, where $\alpha = \frac{1}{2}$ and $\alpha = \frac{2}{3}$, respectively. Empirical evidence suggests that the universe is currently transitioning from the second to the third epoch; observations suggest that the current matter-energy distribution of the universe is approximately 70% dark-energy, 30% matter [29]. It is sometimes assumed that the dark-energy component of the matter-energy distribution will continue to grow; that is, that the cosmological constant is in fact not constant. It is then known as quintessence ([30], [31]). Quintessence is one possible explanation of how dark energy has become the majority of the matter-energy distribution.

In any case, if the current trend continues, the universe will transition to a fully dark-energy-dominated epoch, and the scale factor will not be of the form $a(t) = t^\alpha$. The repulsive forces will be so great that $a(t) \propto e^t$ ([29] and [20]). However, the universe is not in the third epoch yet, and as discussed, observational evidence can be argued to support a model in which $a(t) = t^{8/7}$. This is appealing for multiple reasons, not the least of which is its shared form with the matter- and radiation-dominated epochs.

Finally, a remark on the limit (4.41), which we state again:

$$\lim_{\alpha \to \infty} \rho_{M^r}(\alpha) = 0.$$  \hspace{1cm} (6.1)

As noted above (4.26), the further away an event on a Fermi space slice is (that is, the greater its $\chi$ value) from an observer, the lower that event’s $t_s$ value, until we reach the big bang at $t_s = 0$. What is the big bang? It is a period of extreme expansion. Since “any spacelike geodesic orthogonal to, and with initial point on, the worldline of a comoving observer terminates at the big bang in a finite proper distance” [19], we may say that all events are simultaneous with the big bang; maybe,
then, “period of extreme expansion” should be replaced by the phrase “location of extreme expansion.” Then, we interpret the above limit as saying that the greater the rate of inflation, the shorter the distance to a location of expansion great enough to be the big bang.
Appendix: The Einstein Field Equations

The Einstein Field Equations describe how the metric of spacetime responds to energy and momentum. As they act as a replacement for Newton’s law of gravity, they are in a sense analogous to the Poisson equation for the Newtonian potential:

\[ \nabla^2 \Phi = 4\pi G \rho. \]  

(2)

[3] details the process by which Einstein came to the following:

\[ R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} + \Lambda g_{\mu \nu} = 8\pi G T_{\mu \nu}, \]  

(3)

where \( R_{\mu \nu} \) is the Ricci tensor, \( R \) is the curvature scalar, \( \Lambda \) is a new fundamental constant of nature, the vacuum energy (or cosmological constant), \( G \) is Newton’s gravitational constant, and \( T_{\mu \nu} \) is the energy-momentum tensor. (3) are the Einstein Field Equations; hiding in the subscripts are 10 partial differential equations, hence the use of the plural “equations.”

Briefly, \( T_{\mu \nu} \) gives the flux of the \( \mu^{th} \) component of the momentum 4-vector across a surface with constant \( x^\nu \) coordinate. It generalizes \( \rho \) in (2). The Ricci tensor and curvature scalar describe the curvature of spacetime, and require the introduction of the Riemann curvature tensor, \( R^\rho_{\sigma \mu \nu} \). We develop it by considering the relationship of parallel transport to the curvature of the space: in flat space, parallel transport around a loop is independent of the direction one goes. The Riemann curvature tensor will relate to the degree to which the space under consideration fails to have this property. To do this, we look at the commutator of two covariant derivatives, as in [3], to which we now defer.

The relationship between this and parallel transport around a loop should be evident; the covariant derivative of a tensor in a certain direction measures how much the tensor changes relative to what it would have been if it had been parallel transported, since the covariant derivative of a tensor in a direction along which it is parallel transported is zero. The commutator of two covariant derivatives, then, measures the difference between parallel transporting the tensor first one way and then the other versus the opposite ordering.

Let \( V^\rho \) be some vector field. Then:

\[
[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \\
= \partial_\mu (\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu \sigma} \nabla_\lambda V^\rho - \partial_\nu (\nabla_\mu V^\rho) - \Gamma^\lambda_{\nu \sigma} \nabla_\lambda V^\rho + \Gamma^\rho_{\mu \sigma} \nabla_\mu V^\sigma \\
= (\partial_\mu \Gamma^\rho_{\nu \sigma} - \partial_\nu \Gamma^\rho_{\mu \sigma} + \Gamma^\rho_{\mu \lambda} \Gamma^\lambda_{\nu \sigma} - \Gamma^\rho_{\nu \lambda} \Gamma^\lambda_{\mu \sigma}) V^\sigma - 2 \Gamma^\lambda_{[\mu \nu]} \nabla_\lambda V^\rho \\
= (\partial_\mu \Gamma^\rho_{\nu \sigma} - \partial_\nu \Gamma^\rho_{\mu \sigma} + \Gamma^\rho_{\mu \lambda} \Gamma^\lambda_{\nu \sigma} - \Gamma^\rho_{\nu \lambda} \Gamma^\lambda_{\mu \sigma}) V^\sigma,
\]  

(4)
the last line due to the fact that we consider the Levi-Civita connection, which is torsion-free (see (2.25)). We define the Riemann curvature tensor

\[ R^\rho_{\sigma\mu\nu} := \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}. \]  

(5)

From here, we define the Ricci tensor

\[ R_{\mu\nu} := R^\lambda_{\mu\lambda\nu}, \]  

(6)

and the curvature scalar (or Ricci scalar)

\[ R := g^{\mu\nu} R_{\mu\nu}, \]  

(7)

the trace of the Ricci tensor. Then, since the scalar curvature and Ricci tensor both come directly from the Riemann curvature tensor, which only involves Christoffel symbols and derivatives thereof, and by (2.24), the Christoffel symbols themselves only depend on the metric, we see that the left hand side of (3) involves only the metric and first- and second-order derivatives of the metric (and constants, including \( \Lambda \)).
Bibliography


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