In optimization problems, we are trying to maximize (or minimize) one quantity, while satisfying some restraints on other quantities. We will use techniques for finding minima and maxima that we practiced in the previous section. The variety and complexity of optimization problems is a little overwhelming compared to other types of word problems we have considered thus far. You will see immediately that optimization has applications in many areas.

**Summary of the Method:**

1. Write an equation for the quantity to be maximized (minimized).
2. Write one or more equations representing constraints.
3. Solve the constraint equation for one variable in terms of the other (your choice).
4. Plug this into the function you want to optimize.
5. Examine the situation to decide what interval makes sense for the domain of the new function of one variable.
6. By the Extreme Value Theorem, the function attains a maximum (minimum) somewhere between the endpoints of the chosen interval. Use the first or second derivative test to find the max (min). Don’t forget to check the endpoints of the interval too, but only if they make sense in the problem! (e.g. a square’s side length ranges from 0 to 10. The actual side length could be 10, but can’t be 0.)
7. As always, make sure you give the answer the question asks for. You might have to do some more calculating (e.g. plugging length back into your original equation to find height).

**9.1. Area & Volume**

![Diagram of a building with a field]

**Example 9.1.** We need to enclose a field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so that side won’t need any fencing. Determine the dimensions of the field that will enclose the largest area.
Solution. (1) First write the equation you want to maximize. We are trying to maximize the area. Let’s call the length of the side parallel to the wall $x$, and the length of each of the other sides $y$. This is the equation we are trying to optimize:

$$A = xy$$

(2) Now write the constraint(s). In this problem, there is only a fixed amount of fencing. So the length of the three sides must add up to 500 feet total. This is the only constraint:

$$500 = x + 2y$$

(3) Let’s solve the constraint equation for $x$, because it is a little faster than solving for $y$. You can solve for whichever variable is more convenient.

$$x = 500 - 2y$$

(4) Then plug into the equation you are trying to optimize.

$$A = (500 - 2y)y$$

Now we have a function for the area in terms of $y$ alone:

$$A(y) = 500y - 2y^2$$

(5) What values make sense for $y$? Well, $y$ represents the area of two sides, so the maximum value of $y$ is 250 feet. It doesn’t make sense for $y$ to be zero, but we include it because $y$ can be arbitrarily small. So the interval we are considering is $[0, 500]$.

(6) To find the maximum of $A(y)$ on the interval $[0, 500]$, take the first derivative of the function:

$$A'(y) = 500 - 4y$$

Now set $A'(y) = 0$ and solve for $y$ to find the critical point(s):

$$500 - 4y = 0 \implies y = 125$$

This must be a maximum, since the area at either endpoint of the given interval is zero.

(7) The question is asking for the dimensions, so we need to plug

$$y = 125$$

back into the constraint equation to get the final answer.

$$x = 500 - 2(125) = 250$$

So the dimensions that give the largest area are $250 \times 125$.

Your Turn. (Stewart p 244, #38) A cone-shaped paper drinking cup is to be made to hold 27 $cm^3$ of water. Find the height and radius of the cup that will use the smallest amount of paper.
9.2. Cost functions

**Example 9.2.** We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $10/\text{ft}^2$ and the material used to build the sides cost $6/\text{ft}^2$. If the box must have a volume of $50/\text{ft}^3$ determine the dimensions that will minimize the cost to build the box.

**Solution.**

1. Let $\ell$, $w$, and $h$ represent the length, width, and height of the box respectively. The function to minimize is

$$C(\ell, w, h) = 6(2hw + 2h\ell) + 10(2\ell w) = 12hw + 12h\ell + 12\ell w$$

2. There are two constraint functions. The first dictates the volume of the box:

$$\ell wh = 50$$

The second relates the length and width of the box:

$$\ell = 3w$$

3. Let’s try to get the equation in terms of only $w$. Solve the first constraint function for $h$ in terms of $w$.

$$\ell wh = 50$$

$$3w^2h = 50$$

$$h = \frac{50}{3w^2}$$

4. Plug this into the function we want to optimize.

$$C = 12hw + 12h\ell + 12\ell w$$

$$= 12 \left( \frac{50}{3w^2} \right) w + 12 \left( \frac{50}{3w^2} \right) (3w) + 12(3w)w$$

$$= 60w^2 + \frac{800}{w}$$

5. What values make sense for this equation? Similar to the previous problem, none of the lengths can be zero, but they can be arbitrarily small.

6. Now we need to find the minimum of the function. Start by taking the first derivative and finding critical points. Recall that a critical point is either where $C''(w) = 0$ or $C'(w)$ does not exist.

$$C'(w) = 120w - \frac{800}{w^2}$$

Setting $120w - \frac{800}{w^2} = 0$, we see that $C'(w)$ does not exist at $w = 0$. However, we can throw out this point because we none of the lengths can be equal to zero. To
find the other critical point:

\[ 120w - \frac{800}{w^2} = 0 \]

\[ 120w^3 - 800 = 0 \]

\[ 120w^3 = 800 \]

\[ w = \sqrt[3]{\frac{20}{3}} \]

So \( w = \sqrt[3]{\frac{20}{3}} = 1.8821 \) is the only critical point. Is this the absolute minimum of the function for \( w > 0 \)? If so, the function will be concave up in this region. Check the second derivative:

\[ C''(w) = 120 + \frac{1600}{w^3} \]

Plug in any positive number for the second derivative test: \( C''(1) = 120 + 1600 = 1720 > 0 \). Therefore, the region \((0, \infty)\) is concave up, so \( w = 1.8821 \) must be the absolute minimum.

(7) Finally, we need to find the other two dimensions, \( h \), and \( \ell \).

\[ h = \frac{50}{3w^2} = \frac{50}{3(1.8821)^2} = 4.7050 \]

\[ \ell = 3w = 3(1.8821) = 5.6463 \]

**Your Turn.** (Swokowski p 210) A circular cylindrical metal container, open at the top, is to have a capacity of \( 24\pi \text{in}^3 \). The cost of the material used for the bottom of the container is 15 cents per square inch, and that of the material used for the curved part is 5 cents per square inch. If there is no waste of material, find the dimensions that will minimize the cost of the material.

### 9.3. Smallest distance

**Example 9.3.** Determine the point(s) on \( y = x^2 + 1 \) that are closest to \((0, 2)\).

**Solution.** (1) The function to minimize represents the distance from the point \((x, y)\) on the parabola to \((0, 2)\).

\[ D(x, y) = \sqrt{(x - 0)^2 + (y - 2)^2} \]

(2) The constraint function defines the relationship between \( x \) and \( y \), since we are only considering points on this particular parabola.

\[ y = x^2 + 1 \]
(3) The constraint function was not only given in the problem, but is already solved for $y$!

(4) Now it’s easy to plug this into the function we want to optimize:

$$D(x) = \sqrt{x^2 + ((x^2 + 1) - 2)^2} = \sqrt{x^2 + (x^2 - 1)^2} = \sqrt{x^4 - x^2 + 1}$$

(5) Unlike in the other examples, there are no restrictions on the possible values for $x$ and $y$.

(6) Let’s use the first derivative test to find the minimum of the function. There is a little trick to make taking the derivative of the square root easier. Since the minimal distance will also be the minimal square of the distance, we can take the derivative of the square of the function instead:

$$D^2 = x^4 - x^2 + 1 \implies (D^2)' = 4x^3 - 2x$$

Find the critical values:

$$4x^3 - 2x = 0$$
$$x(4x^2 - 2) = 0$$

$$\implies x = 0 \text{ or } 4x^2 - 2 = 0$$
$$\implies x = 0 \text{ or } x = \pm \frac{1}{\sqrt{2}}$$

Use the second derivative test to decide which points are relative minima (there could be more than one)

$$(D^2)'' = 12x^2 - 2$$

$$(D^2)''(0) = -2 < 0 \quad (D^2)''\left(\frac{1}{\sqrt{2}}\right) = 4 > 0 \quad (D^2)''\left(-\frac{1}{\sqrt{2}}\right) = 4 > 0$$

Therefore, $x = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ are both relative minima, which makes sense since the function is even.

(7) Finally, find the $y$ value for these points. Since the function is even, we only need to calculate once for both points.

$$y = \left(\frac{1}{\sqrt{2}}\right)^2 + 1 = \frac{3}{2}$$

So $\left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right)$ are the two points that are closest to $(0, 2)$.

**Your Turn.** (Swokowski p. 213) A North-South highway intersects and East-West highway at a point $P$. An automobile crosses $P$ at 10:00 a.m. traveling east at a constant speed of 20 mi/hr. At that same instant another automobile is 2 miles north of $P$, traveling south at 50 mi/hr. Find the time at which they are closest to each other and approximate the minimum distance between the two automobiles.
9.4. Inscribed & Circumscribed shapes

Example 9.4. (Swokowski p. 211) Find the maximum volume of a right circular cylinder that can be inscribed in a cone of altitude 12 cm, and base radius 4 cm, if the axes of the cylinder and cone coincide.

Solution. (1) Let \( h \) be the height of the cylinder. Let \( r \) be the radius of the base of the cylinder. The function to maximize is

\[
V(r, h) = \pi r^2 h
\]

(2) The constraint function gives us the relationship between \( r \) and \( h \). This is a little tricky because we need to use the properties of similar triangles. This diagram represents one half of the vertical cross-section of the cone through its axis.

![Diagram of cone and cylinder](image)

Because the small triangle with base \( r \) is similar to the large triangle with base 2, we get the following equation:

\[
\frac{2}{12} = \frac{r}{12 - h}
\]

(3) Let’s solve the constraint function for \( h \):

\[
\frac{2}{12} = \frac{r}{12 - h}
\]

\[
\frac{1}{6} = \frac{r}{12 - h}
\]

now cross-multiply

\[
12 - h = 6r
\]

\[
h = 12 - 6r
\]

(4) Plug this into the function we want to optimize.

\[
V(r) = \pi r^2 (12 - 6r) = \pi (12r^2 - 6r^3)
\]

(5) As usual for volume and area problems, all the quantities can be arbitrarily close to zero, but not equal to zero.

(6) Take the derivative of the function and find critical points.

\[
V'(r) = \pi (24r - 18r^2)
\]
\[ \pi(24r - 18r^2) = 0 \]
\[ r(24 - 18r) = 0 \]
\[ \implies r = 0 \text{ or } 24 - 18r = 0 \]
\[ \implies r = 0 \text{ or } r = \frac{4}{3} \]

Throw out the point \( r = 0 \), leaving us with \( r = \frac{4}{3} \) as the only critical point. Use the second derivative test to confirm that it is a maximum.

\[ V''(r) = \pi(24 - 26r) \]
\[ V'' \left( \frac{4}{3} \right) = -\frac{32}{3} < 0 \]

(7) Finally, we need to plug \( r \) back into the constraint equation to find \( h \) and then into the original equation to find the maximum volume.

\[ h = 12 - 6 \left( \frac{4}{3} \right) = 8 \]

\[ V = \pi \left( \frac{4}{3} \right)^2 (8) = \frac{128}{9} \pi = 14.2 \]

**Your Turn.** Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.