Week 6.

Implicit Differentiation

Let’s say we want to differentiate the equation of a circle:

\[ y^2 + x^2 = 9 \]

Using the techniques we know so far, we need to write the equation as a function of one variable first. If we try to solve for \( y \) in terms of \( x \) (or \( x \) in terms of \( y \)), we end up having to split the equation into two parts, the upper and lower semi-circles:

\[ y(x) = \pm \sqrt{9 - x^2} \]

A better way to do this problem is using what is called “Implicit Differentiation,” which allows us to get information about the derivative of a function without solving the equation. Let’s look at an example.

**Example 6.1.** Find \( y' \) for \( 9x^2 + 4y^2 = 36 \).

**Solution.** There are three steps to follow in the solution:

1) First, write the variable \( y \) as a function of \( x \). That is, substitute \( y = y(x) \);

2) Differentiate both sides of the equation using the Chain Rule;

3) Solve the resulting equation for \( y'(x) \).

Step two is the trickiest. You apply the chain rule every time you see \( y(x) \) in the equation. For example, if we want to differentiate \( \{y(x)\}^2 \), the “outside” function is \( x^2 \), with derivative \( 2x \), and the “inside” function is \( y(x) \), with derivative \( y'(x) \). So, we have:

\[ (\{y(x)\}^2)' = 2(y(x)) \cdot y'(x) \]

Now let’s see how the steps are applied.

1) Substitute \( y = y(x) \)

\[ 9x^2 + 4\{y(x)\}^2 = 36 \]

2) Differentiate using chain rule. Differentiate the variable \( x \) as we would normally, and use the chain rule when you see a \( y(x) \).

\[ 18x + 4 \cdot 2y(x)y'(x) = 0, \]
3) Solve the resulting equation for $y'(x)$
\[
18x + 4 \cdot 2y(x)y'(x) = 0
\]
\[
8y(x)y'(x) = -18x
\]
\[
y'(x) = -\frac{18x}{8y(x)}
\]
\[
y'(x) = -\frac{9x}{4y(x)}.
\]

**Your Turn.** * Find $y'$ for $x^2y = 1 + y^2x$.

**Example 6.2.** * Find equation of the tangent line to the curve given by
\[
x^2y^2 + 4xy = 12y
\]
at point $(2, 1)$.

**Solution.** First, verify that the point is on the graph. Substitute $x = 2$, $y = 1$ into the equation and make sure the left side equals the right side:
\[
2^2 \cdot 1^2 + 4 \cdot 2 \cdot 1 = 12 \cdot 1
\]
Next, evaluate $y'(x)$ using implicit differentiation, following the three step process outlined above.

(1) Substitute $y = y(x)$:
\[
x^2(y(x))^2 + 4xy(x) = 12y(x)
\]
(2) Differentiate using the chain rule:
\[
2xy^2 + x^22yy' + 4y + 4xy' = 12y',
\]
(3) Solve for $y'$:
\[
2xy^2 + x^22yy' + 4y + 4xy' = 12y'
\]
\[
x^22yy' + 4xy' - 12y' = -4y - 2xy^2
\]
\[
y'(x^22y + 4x - 12) = -4y - 2xy^2
\]
\[
y' = \frac{-4y - 2xy^2}{x^22y + 4x - 12}
\]
Now, we can substitute $x = 2$, $y = 1$ to find the derivative at $(2, 1)$.
\[
y'(2) = \frac{-4 \cdot 1 - 2 \cdot 2 \cdot 1^2}{2^2 \cdot 2 \cdot 1 + 4 \cdot 2 - 12} = \frac{-8}{8 + 8 - 12} = -2
\]
Finally, we write the equation of the tangent line. Let’s use point-slope form.
\[
y - y_0 = y'(x_0)(x - x_0),
\]
\[
y - 1 = (-2)(x - 1),
\]
\[
y = -2x + 3.
\]
Your Turn. * Find the equation of the tangent line to the curve given by

\[ \sqrt{y} + xy^2 = 5 \]

at point (4, 1).

6.1. Related rates

Let \( y \) and \( t \) be two variables. Let’s say that we want to calculate \( \frac{dy}{dt} \), the rate of change of variable \( y \) with respect to \( t \). Now suppose that the relationship between \( y \) and \( t \) is either not known or hard to find. Instead, the rates \( \frac{dt}{dx} \) and \( \frac{dy}{dx} \) are given, where \( x \) is some third variable. Is it possible to determine \( \frac{dy}{dt} \) through the related rates \( \frac{dy}{dx} \) and \( \frac{dt}{dx} \)? Yes!

Example 6.3.* If you blow air into a bubble at a rate of 3 cubic inches per second, how fast is the bubble’s radius increasing when its radius is 3 inches? (Assume that a soap bubble retains its spherical shape as it expands).

There are many variations on this problem that are solved in the same way. The radius may be decreasing or increasing, and the object could be a spherical snowball, balloon, bubble, or water droplet.

Solution. First, recall the equation for the volume of a sphere:

\[ V = \frac{4}{3} \pi r^3 \]

Next, differentiate this formula implicitly with respect to \( t \). Both \( V \) and \( r \) are functions of \( t \). We can substitute \( V(t) \) and \( r(t) \) to make this easier:

\[ V(t) = \frac{4}{3} \pi [r(t)]^3 \]

\[ V'(t) = 4\pi [r(t)]^2 r'(t) \]

Especially in the context of these types of problems, the derivative is often written as:

\[ \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \]

The change in volume with respect to change in time is given in the problem: \( \frac{dv}{dt} = 3\text{in}^3 \). Now, all we need to do is substitute this value into the equation. We also have the condition that \( r = 3 \).

\[ 3 = 4\pi(3)^2 \frac{dr}{dt} \],
Finally, solve for \( \frac{dr}{dt} \), which represents change in the radius over change in time, or in other words, the instantaneous rate of change of \( r \) at time \( t \).

\[
\frac{dr}{dt} = \frac{1}{12\pi}.
\]

The next problem is the same mathematically as a problem written about a ladder sliding down or being pushed up a wall.

**Your Turn.** * A woman is standing on a raised dock, pulling in a boat which is tied to a rope. If the dock is raised 10 feet above the surface of the water, and she is pulling at the rate of 2 feet/sec, how fast is the boat approaching the dock, when 25 feet of the rope is still out?

Here are some steps to help you solve a problem. They are very general, and you still need to have some insight into what the problem is asking. Once you get the hang of implicit differentiation, you’ll realize that understanding the problem and writing the equation is the hardest part!

1. Draw a picture. Label the quantities, and convert any units. Decide what you are trying to find.
2. Write an equation (or two) relating the quantities.
3. Differentiate implicitly. (Don’t forget to use the chain rule when necessary)
4. Plug in relevant values given in the problem. Remember these are “related rates” problems, so you need to plug in one rate and one quantity, or two different rates.

Here is another example that requires more challenging geometric thinking, including properties of similar triangles. Other problems which may require similar triangles are conical and pyramidal tanks, triangular troughs. The troughs may be leaking or filling up, and sometimes both! Another common problem requiring the use of similar triangles is about the shadow of a person walking away from a street lamp (see below).

**Example 6.4.** Water is pumped at a uniform rate of 2 liters per minute into a tank shaped like a section of a right circular cone. The tank has altitude 80 centimeters and lower and upper radii of 20 and 40 centimeters, respectively. How fast is the water level rising when the depth of the water is 30 centimeters? (Note: The volume \( v \) of a frustum of a right circular cone of altitude \( h \) and lower and upper radii of \( a \) and \( b \) is \( v = \frac{1}{3}\pi h \cdot (a^2 + ab + b^2) \)).

**Solution.** First we draw a picture.

Next, letting \( h \) be the height of the water in the tank. We want to find \( \frac{dh}{dt} \) when \( h = 30\,\text{cm} \), given that \( \frac{dV}{dt} = 2 \) liters per minute = \( 2000\,\text{cm}^3 \) per minute. We are given in the problem the equation we need. We need to plug in \( a = 20 \), but for \( b \) we want to plug in the upper radius in terms of the height of the water in the tank. For this we need similar triangles. Denote the difference in the the base radius and the radius at height \( h \) by \( x \). Then

\[
\frac{h}{x} = \frac{80}{20}.
\]
So \( x = \frac{1}{4} h \) and \( b = 20 + x = 20 + \frac{1}{4} h \). Now we can plug this into the volume formula:

\[
v = \frac{1}{3} \pi h \cdot \left( 20^2 + 20 \cdot \left( 20 + \frac{1}{4} h \right) + \left( 20 + \frac{1}{4} h \right)^2 \right) = \frac{1}{3} \pi \left( 1200h + 10h^2 + \frac{1}{8} h^3 \right).
\]

That takes care of steps 1 and 2. Now step three is to differentiate implicitly. We’ll make this easier by writing \( v(t) \) for \( v \) and \( h(t) \) for \( h \).

\[
v(t) = \frac{1}{3} \pi \left( 1200h(t) + 10h^2(t) + \frac{1}{8} h^3(t) \right).
\]

Differentiating, we get:

\[
\frac{dv}{dt} = \frac{\pi}{3} \left( 1200 + 20h + \frac{3}{8} h^2 \right) \frac{dh}{dt}.
\]

Now we can solve for \( \frac{dh}{dt} \), which is what we are trying to find:

\[
\frac{dh}{dt} = \frac{3}{\pi} \frac{dv}{dt} \left( 1200 + 20h + \frac{3}{8} h^2 \right).
\]

Then, substituting \( \frac{dv}{dt} = 2000 \) and \( h = 30 \), the given in the problem, we get the final answer:

\[
\frac{dh}{dt} = \frac{3}{\pi} \frac{2000}{\left( 1200 + 20 \cdot 30 + \frac{3}{8} \cdot (30)^2 \right)} = \frac{3}{\pi} \frac{2000}{1200 + 600 + \frac{675}{2}} = \frac{6000}{2137.5\pi} \approx 2.807
\]

Your Turn. †

1. Two people are 50 feet apart. One of them starts walking north at a rate so that the angle shown in the diagram below is changing at a constant rate of 0.01 rad/min. At what rate is distance between the two people changing when radians? (This problem is also sometimes written about particle motion, or reeling in a fish).

2. A light is on the top of a 12 ft tall pole and a 5ft 6in tall person is walking away from the pole at a rate of 2 ft/sec. At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole? (Use similar triangles)