11.1. The Fundamental Theorem of Calculus

Theorem 11.1.

(1) If $f$ is continuous on $[a, b]$, then the function $g(x)$ defined by

$$g(x) = \int_{a}^{x} f(t) \, dt \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on $(a, b)$ and $g'(x) = f(x)$.

(2) If $f$ is continuous on $[a, b]$, then

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$, that is, a function such that $F' = f$.

Example 11.1. Evaluate:

1. $\frac{d}{dx} \int_{1}^{x} \frac{1}{t} \, dt$
2. $\frac{d}{dx} \int_{1}^{x} \sec t \, dt$

Solution. \(\dagger\)

(1) Use the first part of the fundamental theorem, where $a = 1$ and $f(x) = \frac{1}{x}$. Choose $b$ such that $1 \leq b$. Then $f$ is continuous on $[1, b]$, so for every $x \in [1, b]$, we have

$$\frac{d}{dx} \int_{1}^{x} \frac{1}{t} \, dt = \frac{1}{x}$$
(2) Be careful to use chain rule together with the fundamental theorem. Let $u = x^4$
Then $du = 4x^3 \, dx$

$$
\frac{d}{dx} \int_1^{x^4} \sec t \, dt = \frac{d}{dx} \int_1^{u} \sec t \, dt \\
= \frac{d}{dx} \left( \int_1^{u} \sec t \, dt \right) \frac{du}{dx} \\
= \sec \frac{du}{dx} \\
= \sec(x^4) \cdot 4x^3
$$

Your Turn. Evaluate: $\frac{d}{dx} \int_1^{3x} \frac{\sin(3t)}{t} \, dt$

Example 11.2. Evaluate: $\int_1^2 \frac{5}{x^6} \, dx$

Solution. Now we use the second part of the fundamental theorem of calculus. It gives us an easy way to calculate definite integrals.

$$
\int_1^2 \frac{5}{x^6} \, dx = \int_1^2 5x^{-6} \, dx \\
= [-x^{-5}]_1^2 \text{ Take the antiderivative } \\
= (-2^{-5}) - (-1^{-5}) \text{ Evaluate at the end points, subtract } \\
= -\frac{1}{32} + 1 = \frac{31}{32}
$$

Your Turn. Evaluate: $\int_0^1 (x - \sin x) \, dx$

11.2. The Substitution Rule

So far we are very limited in the kinds of anti-derivatives we can calculate. Let’s learn another technique.

Theorem 11.2.

$$
\int f(g(x))g'(x) \, dx = \int f(u) \, du \quad \text{where} \quad u = g(x)
$$

This is rather confusing on its own. An example will help. The substitution rule is used when there is a function “inside” a function (it is something like a “reverse” chain rule).

Example 11.3. $\int x \cos(x^2 + 1) \, dx$
Solution. Since there is more than just a simple $x$ inside of the cosine function, we know we need to use the substitution rule. It’s alright that there is an extra $x$ hanging around outside of the $\cos(x^2 + 1)$ part. Actually, this will be important in a moment. We know how to substitute cosine on its own, so we should choose the substitution

$$u = x^2 + 1$$

Once you know what substitution you are using, take the derivative, and move any coefficients to the other side by multiplying by the reciprocal:

$$du = 2x \, dx$$
$$\frac{1}{2} \, du = x \, dx$$

Finally we can substitute, using both this equation and the original equation for substitution, into the original integral. I’ll do this in two steps so you can see where I substitute each part. You can and should write this as one step, since you can only have one variable in the integrand at a time.

$$\int x \cos(x^2 + 1) \, dx = \int x \cos(u) \, dx = \int \cos(u)(x \, dx) = \int \cos(u) \frac{1}{2} \, du = \frac{1}{2} \int \cos(u) \, du$$

Remember from the previous lecture that you can pull coefficients outside of the integral. It’s a good idea to do this most of the time. It makes calculations easier, because now you can concentrate on finding the $\int \cos(u) \, du$, and then you will multiply your answer at the end by $\frac{1}{2}$. Using the anti-derivative rules we learned in the previous chapter, we get:

$$\frac{1}{2} \int \cos(u) \, du = \frac{1}{2} [\sin u + c] = \frac{1}{2} \sin u + \frac{1}{2} c$$

Since the constant $c$ is arbitrary anyway, multiplying it by a half makes no difference. Let’s replace it by $k$ instead. Finally, we can plug $u = x^2 + 1$ back in to get our complete answer:

$$\frac{1}{2} \sin u + k = \frac{1}{2} \sin(x^2 + 1) + k$$

There is no need to multiply this out. Most of the time it’s actually better to leave it this way so your professor can see that you used the method correctly.

Example 11.4. Evaluate:

$$\int \sin(t) \left( 4 \cos^3(t) + 6 \cos^2(t) - 8 \right) \, dt$$

Solution. Be careful of this notation: $\cos^3 t = (\cos t)^3$. The best substitution to use here is $u = \cos t$ since then the expression in parentheses becomes a polynomial in $u$.

$$u = \cos t \implies du = -\sin t \, dt \implies -du = \sin t \, dt$$
Now plug these into the equation:
\[
\int \sin(t) \left( 4 \cos^3(t) + 6 \cos^2(t) - 8 \right) \, dt = \int (4 \cos^3(t) + 6 \cos^2(t) - 8) \, (\sin t \, dt)
\]
\[
= \int (4u^3 + 6u^2 - 8) \, (- \, du)
\]
\[
= - \int (4u^3 + 6u^2 - 8) \, du
\]
\[
= - \left[ u^4 + 2u^3 - 8u \right]
\]

Finally, substitute back in \( u = \cos t \):
\[
\int \sin(t) \left( 4 \cos^3(t) + 6 \cos^2(t) - 8 \right) \, dt = - \left[ \cos^4 t + 2 \cos^3 t + 8 \cos t \right]
\]

**Your Turn.** Find the following indefinite integrals using substitution.

1. \( \int \sin 2x \, dx \)
2. \( \int x \sin x^2 \, dx \)
3. \( \int \sin^2 x \cos x \, dx \)
4. \( \int \frac{\cos(\sqrt{x})}{\sqrt{x}} \, dx \)
5. \( \int x^2(1 - 5x^3)^4 \, dx \) (You could have calculated this without the substitution rule, but you would have needed to do polynomial multiplication four times!)

**Example 11.5.** Find the following indefinite integral using substitution:
\[
\int x^4 + e^{1-x} \, dx
\]

**Solution.** A common error here is to try to use the substitution \( u = 1 - x \) on the entire integral. Instead, split this up first:
\[
\int x^4 \, dx + \int e^{1-x} \, dx
\]
Remember that the substitution rule is needed to deal with composition of functions. Since \( x^4 \) is not part of the function composition (nor is it multiplied with the composition), then it needs to be dealt with separately.

Use the substitution \( u = 1 - x \implies du = - \, dx \). We get:
\[
\int x^4 \, dx + \int e^{1-x} \, dx = \frac{x^5}{5} + c_1 + \left( - \int e^u \, du \right) = \frac{x^5}{5} + c_1 - e^u + c_2 = \frac{x^5}{5} - e^{1-x} + c
\]
Where \( c = c_1 + c_2 \). Like before where we combined a coefficient together with an arbitrary constant, since \( c_1 \) and \( c_2 \) and both arbitrary, we can combine them.
Your Turn. Integrate:

\[ \int \frac{1}{\sqrt{x}} + \frac{\sqrt{x} - 1}{dx} \]

Example 11.6. Evaluate:

\[ \int_{-2}^{0} 2t^2\sqrt{1 - 4t^3} \, dt \]

Solution. This problem is different from all the previous ones because it involves a definite integral. Let \( u = 1 - 4t^3 \). Then \( du = -12t^2 \, dt \) \( \implies -\frac{1}{12} \, du = t^2 \, dt \). Substitute as usual into the original integral. But because we changed the variable of integration, we can’t use the same limits of integration.

\[ \int_{?}^{?} -\frac{1}{6} \sqrt{u} \, du \]

There are two ways of going about this. The first is to do the indefinite integral, then substitute back in for \( t \), and use the original limits of integration:

\[
\int \frac{-1}{6} \sqrt{u} \, du = \frac{-1}{6} \cdot \frac{2}{3} u^{\frac{3}{2}} \\
= \frac{-1}{9} \left[ (1 - 4t^3)^{\frac{3}{2}} \right]_{-2}^{0} \\
= \frac{-1}{9} \left[ (1 - 4(0)^3)^{\frac{3}{2}} - (1 - 4(-2)^3)^{\frac{3}{2}} \right]
\]

The other option is to change the limits of integration when changing variables:

\[
\int_{t=-2}^{t=0} \frac{-1}{6} \sqrt{u} \, du = \int_{u=1-4(-2)^3}^{u=1-4(0)^3} \frac{-1}{6} \sqrt{u} \, du \\
= \frac{-2}{3} \left[ u^{\frac{3}{2}} \right]_{33}^{1} \\
= \frac{-2}{3} \left[ \frac{1}{33} - \frac{1}{3} \right]
\]

Neither way is necessarily shorter than the other, and they always give the same answer. It’s good to know both methods, so you can have a way to double check your work. After simplifying, the answer is:

\[ \int_{-2}^{0} 2t^2\sqrt{1 - 4t^3} \, dt = \frac{1}{9}(33\sqrt{33} - 1) \]

Your Turn. Evaluate:

\[ \int_{-\pi}^{\pi/2} \cos x \cos(\sin(x)) \]