Week 12.

Applications of Integration

12.1. Areas Between Curves

Example 12.1. Determine the area of the region enclosed by \( y = x^2 \) and \( y = \sqrt{x} \).

Solution. First you need to find the points where the two functions intersect. You do this by setting \( x^2 = \sqrt{x} \) and solving for \( x \):

\[
\begin{align*}
  x^2 &= \sqrt{x} \\
  x^4 &= x \\
  x^4 - x &= 0 \\
  x(x^3 - 1) &= 0 \\
  \implies x &= 0 \text{ or } x^3 - 1 = 0 \\
  \implies x &= 0 \text{ or } x = 1
\end{align*}
\]

So \( x = 0 \) and \( x = 1 \) are the limits of integration. The distance between any point on the blue and the red curves is represented by \( \sqrt{x} - x^2 \). So the integral we need is:

\[
\int_0^1 \sqrt{x} - x^2 \, dx = \left( \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{3} x^3 \right) \bigg|_0^1 = \frac{1}{3}
\]

Your Turn. Determine the area of the region bounded by \( y = 2x^2 + 10 \) and \( y = 4x + 16 \).
Example 12.2. Determine the area of the region bounded by \( y = 2x^2 + 10, \ y = 4x + 16, \ x = -2, \) and \( x = 5. \)

Solution. There are a few things about this problem that are different. First, the limits of integration are determined in the problem. The lower limit is the vertical line \( x = -2 \) and the upper limit is \( x = 5. \) Secondly, as you can see from the picture, the upper and lower functions switch for each green region. So we need to split up the integral into three parts. The first will be red minus blue, the next blue minus red, then finally red minus blue again. To decide where we need to break up the integral, we need to find the points where the two functions intersect, as in the previous example.

\[
2x^2 + 10 = 4x + 16
\]
\[
2x^2 - 4x - 6 = 0
\]
\[
(2x + 2)(x - 3) = 0
\]
\[
\Rightarrow \ x = -1 \text{ or } x = 3
\]

So our integrals start out like this:

\[
\int_{-2}^{-1} ? \, dx + \int_{-1}^{3} ? \, dx + \int_{3}^{5} ? \, dx
\]

Now you can think of this as three separate instances of the first example problem. So for each integral, fill in the upper function minus the lower function.

\[
A = \int_{-2}^{-1} (2x^2 + 10) - (4x + 16) \, dx + \int_{-1}^{3} (4x + 16) - (2x^2 + 10) \, dx + \int_{3}^{5} (2x^2 + 10) - (4x + 16) \, dx
\]
\[
= \int_{-2}^{-1} 2x^2 - 4x - 6 \, dx + \int_{-1}^{3} -2x^2 + 4x + 6 \, dx + \int_{3}^{5} 2x^2 - 4x - 6 \, dx
\]
\[
= \left( \frac{2}{3}x^3 - 2x^2 - 6x \right) \bigg|_{-2}^{-1} + \left( -\frac{2}{3}x^3 + 2x^2 + 6x \right) \bigg|_{-1}^{3} + \left( \frac{2}{3}x^3 - 2x^2 - 6x \right) \bigg|_{3}^{5}
\]
\[
= \frac{14}{3} + \frac{64}{3} + \frac{64}{3} = \frac{142}{3}
\]
Your Turn. Determine the area of the region enclosed by \( y = \sin x, \ y = \cos x, \ x = \frac{\pi}{2}, \) and the \( y \)-axis.

12.2. Volumes

Example 12.3. Determine the volume of the solid obtained by rotating the region bounded by \( y = x^2 - 4x + 5, \ x = 1, \ x = 4, \) and the \( x \)-axis about the \( x \)-axis.

Solution. These problems require a little experience and imagination to visualize. Here is what the figure looks like when you rotate it about the \( x \)-axis, and what a vertical cross section looks like.

We’re going to use the disk method to solve this problem. That means we are going to slice the solid into disks, as in the final picture above, to calculate the volume. Let’s set up the integral. We are given in the problem that the limits of integration are \( x = 1 \) and \( x = 4 \). Inside the integral, we need to put the area of one disk or vertical cross section of the solid. To find the area of a disk (i.e. circle), we only need to know it’s radius. In this problem, the radius is determined by \( y = x^2 + 4x + 5 \). Therefore, the area of each disk is \( \pi (x^2 + 4x + 5)^2 = \pi (x^4 + 8x^3 + 26x^2 + 40x + 25) \). Putting this all together we have:

\[
\int_1^4 \pi (x^4 + 8x^3 + 26x^2 + 40x + 25) \, dx = \pi \left( \frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x \right) \bigg|_1^4 = \frac{78\pi}{5}
\]

Your Turn. Determine the volume of the solid obtained by rotating the portion of the region bounded by \( y = \sqrt{x} \) and \( y = \frac{x}{4} \) that lies in the first quadrant about the \( y \)-axis.
Hint. In this example, your disk has a hole!

This is sometimes called a washer. So the area of the washer, the function you need to integrate, should look like this:

\[ A = \pi \left( (\text{outer radius})^2 - (\text{inner radius})^2 \right) \]

Also notice that the washer is parallel to the \( x \)-axis. The means that the inner and outer radius of the washer will be \( x \)-values, so you need to rewrite the equations in the form \( x = f(y) \) and integrate over a vertical interval.

**Example 12.4.** Determine the volume of the solid obtained by rotating the region bounded by \( y = \sqrt[3]{x} \), \( x = 8 \), and the \( x \)-axis about the \( x \)-axis.

**Solution.** We could use the disk method to solve this problem. However, there is another method of solving volume problems using what we call cylindrical shells.
Since the cylinders are encircling the $x$-axis, the area will be a function of $y$. This means we need to write our equation in $x = f(y)$ form. So $y = \sqrt[3]{x}$ becomes $x = y^3$ by solving for $x$ in terms of $y$. Now, we need to find the surface area of the cylinder, which is given by $2\pi rh$. The radius is the $y$ value and the width is determined by $8 - y^3$. We need to integrate from 0 to 2 since the narrowest cylinder has radius $x = 0$ and the widest one has radius 2. So our integral is:

$$V = \int_0^2 2\pi(y)(8 - y^3) \, dy$$

$$= \int_0^2 2\pi 8y - y^4 \, dy$$

$$= 2\pi \left( 4y^2 - \frac{1}{5}y^5 \right) \bigg|_0^2$$

$$= \frac{96\pi}{5}$$

**Your Turn.** Determine the volume of the solid obtained by rotating the region bounded by $y = (x - 1)(x - 3)^2$ and the $x$-axis about the $y$-axis. Try both the washer method and the cylindrical shell method.

**Hint.**