Week 10.

Integration

10.1. Anti-Derivatives

Definition 10.1. Given a function, \( f(x) \), an anti-derivative of \( f(x) \) is any function \( F(x) \) such that

\[ F'(x) = f(x) \]

Example 10.1. What function did we differentiate to get the function \( f(x) = -\cos x \)?

Solution. It takes a little while to get used to doing these. You need to use the rules for differentiation backwards. Recall that the derivative of \( \sin x \) is \( \cos x \). How can you get the negative? By the multiplication rule, coefficients “carry through” when you differentiate. So the derivative of \( -\sin x \) is \( -\cos x \) which is what we wanted.

Example 10.2. What function did we differentiate to get the function \( f(x) = 12x^3 + x^2 \)?

Solution. Remember the differentiation rule for powers and constants looks like this:

\[ \frac{d}{dx}(ax^n) = anx^{n-1}. \]

We need to find \( a \) and \( n \) such that \((an)x^{n-1} = 12x^3\). This gives us two easy-to-solve equations: \( n - 1 = 3 \implies n = 4 \), and \( 4a = 12 \implies a = 3 \). Plugging this back into our formula gives us: \( \frac{d}{dx}(3x^4) = 12x^3 \). In other words, the function we differentiated to get \( 12x^3 \) is \( 3x^4 \).

Although the work we did is perfectly correct, you will find that guess-and-check often faster. Here’s the intuitive way of finding of solving these problems that, with practice, you will likely prefer. This time we’ll take the second term: \( x^2 \). Recall that when you take a derivative, you “lose” one power in the exponent. This tells use that we are looking for a function with a power of three in it, if its derivative has a power of two. Let’s check. \( \frac{d}{dx}(x^3) = 3x^2 \). Now we ask ourselves, what number do I need to multiply by to get the answer I want, which is \( x^2 \) with a coefficient of 1? The answer is \( \frac{1}{3} \). So I think that my antiderivative should be \( \frac{1}{3}x^3 \). Double checking, we have: \( \frac{d}{dx} \left( \frac{1}{3}x^3 \right) = 3 \cdot \frac{1}{3}x^2 = x^2 \), which is what we wanted.

The final answer is \( F(x) = 3x^4 + \frac{1}{3}x^3 \), because \( F'(x) = 12x^3 + x^2 \).
Your Turn. What functions can be differentiated to get the following functions?

(1) \( f(x) = x^5 - 2x^3 + 5x^2 \)
(2) \( f(x) = \sin x \)
(3) \( f(x) = 3x^4 + x^{-3} \)
(4) \( f(x) = \sqrt{x^4} \)

10.2. The Indefinite Integral

Definition 10.2. Given a function, \( f(x) \), an anti-derivative of \( f(x) \) is any function \( F(x) \) such that

\[ F'(x) = f(x) \]

All of the solutions we’ve found so far in this section are examples of anti-derivatives. But consider the following two functions:

\[ f(x) = x^2 + 3 \quad \text{and} \quad g(x) = x^2 + 10 \]
\[ f'(x) = 2x \quad \text{and} \quad g'(x) = 2x \]

This means that if I ask “what function can I differentiate to get \( 2x \)” (i.e. what is the anti-derivative of \( 2x \)), then there is not a unique answer. You could say \( x^2 + 1000 \) or \( x^2 + \pi \) and still be correct. So we make a more general definition.

Definition 10.3. If \( F(x) \) is any anti-derivative of \( f(x) \) then the most general anti-derivative of \( f(x) \) is called an indefinite integral and denoted,

\[ \int f(x) \, dx = F(x) + c \]

In this definition the \( \int \) is called the integral symbol, \( f(x) \) is called the integrand, \( x \) is called the integration variable and \( c \) is called the constant of integration.

The method for finding an indefinite integral is exactly the same as finding an anti-derivative, except that we add \( c \) at the end of every answer. Here are some examples:

Example 10.3. Find \( \int 12x^3 + x^2 \, dx \).

Solution. We already found that one of the anti-derivatives of \( 12x^3 + x^2 \) is \( 3x^4 + \frac{1}{3}x^3 \).

All we need to do is add \( c \):

\[ \int 12x^3 + x^2 \, dx = 3x^4 + \frac{1}{3}x^3 + c \]

We can always check by differentiating our answer. When we do, the constant \( c \) always disappears.

Here are some rules for indefinite integrals that you need to know.

Theorem 10.1.

(1) \( \int kf(x) \, dx = k \int f(x) \, dx \).
(2) \( \int -f(x)\,dx = -\int f(x)\,dx \). (This is the same as the first property where \( k = -1 \)).

(3) \( \int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx \)

**Your Turn.** Evaluate the following indefinite integrals:

1. \( \int w^4 - 9\,dw \)
2. \( \int \,dx \)
3. \( \int \sin x\,dx \)
4. \( \int \sqrt{t} + \frac{3}{t^2}\,dt \)

Did you notice that there is no rule dealing with products or quotients? It is very important to remember that the integral of a product is NOT the product of two integrals. In symbols:

\[
\int f(x)g(x)\,dx \neq \left( \int f(x)\,dx \right) \left( \int g(x)\,dx \right)
\]

**Example 10.4.** Evaluate \( \int (x + 1)(x^2 - 5)\,dx \)

**Solution.** How can we do this without a product rule? By multiplying everything out first (FOIL in this case).

\[
\int (x + 1)(x^2 - 5)\,dx = \int x^3 + x^2 - 5x - 5\,dx
\]

\[
= \int x^3\,dx + \int x^2\,dx - \int 5x\,dx - \int 5\,dx
\]

\[
= \frac{x^4}{4} + \frac{x^3}{3} - \frac{5x^2}{2} - 5x + c
\]

**Your Turn.** Evaluate \( \int \frac{x^5 + 3x^2 + x}{x^2}\,dx \) [Hint: We don’t have a quotient rule for integrals either! Split this up into three smaller fractions first.]

### 10.3 Riemann Sums

**Notation.** The symbol \( \sum \) represents a sum. When we have an indexed bunch of things, say \( x_1, x_2, x_3, \ldots, x_n \), if we want to add them all together, we use this symbol. The sum is of all the \( x \)'s from \( x_1 \) through \( x_n \) is represented like this:

\[
\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_{n-1} + x_n
\]

**Definition 10.4.** If \( f(x) \) is a function that is continuous on the interval \([a, b]\) (with the exception of finitely many jump discontinuities), then we divide the interval into \( n \) subintervals of equal width, \( \Delta x \), and choose a point \( x^*_n \) from each interval. Let \( A \) be the area under the curve. Then the sum

\[
A = \sum_{i=1}^{n} f(x^*_n)\Delta x
\]
is called a Riemann sum, and it can be used to approximate the area under a curve to any desired degree of accuracy, by using large enough $n$.

This is a lot of information to parse all at once, but this figure shows intuitively what the definition describes mathematically. In the picture, rectangles of a fixed width are used to estimate the area under the curve. As the width of the bars gets smaller, the estimate for the area gets closer and closer to the actual area under the curve.

Here’s what you need to know from the definition:

- $n =$ number of subintervals of equal width, i.e. number of rectangles.
- $\Delta x =$ the width of each rectangle.
- $x^*_{n} =$ a point chosen from the $n^{th}$ subinterval, that is used as the height of the rectangle.
- $A =$ the estimated area under the curve $f(x)$, i.e. the total area of all the rectangles.

Let’s get some practice with these ideas and symbols before moving on.

**Example 10.5.** Let $f(x) = x^3$.

1. We know that $x^3$ is continuous on the interval $[0, 2]$. If $n = 8$, what is $\Delta x$?
2. Consider the interval $[3, 4]$. If $\Delta x = \frac{1}{10}$, what is $n$?
3. What are the subintervals of $[0, 2]$, if $n = 4$?
4. Use the interval and subintervals in part (3). Let the $x^*_{n}$’s be the left hand points of the intervals. What are all the $x^*_{n}$’s? And what are all the values of $f(x^*_{n})$?
Solution.  

(1) We can use the formula:

$$\Delta x = \frac{b - a}{n}$$

To make sense of this, we are taking the length of the entire interval, $b - a$, and dividing it (literally) into $n$ equal parts. This gives us the length of each subinterval:

$$\Delta x = \frac{2 - 0}{8} = \frac{1}{4}$$

(2) We can switch our formula around by solving for $n$:

$$n = \frac{b - a}{\Delta x}$$

We get:

$$n = \frac{4 - 3}{\frac{1}{10}} = 10$$

(3) If we divide up the interval from 0 to 2 into four equal parts, we get subintervals of length $\frac{1}{2}$. Those subintervals are:

$$[0, .5], [.5, 1], [1, 1.5], [1.5, 2]$$

Notice that all of your subintervals should be closed.

(4) The left-hand points of each interval, which we are calling the $x^*_n$'s are

$$x_1^* = 0, x_2^* = .5, x_3^* = 1, x_4^* = 1.5$$

Now take the value of the function at each point:

$$f(x_1^*) = f(0) = 0, f(x_2^*) = f(.5) = \frac{1}{8}, f(x_3^*) = f(1) = 1, f(x_4^*) = f(1.5) = \frac{27}{8}$$

Now let’s use what we’ve practiced to calculate a complete Riemann sum.

Example 10.6. Use the following methods to approximate the area under the curve of

$$f(x) = 16 - x^2$$

from $x = 0$ to $x = 3$.

(1) Riemann sum where $x_n^*$ are the right end points of each interval, $\Delta x = 1$
(2) Riemann sum where $x_n^*$ are the midpoints of each interval, $n = 6$

Solution.  

(1) First, we’ll do the right-hand Riemann sum. Let’s write our $x_n^*$’s as $x_n^R$ and the estimated area by $A_R$.

First, let’s figure out $n$, given that $\Delta x = 1$.

$$n = \frac{3 - 0}{1} = 3$$

This means the subintervals are

$$[0, 1], [1, 2], [2, 3]$$
Now take the formula given in the definition of the Riemann sum and plug in $n = 3$ and $\Delta x = 1$.

$$A = \sum_{i=1}^{3} f(x_i^R)(1) = \sum_{i=1}^{3} f(x_i^R)$$

We can break up this sum by letting $i$ range from 1 to 3 and writing out each term separately.

$$A_R = \sum_{i=1}^{3} f(x_i^R) = f(x_1^R) + f(x_2^R) + f(x_3^R)$$

$$= f(1) + f(2) + f(3) \text{ choose the right side of each interval}$$

$$= (16 - 1^2) + (16 - 2^2) + (16 - 3^2)$$

$$= 34$$

Because the graph is concave downwards, notice that this is an under estimate. We expect to get a larger number if we make $n$ larger (i.e. $\Delta x$ smaller) or if we use some other point in each interval instead. In the next part, we do both.

(2) Now call the midpoints $x_n^M$ and the area $A_M$. This time, we want $n = 6$. This means that $\Delta x = 1/2$, and we have six subintervals:

$$\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 2\right], \left[2, \frac{5}{2}\right], \left[\frac{5}{2}, 3\right]$$

We need to find the midpoint of each interval. You can find them using the formula

$$\text{midpoint}_{[a,b]} = \frac{a + b}{2}$$

Our first few midpoints are

$$\frac{0 + \frac{1}{2}}{2}, \frac{\frac{1}{2} + 1}{2}, \frac{1 + \frac{3}{2}}{2}, \ldots = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots$$

You can calculate the next three. Now plug what we have into the formula for the Riemann sum. Separate the sum into six separate parts and plug in the midpoints we’ve found. Finally, evaluate the function at each midpoint and add.

$$A_M = \sum_{i=1}^{6} f(x_i^M) \left(\frac{1}{2}\right)$$

$$= \frac{1}{2} \left[ f(x_1^M) + f(x_2^M) + f(x_3^M) + f(x_4^M) + f(x_5^M) + f(x_6^M) \right]$$

$$= \frac{1}{2} \left[ f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) \right]$$

$$= \frac{1}{2} \left[ \left(16 - \frac{1}{16}\right) + \left(16 - \frac{9}{16}\right) + \left(16 - \frac{25}{16}\right) + \left(16 - \frac{49}{16}\right) + \left(16 - \frac{81}{16}\right) + \left(16 - \frac{121}{16}\right) \right]$$

$$= \frac{1}{2} (78.125) = 39.0625$$
Your Turn. Approximate the area, \( A \), under the graph of \( f(x) = x + 2 \) by dividing \([-1, 4]\) into subintervals of equal length \( \Delta x = 1 \). Use a Riemann sum where the \( x^*_n \)'s are the midpoints of each interval.

10.4. The Definite Integral

Definition 10.5. If \( f(x) \) is a function that is continuous on the interval \([a, b]\) (with the exception of finitely many jump discontinuities), then we divide the interval into \( n \) subintervals of equal width, \( \Delta x \), and choose a point \( x^*_n \) from each interval.

Then the definite integral of \( f(x) \) from \( a \) to \( b \) is given by:

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_n) \Delta x
\]

Mathematically, this definition asks us to divide the interval \([a, b]\) into \( n \) equal pieces, just as with a Riemann sum. As before, the larger \( n \) is, the smaller the pieces are. The difference is that now we take the limit as \( n \) goes to infinity. Another way of visualizing this is to think of taking the limit as the width of the subintervals goes to zero. Now, instead of an approximation, we are calculating the true area under the given curve.

Example 10.7. Evaluate using the definition of the definite integral:

\[
\int_{0}^{3} 16 - x^2 \, dx
\]

Solution. First, we know that \( \Delta x = \frac{3 - 0}{n} = \frac{3}{n} \), even though we don’t know what \( \Delta x \) or \( n \) are. So this means that the first subinterval is \([0, \Delta x]\), the next is \([\Delta x, 2\Delta x]\), and so on. In general we can write the \( i^{th} \) interval as \([ (i - 1)\Delta x, i\Delta x] \).

Next, we choose what point to use from each interval (it doesn’t matter what we choose). Let’s use the right-hand side of each interval, so that for the \( i^{th} \) interval, we are using \( i\Delta x \) as our point in the the interval. Put these two together to get that

\[
x_i = i\Delta x = \frac{3i}{n}
\]

And so

\[
f(x_i) = 16 - \left( \frac{3i}{n} \right)^2 = 16 - \frac{9i^2}{n^2}
\]
Now consider the Riemann sum:

\[\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(16 - \frac{9i^2}{n^2}\right) \frac{3}{n}\]

\[= \frac{3}{n} \sum_{i=1}^{n} \left(16 - \frac{9i^2}{n^2}\right)\]

\[= \frac{3}{n} \left[\sum_{i=1}^{n} 16 - \sum_{i=1}^{n} \frac{9i^2}{n^2}\right]\]

\[= \frac{3}{n} \left[16n - \frac{9}{n^2} \sum_{i=1}^{n} i^2\right] \text{ See remark below}\]

\[= \frac{3}{n} \left[16n - \frac{9}{n^2} \left(\frac{n(n+1)(2n+1)}{6}\right)\right]\]

\[= 48 - \frac{9}{2} \left(\frac{2n^3 + 3n^2 + n}{n^3}\right)\]

Now take the limit as \(n \to \infty\):

\[\lim_{n \to \infty} 48 - \frac{9}{2} \frac{2n^3 + 3n^2 + n}{n^3} = 48 - \frac{9}{2}(2) = 39\]

So the exact area below the curve is 39 units square.

**Your Turn.** Evaluate using the definition of the definite integral:

\[\int_{0}^{4} 2x + 3 \, dx\]

**Remark 10.1.** You should memorize these formulas. They will come in handy (like in the previous example) for calculating definite integrals.

\[(1)\]

\[\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}\]

\[(2)\]

\[\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}\]

\[(3)\]

\[\sum_{i=1}^{n} i^3 = \left[\frac{n(n + 1)}{2}\right]^2\]