HAMILTONIAN FORMULATION FOR WATER WAVES

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

By

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August 2013
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ABSTRACT

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The system of equations describing incompressible inviscid fluid in a domain with a free top surface can be shown to be a Hamiltonian system with respect to the two functions that define the wave height and the velocity potential along the top boundary. The system is described by the Hamiltonian which is expressed as a function of the above variables using the Dirichlet-Nuemann operator over the fluid domain. Taylor expansion of this operator around the flat surface solution and the Fourier transform of the Hamiltonian then give rise to a solution in terms of the Fourier coefficients that can be implemented numerically.

In this work we derive the Hamiltonian formulation and implement numerical solution of the problem involving terms up to quadratic order in the amplitude of the waves. This work can be used as a basis for development of fast and accurate numerical method for solution of the water wave problem.
The propagation of water waves has always been a source of intrigue and fascination in 
humans. Our innate ability to spot and recognize patterns naturally draws our attention 
to the periodic movements of water waves in large bodies of water or to ripples of waves 
that radiate outwards when a standing body of water is disturbed by an external stimulus. 
The origin of the modern mathematical approach to solving the water wave problem can be 
 traced back to Leonhard Euler, who in 1755 successfully described the behavior of an ideal 
fluid using mathematical equations. These equations, known as Euler’s Equations, are the 
 foundation of modern fluid dynamics and provide the starting point of this thesis.

1.1 Lagrangian and Eulerian Variables

Let $D$ be a fluid domain that is continuously deformed over time $t$. We will denote the 
initial position of a particle $\vec{X}$ at $t = 0$ to be the Lagrangian coordinate. At the same time, 
we denote the current position, $\vec{x}$ at time $t$, to be the Eulerian coordinate. Here $\vec{X}$ and $\vec{x}$ 
can be considered as vectors in two or three dimensions, $(x, y) \in \mathbb{R}^2$ or $(x, y, z) \in \mathbb{R}^3$ 
respectively.

The correspondence between the initial and current coordinates of a point gives us a 
vector transformation $\vec{x} = \vec{x}(\vec{X}, t)$. Now if we make the physically natural assumption that 
$D$ deforms continuously then the mapping $\vec{x} \mapsto \vec{X}(\vec{x}, t)$ is a homeomorphism (continuous, 
with a continuous inverse). Clearly, a particle cannot move instantaneously around in $D$ 
and it could not occupy more than one initial position. The assumption that the mapping is 
onto corresponds to the absence of vacuum (or cavitation) at all times. If we let $J$ be the 
Jacobian of the transformation then a sufficient condition guaranteeing that the mapping is 
a homeomorphism is that $0 < J < \infty$ [10].
1.2 Convective Derivative

Two types of derivatives naturally arise when studying a deforming fluid domain. The first one is the rate of change with respect to time at a fixed position in the domain, while the second one is the rate of change with respect to time for a fixed particle as it moves in the fluid domain. The former is known as the Eulerian time derivative while the latter is known as the Lagrangian or convective time derivative. So let

\[ \frac{\partial}{\partial t} := \frac{\partial}{\partial t} \bigg|_{\bar{x}} \] = rate of change when \( \bar{x} \) is held constant
denote the Eulerian time derivative and

\[ \frac{D}{Dt} := \frac{\partial}{\partial t} \bigg|_{\bar{X}} = \text{rate of change when } \bar{X} \text{ is held constant} \]
denote the Lagrangian time derivative.

Let the mapping \((\bar{X}, t) \mapsto \bar{x}(\bar{X}, t)\) be specified; then the velocity \( \bar{u} = (u_1, u_2, u_3) \) is a function of \( \bar{x} = (x, y, z) \) and \( t \) defined as

\[ \bar{u} = \frac{D\bar{x}}{Dt}. \]

If \( \bar{u} \) does not depend on \( t \) when \( \bar{x} \) is fixed then we say that the flow is steady. Now let \( f(x, y, z, t) \) be any differentiable function. Applying the chain rule we get that

\[ \frac{\partial f}{\partial t} \bigg|_X = \frac{\partial f}{\partial t} \bigg|_x + \frac{\partial x}{\partial t} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial f}{\partial y} + \frac{\partial z}{\partial t} \frac{\partial f}{\partial z} \tag{1.1} \]

\[ = \frac{\partial f}{\partial t} + u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} + u_3 \frac{\partial f}{\partial z} \tag{1.2} \]

\[ = \frac{\partial f}{\partial t} \bigg|_x + (\bar{u} \cdot \nabla)f. \tag{1.3} \]

Note that \( \nabla \) is the gradient with respect to \( \bar{x} \) and not with respect to \( t \). This convention
will be used for the remainder of the paper. Equation (1.3) leads to the expression for the Lagrangian derivative $D/Dt$ as a differential operator in Eulerian coordinates:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u} \cdot \nabla).$$  \hspace{1cm} (1.4)

Here, $(\vec{u} \cdot \nabla)$ is a notation for the differential operator $u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + u_3 \frac{\partial}{\partial z}$. Furthermore, since acceleration is the rate of change of the velocity with respect to time we can apply the operator above to $\vec{u}$ itself. We get that the acceleration of the fluid is given by

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}.$$  \hspace{1cm} (1.5)

1.3 Reynolds’ Transport Theorem

We now introduce a famous theorem in fluid dynamics which will be used several times in the sections to follow.

Theorem (Reynolds’ Transport Theorem -1903), [10]. Let $D(t)$ be a time-dependent volume that is convected by the fluid, such that the volume always consists of the same fluid elements. Then for any function $f(\vec{x}, t)$ that is continuously differentiable with respect to all its arguments,

$$\frac{d}{dt} \int\int\int_{D(t)} f \, dx \, dy \, dz = \int\int\int_{D(t)} \frac{\partial f}{\partial t} \, dx \, dy \, dz + \nabla \cdot (f \vec{u}) \, dx \, dy \, dz.$$ \hspace{1cm} (1.6)

Proof: To evaluate the integral on the left-hand side we first transform it in terms of its Lagrangian variables

$$I(t) := \int\int\int_{D(t)} f \, dx \, dy \, dz = \int\int\int_{D(0)} f J \, dX \, dY \, dZ,$$ \hspace{1cm} (1.7)

where $J$ denotes the Jacobian of the transformation $\vec{X} \mapsto \vec{x}(\vec{X}, t)$:
\[ J = \frac{\partial \vec{x}}{\partial \vec{X}} = \begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{pmatrix}. \]

The integral on the left-hand side of (1.7) is taken over the fixed initial domain of the particles which end up in \( D \) at time \( t \). We can therefore differentiate under the integral sign:

\[
\frac{dI}{dt} = \iint_{D(0)} \frac{D}{Dt}(fJ) dX dY dZ
= \iint_{D(0)} Df \frac{J}{Dt} + f \frac{DJ}{Dt} dX dY dZ
= \iint_{D(0)} Df \frac{J}{Dt} + f(J \nabla \cdot \vec{u}) dX dY dZ
= \iint_{D(0)} Df \frac{J}{Dt} + f(\nabla \cdot \vec{u}) J dX dY dZ
= \iint_{D(t)} \frac{Df}{Dt} + f \nabla \cdot \vec{u} dxdydz
= \iint_{D(t)} \frac{Df}{Dt} + \nabla \cdot (f \vec{u}) dxdydz.
\]

In the second step we used Euler’s identity \( DJ/Dt = J \nabla \cdot \vec{u} \), the proof of which can be found in Appendix A. \( \square \)

1.4 Conservation of Mass

Let \( M \) be the mass of an arbitrary volume of fluid \( D(t) \) that moves with the fluid so that \( D(t) \) consists of the same material elements as \( D(0) \). \( M \) can be expressed as

\[
M = \iiint_{D(t)} \rho(x, y, z, t) dxdydz, \quad (1.8)
\]
where $\rho$ is the density. Conservation of mass dictates that matter is neither created or destroyed in $D(t)$, thus the net mass must be constant for all $t$, that is

$$0 = \frac{dM}{dt} = \frac{d}{dt} \iiint_{D(t)} \rho(x, y, z, t) \, dx \, dy \, dz$$

$$= \iiint_{D(t)} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) \, dx \, dy \, dz,$$

where the moving volume $D(t)$ is arbitrary. Thus, the integrand has to vanish identically, assuming that it is continuous. Furthermore, expansion of the integrand yields:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \rho u_1 + \frac{\partial}{\partial y} \rho u_2 + \frac{\partial}{\partial z} \rho u_3 = \frac{\partial \rho}{\partial t} + u_1 \frac{\partial \rho}{\partial x} + u_2 \frac{\partial \rho}{\partial y} + u_3 \frac{\partial \rho}{\partial z} + \rho \left( \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right)$$

$$= \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho + \rho(\nabla \cdot \vec{u})$$

$$= \frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{u}).$$

Thus, we obtain the equation of conservation of mass:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{u}) = 0. \quad (1.9)$$

Remark: The fact that we used the moving volume of fluid rather than a fixed volume in physical space does not affect the transition from (1.8) to (1.9). Indeed, since we assumed that the mapping $\vec{X} \mapsto \vec{x}(\vec{X}, t)$ is one-to-one and onto (no cavitation), then given a value of $t$ an arbitrary volume $D(t)$ can be obtained starting from a certain volume $D(0)$.  

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1.5 Newton’s Second Law

Newton’s second law of motion dictates that the net force on an object is equal to the rate of change of its linear momentum. Momentum is equal to mass times velocity, thus the net momentum of a of a moving volume of fluid can be expressed as

$$\iiint_D \rho \vec{u} \, dx \, dy \, dz. \quad (1.10)$$

The forces acting on the fluid can be split into two categories: *internal forces* and *external forces*. We denote the external force per unit mass to be $\vec{g}$ (usually it has the meaning of gravity force, constant in magnitude, in the downward direction). Therefore the total external force on the volume of fluid is

$$\iiint_D \rho \vec{g} \, dx \, dy \, dz.$$

For internal forces we consider the forces exerted on $D(t)$ by the surrounding fluid in the form of *pressure*, $p$. Now pressure acts on the boundary of $D(t)$ in an inward normal direction; thus the net internal force can be expressed as

$$\iint_{\partial D(t)} -p \vec{n} \, dS = \iiint_{D(t)} -\nabla p \, dx \, dy \, dz \quad (1.11)$$

where $dS$ denotes the surface measure on the boundary $\partial D(t)$, and the equality comes from applying the Divergence Theorem.

We now assume our flow to be *inviscid*. An *inviscid* flow is the flow of an ideal fluid that has no *viscosity*. Viscosity is the resistance of a fluid to deformation caused by shear or tensile strength. Viscous fluids or elastic solids can contribute tangential as well as normal internal forces. Thus the restriction to the category of inviscid flows is made to simplify our
work. Although no fluid is completely ideal, an assumption that the viscosity is sufficiently small is applicable in many problems of practical importance.

Going back to Newton’s second law we may now differentiate the net momentum equation (1.10) with respect to time and set it equal to the sum of the internal and external forces:

\[
\begin{align*}
\frac{d}{dt} \int \int \int_{D(t)} \rho \vec{u} \, dx \, dy \, dz &= \int \int \int_{D(t)} -\nabla p \, dx \, dy \, dz + \int \int \int_{D(t)} \rho \vec{g} \, dx \, dy \, dz.
\end{align*}
\]

(1.12)

Applying Reynolds’ Transport Theorem we get that the left-hand side of (1.12) is

\[
\begin{align*}
\frac{d}{dt} \int \int \int_{D(t)} \rho \vec{u} \, dx \, dy \, dz &= \int \int \int_{D(t)} \frac{\partial}{\partial t} \rho \vec{u} + \nabla \cdot \left( \rho \vec{u} \vec{u} \right) \, dx \, dy \, dz \\
&= \int \int \int_{D(t)} \rho \frac{\partial \vec{u}}{\partial t} + \vec{u} \frac{\partial \rho}{\partial t} + \vec{u} \nabla \cdot (\rho \vec{u}) + \rho (\vec{u} \cdot \nabla) \vec{u} \, dx \, dy \, dz \\
&= \int \int \int_{D(t)} \rho \left( \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) + \vec{u} \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) \right) \, dx \, dy \, dz \\
&= \int \int \int_{D(t)} \rho \frac{D\vec{u}}{Dt} \, dx \, dy \, dz.
\end{align*}
\]

Substituting this expression into (1.12) then yields

\[
\int \int \int_{D(t)} \rho \frac{D\vec{u}}{Dt} + \nabla p - \rho \vec{g} \, dx \, dy \, dz = 0,
\]

which holds for any volume of fluid \( D(t) \). Thus, assuming continuity of the integrand, it has to vanish identically:

\[
\rho \frac{D\vec{u}}{Dt} + \nabla p - \rho \vec{g} = 0,
\]

or equivalently

\[
\rho \frac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{g},
\]

(1.13)

which is known as the \textit{momentum equation} for ideal fluid.
1.6 Incompressible Fluid

A fluid is said to be incompressible if the density $\rho$ is constant. In that case, it follows immediately from equation (1.9) that

$$\rho(\nabla \cdot \vec{u}) = 0,$$

which implies $\nabla \cdot \vec{u} = 0$. Next, from the momentum equation (1.13) we get that

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho} \nabla p + \vec{g},$$

and recalling the definition of the convective derivative, this takes the form

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla p + \vec{g}.$$ 

Applying the vector identity, see [10, p. 12],

$$(\vec{u} \cdot \nabla) \vec{u} = \nabla \left( \frac{1}{2} |\vec{u}|^2 \right) + (\nabla \times \vec{u}) \times \vec{u} \quad (1.14)$$

and taking the gravity force as $\vec{g} = -g\vec{k}$ where $g$ is the scalar acceleration of gravity, we obtain

$$\frac{\partial \vec{u}}{\partial t} + \nabla \left( \frac{1}{2} |\vec{u}|^2 \right) + (\nabla \times \vec{u}) \times \vec{u} = \frac{1}{\rho} \nabla p - g\vec{k}.$$ 

Moving the terms that have form of a gradient of a scalar function to the right-hand side we arrive at the equation in the form

$$\frac{\partial \vec{u}}{\partial t} + (\nabla \times \vec{u}) \times \vec{u} = -\nabla \left( \frac{p}{\rho} + \frac{1}{2} |\vec{u}|^2 + g\vec{k} \right). \quad (1.15)$$
1.7 Vorticity

The *vorticity* of a fluid is a measure of the local rotation of the flow. Quantitatively the vorticity \( \vec{\omega} \) is defined to be the curl of the velocity field:

\[
\vec{\omega} = \nabla \times \vec{u}.
\]

Recalling that the curl of the gradient of any scalar field is zero, substituting \( \omega \) and taking the curl of (1.15) we get

\[
\begin{align*}
\nabla \times \left( \frac{\partial \vec{u}}{\partial t} + \vec{\omega} \times \vec{u} \right) &= \nabla \times \left( -\nabla \left( \frac{p}{\rho} + \frac{1}{2} |\vec{u}|^2 + \vec{g} \right) \right), \\
\nabla \times \frac{\partial \vec{u}}{\partial t} + \nabla \times (\vec{\omega} \times \vec{u}) &= 0, \\
\n\frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega} \times \vec{u}) &= 0.
\end{align*}
\]

The last result is known as the *Helmholtz Equation*. We then use the vector identity, see [10, p. 18],

\[
\nabla \times (\vec{a} \times \vec{b}) = (\nabla \cdot \vec{b}) \vec{a} - (\nabla \cdot \vec{a}) \vec{b} + (\vec{b} \cdot \nabla) \vec{a} - (\vec{a} \cdot \nabla) \vec{b},
\]

the condition of incompressibility \( \nabla \cdot \vec{u} = 0 \) and the fact that \( \text{div} \, \text{curl} \equiv 0 \) to obtain

\[
\nabla \times (\vec{\omega} \times \vec{u}) = (\nabla \cdot \vec{u}) \vec{\omega} - (\nabla \cdot \vec{\omega}) \vec{u} + (\vec{u} \cdot \nabla) \vec{\omega} - (\vec{\omega} \cdot \nabla) \vec{u}
\]

\[
= (\vec{u} \cdot \nabla) \vec{\omega} - (\vec{\omega} \cdot \nabla) \vec{u}.
\]
Thus substituting (1.17) into the Helmholtz equation leads to the vorticity equation in the form

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla) \vec{\omega} = (\vec{\omega} \cdot \nabla) \vec{u}.$$  \hspace{1cm} (1.18)

A very natural question arises when considering the vorticity of a fluid domain and that is how the vorticity of a particle changes as it convects through the fluid domain. Luckily, the following famous theorem answers this question for us.

![Figure 1.1: Circulation around a closed path $C(t)$.](image)

**Theorem [Kelvin’s Circulation Theorem - 1869].** Let $\vec{u}$ be continuously differentiable vector field satisfying the incompressible Euler equations, with potential external force field. Take an arbitrary simple closed loop $C(t)$ in the fluid domain and define the circulation around this loop,

$$\Gamma = \oint_{C(t)} \vec{u} \cdot d\vec{x}.$$

Then the circulation around the loop is conserved, that is

$$\frac{d\Gamma}{dt} = 0.$$
Proof: Switching to Lagrangian coordinates and using the chain rule we obtain

\[ \Gamma = \oint_{C(t)} \vec{u} \cdot d\vec{x} = \oint_{C(0)} \sum_{i,j} u_i \frac{\partial x_i}{\partial X_j} dX_j. \]

In Lagrangian variables, the integration is around a fixed curve \( C(0) \). Therefore, we can differentiate though the integration:

\[ \frac{d\Gamma}{dt} = \frac{d}{dt} \oint_{C(0)} \sum_{i,j} u_i \frac{\partial x_i}{\partial X_j} dX_j \]

\[ = \oint_{C(0)} \frac{D}{Dt} \left( \sum_{i,j} u_i \frac{\partial x_i}{\partial X_j} \right) dX_j \]

\[ = \oint_{C(0)} \sum_{i,j} \left( \frac{Du_i}{Dt} \frac{\partial u_i}{\partial X_j} + u_i \frac{\partial u_i}{\partial X_j} \right) dX_j. \]

Reverting back to Eulerian coordinates we obtain the equations:

\[ \frac{d\Gamma}{dt} = \oint_{C(t)} \frac{D\vec{u}}{Dt} \cdot d\vec{x} + \oint_{C(t)} \frac{1}{2} \nabla |\vec{u}|^2 \cdot d\vec{x} \]

\[ = \oint_{C(t)} \left( -\frac{1}{\rho} \nabla p + \vec{g} \right) \cdot d\vec{x} + \oint_{C(t)} \frac{1}{2} \nabla |\vec{u}|^2 \cdot d\vec{x} \]

\[ = \oint_{C(t)} \nabla \left( -\frac{p}{\rho} - g + \frac{1}{2} |\vec{u}|^2 \right) \cdot d\vec{x} = 0 \]

where the last equality holds true since we are integrating the gradient of a scalar function over a closed loop.

A particular consequence of Kelvin’s Circulation theorem is that if the vorticity is initially zero then it will remain zero for all times. Indeed, suppose at a point \( \vec{x} \) the vorticity is initially zero and at some time \( t \) it takes on a non-zero value. Vorticity is the local measure of rotation so we can define a closed path around \( \vec{x} \) such that \( \Gamma_0 = 0 \) and \( \Gamma_t \neq 0 \). Kelvin’s circulation theorem states that \( \Gamma \) is independent of time, therefore we arrive at a contradiction. This allows us to consider a particular class of fluid flows characterized by
the condition of vanishing vorticity.

1.8 Irrotational Flow

A flow is said to be \textit{irrotational} if the curl of the velocity vector field is identically zero, that is

\[ \vec{\omega} = \nabla \times \vec{u} \equiv 0. \]

Intuitively this means that the fluid particles are not rotating in the fluid domain. This assumption allows us to simplify the Euler equation since the non-linear term \( (u \cdot \nabla) u \) is then the gradient of a scalar function \( \frac{1}{2} \| \vec{u} \|^2 \). Thus, if we assume the flow to be irrotational

the momentum equation simplifies to

\[ \frac{\partial \vec{u}}{\partial t} = -\nabla \left( \frac{p}{\rho} + \frac{1}{2} \| \vec{u} \|^2 + g\vec{k} \right). \]

In particular, if the flow is steady, the left-hand side is zero, thus \( \frac{p}{\rho} + \frac{1}{2} \| \vec{u} \|^2 + g\vec{k} \) must be constant. This is known as \textit{Bernoulli’s Theorem for steady irrotational flow}.

If the velocity field is continuously differentiable in simply connected fluid domain, the irrotational flow condition allows us to define the velocity potential function \( \varphi(\vec{x}, t) \) such that

\[ \vec{u} = \nabla \varphi. \]

Indeed, taking a point \( \vec{x}_0 = (x_0, y_0, z_0) \) in the domain, define

\[ \varphi(\vec{x}, t) = \varphi_0(t) + \int_C \vec{u} \cdot d\vec{x} \]

where \( \varphi_0 \) is arbitrary and \( C \) is any path connecting \( \vec{x}_0 \) to \( \vec{x} \). Notice that the definition of \( \varphi \)
is independent of the path $C$: if $C_1$ and $C_2$ are two paths from $\vec{x}_0$ to $\vec{x}$ then

$$\int_{C_1} \vec{u} \cdot d\vec{x} - \int_{C_2} \vec{u} \cdot d\vec{x} = \oint_{C_1-C_2} \vec{u} \cdot d\vec{x} = \iint_S (\nabla \times \vec{u}) \cdot \vec{n} dS = 0,$$

where $C_1 - C_2$ is a closed loop obtained by joining the paths $C_1$ and $C_2$, $S$ is an arbitrary smooth surface that has $C_1 - C_2$ as its boundary, and we use the Stokes theorem and the irrotational flow condition to obtain our conclusion. Thus, we have that

$$\int_{C} \vec{u} \cdot d\vec{x}$$

is path independent. Therefore, we get the equivalent condition that $\vec{u}$ is conservative vector field, see Appendix B. That is, there exist a scalar field $\varphi$ such that

$$\vec{u} = \nabla \varphi. \quad (1.19)$$

The class of flows where the velocity field can be described as the gradient of a potential is called potential flows. We saw earlier that incompressibility implies that $\nabla \cdot \vec{u} = 0$. If the flow is potential, $\nabla \cdot \vec{u} = \nabla \cdot \nabla \varphi = \Delta \varphi$, we obtain that the velocity potential function must satisfy Laplace’s equation

$$\Delta \varphi = 0.$$

Now if we take (1.15) and let $\vec{\omega} = 0$, then

$$\frac{\partial \vec{u}}{\partial t} + \nabla (\frac{1}{2} |\vec{u}|^2) = \frac{1}{\rho} \nabla p + g\vec{k}. \quad (1.20)$$
We then substitute the gradient of the velocity potential for $\vec{u}$:

$$\frac{\partial}{\partial t}(\nabla \varphi) + \nabla (\frac{1}{2} |\nabla \varphi|^2) = \frac{1}{\rho} \nabla p + g \vec{k}.$$ 

Integrating with respect to $\vec{x}$ we obtain

$$\frac{p - p_0}{\rho} = B(t) - \varphi_t - \frac{1}{2} |\nabla \varphi|^2 - gz,$$

where $p_0$ is an arbitrary constant and $B(t)$ is an arbitrary function. (In particular, in the problem of surface water waves it is convenient to choose $p_0$ to be the air pressure at the surface of the wave.) Since $\varphi$ is not unique, $B(t)$ can be absorbed into $\varphi$ by defining $\varphi' = \varphi - \int B(t) \, dt$ as a new potential function. In that case we have

$$\frac{p - p_0}{\rho} = -\varphi_t - \frac{1}{2} |\nabla \varphi|^2 - gz. \quad (1.21)$$

1.9 Formulation of the Water Wave Problem

The interface of a fluid domain is taken to be the boundary of the fluid domain where it contacts the air. Consider the equation of the interface $f(\vec{x}, t) = 0$. For fluid particles on the interface, $f(\vec{x}, (\vec{X}, t), t) = 0$ differentiating with respect to $t$ yields

$$\frac{d}{dt} f(\vec{x}, (\vec{X}, t), t) = \frac{\partial f}{\partial t} + u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} + u_3 \frac{\partial f}{\partial z} = 0. \quad (1.22)$$

A convenient way to describe the surface is by $z = \eta(x, y, t)$, where the function $\eta(x, y, t)$ defines the top surface of the fluid domain, which corresponds to the choice $f(\vec{x}, t) = \eta(x, y, t) - z$. Substituting this into the previous equation yields

$$\frac{\partial \eta}{\partial t} + u_1 \frac{\partial \eta}{\partial x} + u_2 \frac{\partial \eta}{\partial y} - u_3 = 0,$$
or equivalently,

\[ \frac{\partial \eta}{\partial t} = u_3 - u_1 \frac{\partial \eta}{\partial x} - u_2 \frac{\partial \eta}{\partial y}. \]

In the case of potential flow the condition on the interface takes the form

\[ \frac{\partial \eta}{\partial t} = \frac{\partial \varphi}{\partial z} - \frac{\partial \varphi}{\partial x} \frac{\partial \eta}{\partial x} - \frac{\partial \varphi}{\partial y} \frac{\partial \eta}{\partial y}. \] (1.23)

In addition, the pressure on the surface of the fluid is assumed to be equal to the air pressure. That is, \( p = p_0 \) when \( z = \eta(x, y, t) \). Thus referring back to (1.21) and substituting \( z = \eta(x, y, t) \) we obtain

\[ \frac{\partial \varphi}{\partial t} = -g \eta - \frac{1}{2} |\nabla \varphi|^2. \] (1.24)

We can therefore set up the basic system of differential equations for the water wave problem:

\[ \eta_t = \varphi_x - \eta_x \varphi - \eta_y \varphi_y, \quad z = \eta(x, y, t) \] (1.25)
\[ \varphi_t = -g \eta - (\varphi_x)^2 - (\varphi_y)^2, \quad z = \eta(x, y, t) \] (1.26)
\[ \varphi_{xx} + \varphi_{yy} + \varphi_z = 0, \quad -h < z < \eta(x, y, t). \] (1.27)

Various classical water wave problems can now be posed by imposing various boundary conditions on the fluid domain. Generally, the bottom boundary is taken to be an impermeable solid. However, there are different options for the side boundaries. Commonly imposed side boundaries are periodic, solid or their combination. An example of a typical fluid domain is shown in Figure 1.9.
\[ z = \eta(x, y, t) \]

Figure 1.2: Fluid domain for the water wave problem.
Chapter 2

Water Wave Problem

With the derivation of the boundary conditions now established we will examine one of the classical problems for an ideal fluid with a free surface being acted upon by gravity. For the rest of the paper we will restrict our dimension to two, but the theory can be applied to three dimensions as well.

\[
\begin{align*}
\Delta \varphi &= 0 \\
\eta(x, t) &= \eta(x + L, t) \\
\varphi(x, t) &= \varphi(x + L, t)
\end{align*}
\]

Figure 2.1: Fluid domain with periodic side boundaries

We begin by defining the fluid domain to be the region enclosed by a top surface defined by \( y = \eta(x, t) \), an average bottom depth \(-h\) (which may even be infinite, \( h = \infty \)), and side boundaries at \( x = 0 \) and \( x = L \). That is, our fluid domain is

\[
\Gamma(\eta) = \{(x, y) \in \mathbb{R}^2 \mid -h < y < \eta(x, t), \ 0 < x < L \}. \tag{2.1}
\]

For convenience sake, we take \( L = 2\pi \) from this point on.

The fluid is assumed to be incompressible and irrotational which gives rise to a potential flow as the previous section showed. Thus there exist velocity potential \( \varphi \) such that \( u = \nabla \varphi \) and \( \Delta \varphi = 0 \) for \((x, y) \in \Gamma(\eta)\).

We now impose boundary conditions on \( \partial \Gamma \). The bottom boundary is taken to be a
solid, impermeable boundary, that is $\partial_n \varphi = 0$. Intuitively this means that the particles at
the bottom can not cross the bottom boundary nor can they leave the bottom boundary.
Side boundary conditions are taken to be $2\pi$ periodic, that is $\eta(x + 2\pi, t) = \eta(x, t)$ and
$\varphi(x + 2\pi, t) = \varphi(x)$.

The boundary conditions on the top surface, as derived in an earlier section, introduce
the time dependence and nonlinearity to the problem. In two dimensions they are:

$$\partial_t \eta = \partial_y \varphi - \partial_x \eta \partial_x \varphi, \quad y = \eta(x, t)$$  \hspace{1cm} (2.2)

$$\partial_t \varphi = -g \eta - \frac{1}{2} |\nabla \varphi|^2, \quad y = \eta(x, t).$$  \hspace{1cm} (2.3)

2.1 Hamiltonian Systems

As the title of this thesis indicates, we will show that the water wave problem has
the structure of an infinite-dimensional Hamiltonian system and use this approach in the
numerical solution of the water wave problem. We first give an introduction to Hamiltonian
systems in the case of finitely many variables.

Definition: A system of differential equations is called a Hamiltonian system in canonical
form if there exists a real-valued function $H(\vec{x}, \vec{y})$ where $\vec{x}, \vec{y} \in \mathbb{R}^n$ such that

$$\frac{dx_k}{dt} = \frac{\partial H}{\partial y_k},$$  \hspace{1cm} (2.4)

$$\frac{dy_k}{dt} = -\frac{\partial H}{\partial x_k}$$

for all $k \in \{1, ..., n\}$. The function $H$ is called the Hamiltonian function (or simply the
Hamiltonian) for the system.

Proposition: The Hamiltonian function of a Hamiltonian system is a conserved quantity.
That is, given any solution \((\vec{x}(t), \vec{y}(t))\),

\[
\frac{d}{dt} H(\vec{x}(t), \vec{y}(t)) = 0.
\]

Proof: The proof is straightforward using the chain rule and then substituting in equations (2.4):

\[
\frac{d}{dt} H(\vec{x}(t), \vec{y}(t)) = \sum_{k=1}^{n} \frac{\partial H}{\partial x_k} \frac{dx_k}{dt} + \frac{\partial H}{\partial y_k} \frac{dy_k}{dt}
= \sum_{k=1}^{n} \frac{\partial H}{\partial x_k} \frac{\partial H}{\partial y_k} - \frac{\partial H}{\partial y_k} \frac{\partial H}{\partial x_k} = 0.
\]

Example: Consider the following system of differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= x - x^2.
\end{align*}
\]

Then the real-valued function

\[
H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3
\]

is a conserved quantity since

\[
\frac{d}{dt} H(x(t), y(t)) = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt}
= (-x + x^2)y + y(x - x^2)
= 0.
\]

Furthermore, the above system is a Hamiltonian system since the condition in (2.4) is
satisfied by $H$.

Next, we examine how a change of variables affects a Hamiltonian system. Let $H$ be the Hamiltonian for a system with $\vec{x}(t), \vec{y}(t) \in \mathbb{R}^n$, that is $H$ satisfies

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

Let $\vec{z} = (\vec{x}, \vec{y}) \in \mathbb{R}^{2n}$ and let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a transformation such that

$$\vec{z} = \varphi(\vec{z}).$$

Applying the chain rule we get

$$\frac{d\tilde{z}_i}{dt} = \sum_{j=1}^{2n} A_{ij} J_{ji} \frac{\partial H}{\partial z_i}.$$

Now let $A$ be a $2n$-square matrix whose $i$-th, $j$-th element is

$$A^i_j = \frac{\partial \varphi_i}{\partial z_j}$$

and $J$ be a $2n$ by $2n$ block matrix with an $I_n$ matrix in the top right, $-I_n$ in the lower left and zeros elsewhere. Thus,

$$\frac{d\tilde{z}_i}{dt} = \sum_{j=1}^{2n} A^i_j J^i_j \frac{\partial H}{\partial z_i}$$

or

$$\tilde{z}^T = AJdH$$
where \( dH \) is the column vector \( \left( \frac{\partial H}{\partial z_i} \right) \). Next, applying the chain rule to \( H \) we get that

\[
\frac{\partial H}{\partial z_i} = 2^n \sum_{j=1}^{2^n} \frac{\partial \varphi_j}{\partial z_i} \frac{\partial \tilde{H}}{\partial z_i} = \sum_{j=1}^{2^n} A_{ij} \frac{\partial \tilde{H}}{\partial z_i}.
\]

Thus,

\[
\frac{d\tilde{z}^T}{dt} = AJA^T d\tilde{H}.
\]

Therefore, if

\[
AJA^T = J \tag{2.5}
\]

then the transformed variables \((\tilde{x}, \tilde{y})\) satisfy equations in canonical form,

\[
\begin{align*}
\frac{d\tilde{x}_i}{dt} &= \frac{\partial H}{\partial \tilde{y}_i}, \\
\frac{d\tilde{y}_i}{dt} &= -\frac{\partial H}{\partial \tilde{x}_i}.
\end{align*}
\]

A transformation \( \tilde{z} = \varphi(z) \) that satisfies condition (2.5) is called a canonical transformation.

2.2 Functional Derivative

Calculus of Variations studies functionals, which is a name for real-valued functions defined on function spaces, in much the same way that Calculus studies the behavior of functions. In practical applications, as in the water wave problem, functionals usually involve integrals.

In Calculus if \( f(x_1, \ldots, x_n) \) is a function from \( \mathbb{R}^n \to \mathbb{R} \), then its differential is a linear function

\[
\langle df, d\vec{x} \rangle = \frac{\partial f}{\partial x_1} dx_1 + \ldots + \frac{\partial f}{\partial x_n} dx_n.
\]

In Calculus of Variations, if \( I(f) \) is a functional and \( f + \varepsilon g \) is a small perturbation of \( f \),
then the analogous concept to the differential $df$ is referred to as the (first) variation $\delta I$.

$$\delta I : g(x) \mapsto \int \frac{\delta I}{\delta f} g(x) \, dx,$$

where $\frac{\delta I}{\delta f}$ is a function, known as the variational derivative of $I(f)$. Thus, $\frac{\delta I}{\delta f}$ is defined by the property that for every $g(x)$ satisfying suitable conditions,

$$\langle \frac{\delta I}{\delta f}, g \rangle = \lim_{\varepsilon \to 0} \frac{I(f + \varepsilon g) - I(f)}{\varepsilon} = \frac{d}{d\varepsilon} I(f + \varepsilon g) \bigg|_{\varepsilon=0}$$

where $\langle f, g \rangle$ is the standard $L^2$ pairing of the functions $f$ and $g$:

$$\langle f, g \rangle = \int f(x)g(x) \, dx.$$

### 2.3 An Example of an Infinite-Dimensional Hamiltonian System

Consider the problem of small transversal vibrations of an elastic membrane. The shape of the membrane is given by the region $D \subset \mathbb{R}^2$, the boundary condition corresponds to the membrane attached at the edge:

$$\partial_t u = \partial_x^2 u + \partial_y^2 u, \quad (x, y) \in D \quad (2.6)$$

$$u = 0, \quad (x, y) \in \partial D. \quad (2.7)$$

We will use the notation $\Delta$ for the Laplace operator $\partial_x^2 + \partial_y^2$. If we set $p = \partial_t u$ then the wave equation (2.6) implies $\partial_t p = \Delta u$. We will show that the problem has a structure of an infinite-dimensional Hamiltonian system. To this end, define the Hamiltonian function

$$H(u, p) = \iint_D \frac{1}{2} p^2 + \frac{1}{2} |\nabla u|^2 \, dx \, dy.$$
Let $p + \varepsilon q$ be a perturbation of $p$, then

\[
H(u, p + \varepsilon q) = \iint_D \frac{1}{2} (p + \varepsilon q)^2 + \frac{1}{2} |\nabla u|^2 \, dx \, dy \\
= \iint_D \frac{1}{2} p^2 + \frac{1}{2} |\nabla u|^2 \, dx \, dy + \varepsilon \iint_D p \, q \, dx \, dy + o(\varepsilon),
\]

or equivalently

\[
H(u, p + \varepsilon q) - H(u, p) = \varepsilon \iint_D p \, q \, dx \, dy + o(\varepsilon).
\]

Dividing by $\varepsilon$ and taking the limit $\varepsilon \to 0$ we obtain

\[
\langle \frac{\delta H}{\delta p}, q \rangle = \iint_D p \, q \, dx \, dy,
\]

where $q$ is arbitrary, so that

\[
\frac{\delta H}{\delta p} = p.
\]

Let $u + \varepsilon v$ be a perturbation of $u$ satisfying the boundary conditions (2.7), then

\[
|\nabla (u + \varepsilon v)|^2 = \nabla (u + \varepsilon v) \cdot \nabla (u + \varepsilon v) \\
= \nabla u \cdot \nabla u + 2\varepsilon \nabla u \cdot \nabla v + o(\varepsilon).
\]

Thus,

\[
H(u + \varepsilon v, p) = \iint_D \frac{1}{2} p^2 + \frac{1}{2} |\nabla (u + \varepsilon v)|^2 \, dx \, dy \\
= \iint_D \frac{1}{2} p^2 + \frac{1}{2} |\nabla u|^2 \, dx \, dy + \varepsilon \iint_D \nabla u \cdot \nabla v \, dx \, dy + o(\varepsilon).
\]
Applying Green’s first identity yields

\[
\int \int_D \nabla u \cdot \nabla v \, dx \, dy = \int_{\partial D} v \nabla u \cdot n \, dS - \int \int_D v \Delta u \, dx \, dy = -\int \int_D v \Delta u \, dx \, dy,
\]

since }u\text{ vanishes on the boundary of }D\text{. Thus, }H(u + \varepsilon v, p) - H(u, p) = -\varepsilon \int \int_D v \Delta u \, dx \, dy + o(\varepsilon),

which implies

\[
\langle \frac{\partial H}{\partial u}, v \rangle = -\int \int_D v \Delta u \, dx \, dy,
\]

for every }v\text{ twice continuously differentiable and vanishing on }\partial D\text{. In this way we obtain

\[
\frac{\delta H}{\delta u} = -\Delta u,
\]

Therefore, the wave equation is indeed a Hamiltonian system since

\[
\partial_t u = \frac{\delta H}{\delta p} = p,
\]

\[
\partial_p p = -\frac{\delta H}{\delta u} = \Delta u.
\]

2.4 Hamiltonian Function for the Water Wave Problem

We will now show that the water wave problem has the structure of a Hamiltonian system with a suitable choice of canonical variables. The first step is to quantitatively describe the total energy of the system as the sum of kinetic energy and potential energy. The total energy will provide the Hamiltonian function for the system. Conservation of energy implies that the time derivative of the total energy is zero. The crucial step in the
derivation is finding the correct canonical variables such that (2.4) is satisfied.

To simplify the notation and allow for more compact representation of calculations, from now on we adopt the subscript notation on a function to indicate the partial derivative with respect to the subscript variables; thus $\partial_x \varphi = \varphi_x$, etc.

The total energy of the system can easily be computed using the following two expressions for the kinetic energy $K$ and potential energy $P$:

$$K = \int_0^{2\pi} \int_{-h}^{\eta(x)} \frac{1}{2} |\vec{u}|^2 \, dy \, dx = \int_0^{2\pi} \int_{-h}^{\eta(x)} \frac{1}{2} |\nabla \varphi|^2 \, dy \, dx,$$  

(2.8)

$$P = \int_0^{2\pi} \int_{-h}^{\eta(x)} g y \, dy \, dx = \int_0^{2\pi} \frac{1}{2} g \eta^2(x) \, dx - \pi gh^2.$$  

(2.9)

Therefore, the following functional expresses the total energy (up to a constant in the potential energy, which is not significant) and is a candidate for a Hamiltonian:

$$H = \int_0^{2\pi} \int_{-h}^{\eta(x)} \frac{1}{2} |\nabla \varphi|^2 \, dy \, dx + \int_0^{2\pi} \frac{1}{2} g \eta^2(x) \, dx.$$  

(2.10)

Expression (2.8) can be further simplified by “integrating by parts” by means of Green’s first identity:

$$K = \int_0^{2\pi} \int_{-h}^{\eta(x)} \frac{1}{2} |\nabla \varphi|^2 \, dy \, dx$$

$$= \int_0^{2\pi} \int_{-h}^{\eta(x)} \frac{1}{2} \nabla \varphi \cdot \nabla \varphi \, dy \, dx$$

$$= \int_{\partial \Gamma} \frac{1}{2} \varphi \nabla \varphi \cdot \hat{n} \, dS - \int_{\Gamma} \frac{1}{2} \varphi \Delta \varphi \, dy \, dx,$$

where $dS$ is the surface measure on $\partial \Gamma$. The first term vanishes on the side boundary by applying the periodic boundary conditions. In addition, it also vanishes on the bottom boundary as well since $\nabla \varphi \cdot \vec{n} = 0$ for $y = -h$. The second term also vanishes since
$\Delta \varphi = 0$ in $\Gamma$. Thus

$$K = \int_{y=\eta(x)} \frac{1}{2} \varphi \nabla \varphi \cdot \vec{n} \, dS = \int_0^{2\pi} \frac{1}{2}(\varphi_y - \eta_x \varphi_x) \, dx,$$

where we used that

$$dS = \sqrt{1 + \eta_x^2} \, dx \quad \text{and} \quad \nabla \varphi \cdot \vec{n} = \frac{-\eta_x \cdot \varphi_x + \varphi_y}{\sqrt{1 + \eta_x^2}}.$$

In addition, we recognize that $\nabla \varphi \cdot \vec{n}$ is the normal derivative of $\varphi$. The normal derivative is expressed in terms of the boundary data for the potential by means of the following Dirichlet-Neumann operator.

**Definition (Dirichlet-Neumann operator):** For the fluid domain $\Gamma(\eta)$ defined by (2.1), we specify boundary values $\xi(x)$ on the free surface $y = \eta(x)$, and consider their harmonic extension $\varphi(x,y)$:

$$\Delta \varphi = 0, \quad (x,y) \in \Gamma(\eta) \quad (2.11)$$

with the Neumann bottom boundary condition $\partial_n \varphi$ for $y = -h$ and $2\pi$ periodic in the $x$-variable. The Dirichlet-Neumann operator $\eta \mapsto G(\eta)$ is defined with the normal derivative of $\varphi$ on the free surface,

$$G(\eta)\xi(x) = (\varphi_y - \eta_x \varphi_x)(x, \eta(x)) = R \nabla \varphi \cdot \vec{n} (x, \eta(x)) \quad (2.12)$$

where $R = \sqrt{1 + |\partial_x \eta|^2}$ is a normalization factor, which is natural if we wish to work with integrals with respect to $x$ rather than the arc length measure $dS$.

**Proposition:** The operator $G(\eta)$ is positive and self-adjoint operator on $L^2(0,2\pi)$.  

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Proof: Let $\xi$ and $\zeta$ be two functions $2\pi$-periodic in $x$, and $\varphi$ and $\psi$ denote their harmonic extensions to $\Gamma(\eta)$, with the homogeneous Neumann boundary condition for $y = -h$.

We compute

\[
\int_0^{2\pi} \zeta G(\eta) \xi \, dx = \int_0^{2\pi} \zeta (\varphi_y - \eta_x \varphi_x)(x, \eta(x)) \, dx
\]

\[
= \int_{\partial \Gamma} \psi \nabla \varphi \cdot n \, dS
\]

\[
= \iint_{\Gamma} \nabla \psi \cdot \nabla \varphi \, dx \, dy + \iint_{\Gamma} \psi \Delta \varphi \, dx \, dy
\]

\[
= \iint_{\Gamma} \nabla \varphi \cdot \nabla \psi \, dx \, dy
\]

\[
= \int_{\partial \Gamma} \varphi \nabla \psi \cdot n \, dS - \iint_{\Gamma} \varphi \Delta \psi \, dx \, dy
\]

\[
= \iint_{\Gamma} \varphi \nabla \psi \cdot n \, dS
\]

\[
= \int_0^{2\pi} \xi (\psi_y - \eta_x \psi_x)(x, \eta(x)) \, dx
\]

\[
= \int_0^{2\pi} \xi G(\eta) \zeta \, dx.
\]

The positivity follows from a similar calculation:

\[
\int_0^{2\pi} \xi G(\eta) \xi \, dx = \int_0^{2\pi} \xi (\varphi_y - \eta_x \varphi_x)(x, \eta(x)) \, dx
\]

\[
= \int_{\partial \Gamma} \varphi \nabla \varphi \cdot n \, dS
\]

\[
= \iint_{\Gamma} \nabla \varphi \cdot \nabla \varphi \, dx \, dy + \iint_{\Gamma} \varphi \Delta \varphi \, dx \, dy
\]

\[
= \iint_{\Gamma} \nabla \varphi \cdot \nabla \varphi \, dx \, dy
\]

\[
= \iint_{\Gamma} |\nabla \varphi|^2 \, dx \, dy \geq 0.
\]

Using the definition of the operator $G(\eta)$ the total energy of the system can now be
expressed as

\[ H = \int_{0}^{2\pi} \frac{1}{2} \xi G(\eta)\xi + \frac{1}{2} g\eta^2 \, dx. \] (2.13)

We will now show that our top boundary conditions (2.2) and (2.3) can be expressed explicitly in terms of \( \eta, \xi \), their derivatives and the operator \( G(\eta)\xi \).

The first surface boundary condition (2.2) follows directly from the definition of \( G(\eta)\xi \):

\[ \eta_t = G(\eta)\xi. \] (2.14)

Next we want to express the second surface boundary condition (2.3) in terms of \( \eta \) and \( \xi \). We start by noting that \( \xi \) is the velocity potential at the surface, that is \( \xi(x, \eta(x), t) = \varphi(x, \eta(x), t) \). Taking the partial derivative with respect to \( x \) and \( t \) we get that

\[ \xi_x = \varphi_x + \eta_x \varphi_y \] (2.15)

and

\[ \xi_t = \varphi_t + \varphi_y \eta_t. \] (2.16)

Equations (2.12) and (2.15) results in a simple system of equations that will allow us to express \( \varphi_x \) and \( \varphi_y \) in terms of \( \eta \) and \( \xi \). Namely, solving

\[-\eta_x \varphi_x + \varphi_y = G(\eta)\xi, \]

\[ \varphi_x + \eta_x \varphi_y = \xi_x, \]

we obtain

\[ \varphi_x = \frac{\xi_x - \eta_x G(\eta)\xi}{1 + \eta_x^2} \]
and
\[ \varphi_y = \frac{\xi_x \eta_x + G(\eta)\xi}{1 + \eta_x^2}. \]

In addition, (2.16) implies
\[
\varphi_t = \xi_t - \varphi_y \eta_t
= \xi_t - \varphi_y G(\eta)\xi.
\] (2.17)

Next, substituting (2.17) into the left-hand side of (2.3) yields
\[
\xi_t - \varphi_y G(\eta)\xi = -\frac{1}{2} |\nabla \varphi|^2 - g\eta,
\] (2.18)
and we can further compute

\begin{align*}
\xi_t &= \varphi_y G(\eta)\xi - \frac{1}{2} |\nabla \varphi|^2 - g\eta \\
&= \frac{\xi_x \eta_x + G(\eta)\xi}{1 + \eta_x^2} G(\eta)\xi \\
&- \frac{1}{2} \left( (\xi_x - \eta_x \varphi_y)^2 + \varphi_y^2 \right) - g\eta \\
&= \frac{\xi_x \eta_x G(\eta)\xi + (G(\eta)\xi)^2}{1 + \eta_x^2} \\
&- \frac{\xi_x^2 - 2\xi_x \eta_x \varphi_y + (1 + \eta_x^2) \varphi_y^2}{2(1 + \eta_x^2)} - g\eta \\
&= \frac{\xi_x \eta_x G(\eta)\xi + (G(\eta)\xi)^2}{1 + \eta_x^2} \\
&- \frac{\xi_x^2 + 2\xi_x \eta_x G(\eta)\xi + (G(\eta)\xi)^2}{2(1 + \eta_x^2)} - g\eta.
\end{align*}

Therefore our original top boundary conditions (2.2) and (2.3) can be expressed in terms of \( \eta \) and \( \xi \) as

\begin{align*}
\eta_t &= G(\eta)\xi \\
\xi_t &= \frac{-\xi_x^2 + 2\xi_x \eta_x G(\eta)\xi + (G(\eta)\xi)^2}{2(1 + \eta_x^2)} - g\eta.
\end{align*}

We next show that the above two boundary conditions take the canonical form

\begin{equation}
\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \xi}
\end{equation}
\[ \frac{\partial \xi}{\partial t} = -\frac{\delta H}{\delta \eta}. \] (2.22)

This result was first obtained by Zakharov [14], see also [15], using a different method based on an infinite-dimensional analog of Legendre’s transform for the Lagrangian functional, see also [3]. The approach presented here is more direct and has the advantage of using only basic techniques of variational derivatives.

2.5 Functional Derivative of the Hamiltonian

We will now show that the water wave problem has the structure of a Hamiltonian system by proving that \( H \) satisfies (2.21) and (2.22). Recall that

\[ H(\eta, \xi) = \int_0^{2\pi} \frac{1}{2} \xi G(\eta) \xi + \frac{1}{2} g \eta^2 \, dx, \]

and let \( \xi + \varepsilon \tilde{\xi} \) be a small perturbation of \( \xi \). We compute

\[
H(\eta, \xi + \varepsilon \tilde{\xi}) = \int_0^{2\pi} \frac{1}{2} (\xi + \varepsilon \tilde{\xi}) G(\eta) (\xi + \varepsilon \tilde{\xi}) + \frac{1}{2} g \eta^2 \, dx \\
= \int_0^{2\pi} \frac{1}{2} \xi G(\eta) \xi + \frac{1}{2} g \eta^2 \, dx \\
+ \int_0^{2\pi} \frac{1}{2} \xi G(\eta) \varepsilon \tilde{\xi} + \frac{1}{2} \varepsilon \tilde{\xi} G(\eta) \xi \, dx \\
+ \int_0^{2\pi} \frac{1}{2} \varepsilon \tilde{\xi} G(\eta) \varepsilon \tilde{\xi} \, dx.
\]

The mixed terms involving \( \xi \) and \( \tilde{\xi} \) above are equal since \( G(\eta) \) is a symmetric operator. Furthermore, the third term is of order \( \varepsilon^2 \) and does not contribute to the first-order derivative. Thus,

\[
\left\langle \frac{\delta H}{\delta \xi}, \tilde{\xi} \right\rangle = \lim_{\varepsilon \to 0} \frac{H(\eta, \xi + \varepsilon \tilde{\xi}) - H(\eta, \xi)}{\varepsilon} = \int_0^{2\pi} \tilde{\xi} G(\eta) \xi \, dx,
\]

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where $\tilde{\xi}$ is arbitrary. Therefore,

$$\delta H = H(\xi + \varepsilon \tilde{\xi}) - H(\xi) = \varepsilon \int_{0}^{2\pi} \tilde{\xi} G(\eta) \xi \, dx.$$ 

Therefore,

$$\frac{\delta H}{\delta \xi} = G(\eta) \xi.$$ 

Comparing this to (2.19) leads to

$$\frac{\delta \eta}{\delta t} = \frac{\delta H}{\delta \xi}.$$ 

In order to compute the functional derivative of $H$ with respect to $\eta$ we revert back to the original form of the Hamiltonian

$$H = \int_{0}^{2\pi} \int_{-h}^{\eta(x)} \frac{1}{2} |\nabla \varphi|^2 \, dy \, dx + \int_{0}^{2\pi} \frac{1}{2} g \eta^2 \, dx. \quad (2.23)$$

Recalling the original boundary conditions (2.2) and (2.3), substituting these into (2.16) leads to the expression

$$\xi_t = \varphi_y (\varphi_y - \eta_x \varphi_x) - \frac{1}{2} |\varphi|^2 - g \eta. \quad (2.24)$$

We will show that

$$- \frac{\partial H}{\partial \eta} = \varphi_y (\varphi_y - \eta_x \varphi_x) - \frac{1}{2} |\varphi|^2 - g \eta,$$

where the first two terms correspond to the derivative of the kinetic energy, and the third term is the derivative of the potential energy.

Computing the functional derivative of $H$ with respect to $\eta$ is straightforward. Let
\( \eta(x) + \varepsilon \rho(x) \) be a small perturbation of the surface, then

\[
P(\eta + \varepsilon \rho) = \int_{0}^{2\pi} \frac{1}{2} g(\eta + \varepsilon \rho)^2 \, dx
= \int_{0}^{2\pi} \frac{1}{2} g(\eta^2 + 2\varepsilon \eta \rho + \varepsilon^2 \rho^2) \, dx
= \int_{0}^{2\pi} \frac{1}{2} g\eta^2 \, dx + \varepsilon \int_{0}^{2\pi} g\eta \rho \, dx + o(\varepsilon).
\]

Thus

\[
P(\eta + \varepsilon \rho) - P(\eta) = \varepsilon \int_{0}^{2\pi} g\eta \rho \, dx + o(\varepsilon).
\]

Dividing by \( \varepsilon \) and taking the limit as \( \varepsilon \to 0 \) yields

\[
\left< \frac{\delta P}{\delta \eta}, \rho \right> = \int_{0}^{2\pi} g\eta \rho \, dx,
\]

therefore

\[
\frac{\delta P}{\delta \eta} = g\eta.
\] (2.25)

Now consider the effect of the perturbation on the kinetic energy.

\[
K(\eta + \varepsilon \rho, \xi) = \int_{0}^{2\pi} \int_{-h}^{\eta(x) + \varepsilon \rho(x)} \frac{1}{2} |\nabla \varphi_\varepsilon|^2 \, dy \, dx,
\]

where \( \varphi_\varepsilon \) is a harmonic extension of \( \xi \) corresponding to the top boundary \( y = \eta(x) + \varepsilon \rho(x) \).

We have

\[
K(\eta + \varepsilon \rho, \xi) = K(\eta, \xi) + \int_{0}^{2\pi} \int_{-h}^{\eta + \varepsilon \rho} \frac{1}{2} |\nabla \varphi_\varepsilon|^2 \, dy \, dx - \int_{0}^{2\pi} \int_{-h}^{\eta} \frac{1}{2} |\nabla \varphi_\varepsilon|^2 \, dy \, dx
\]

\[
= \int_{0}^{2\pi} \int_{-h}^{\eta + \varepsilon \rho} \frac{1}{2} |\nabla \varphi_\varepsilon|^2 \, dy \, dx - \int_{0}^{2\pi} \int_{-h}^{\eta} \frac{1}{2} |\nabla \varphi_\varepsilon|^2 \, dy \, dx
\]

\[
+ \int_{0}^{2\pi} \int_{-h}^{\eta} \frac{1}{2} |\nabla \varphi_\varepsilon|^2 \, dy \, dx - \int_{0}^{2\pi} \int_{-h}^{\eta} \frac{1}{2} |\nabla \varphi|^2 \, dy \, dx.
\] (2.26)
Now consider (2.26),

\[
\int_0^{2\pi} \int_{-\eta}^{\eta(x) + \varepsilon \rho(x)} \frac{1}{2} |\nabla \varphi_\varepsilon|^2 \, dy \, dx - \int_0^{2\pi} \int_{-\eta}^{\eta(x)} \frac{1}{2} |\nabla \varphi_\varepsilon|^2 \, dy \, dx = \int_0^{2\pi} \int_{\eta(x)}^{\eta(x) + \varepsilon \rho(x)} \frac{1}{2} |\nabla \varphi_\varepsilon|^2 \, dy \, dx.
\]

(2.28)

(2.29)

Applying the mean value theorem yields,

\[
\int_{\eta}^{\eta(x) + \varepsilon \rho(x)} \frac{1}{2} |\nabla \varphi|^2 \, dy = \varepsilon \rho(x) \frac{1}{2} |\nabla \varphi_\varepsilon|^2(x, \eta(x) + \varepsilon \tilde{\rho}(x))
\]

where \(|\tilde{\rho}(x)| \leq |\rho(x)|\).

We assume that \(\varphi_\varepsilon\) and \(\nabla \varphi\) have Taylor expansions

\[
\varphi_\varepsilon(x, y) = \varphi(x, y) + \varepsilon \varphi^{(1)}(x, y) + o(\varepsilon)
\]

(2.30)

and

\[
\nabla \varphi_\varepsilon(x, y) = \nabla \varphi(x, y) + \varepsilon \nabla \varphi^{(1)}(x, y) + o(\varepsilon).
\]

(2.31)

Then

\[
\frac{1}{2} |\nabla \varphi_\varepsilon|^2(x, \eta(x) + \varepsilon \tilde{\rho}(x)) = \frac{1}{2} |\nabla \varphi|^2(x, \eta(x)) + \frac{1}{2} |\nabla \varphi^{(1)}|^2(x, \eta(x)) + \varepsilon \nabla \varphi \cdot \nabla \varphi^{(1)}(x, \eta(x)) + o(\varepsilon).
\]

This yields

\[
\int_{\eta(x)}^{\eta(x) + \varepsilon \rho(x)} \frac{1}{2} |\nabla \varphi_\varepsilon|^2 \, dy = \varepsilon \rho(x) \frac{1}{2} |\nabla \varphi|^2(x, \eta(x)) + o(\varepsilon).
\]

Thus,

\[
\int_0^{2\pi} \int_{\eta(x)}^{\eta(x) + \varepsilon \rho(x)} \frac{1}{2} |\nabla \varphi_\varepsilon|^2 \, dy \, dx = \varepsilon \int_0^{2\pi} \rho(x) \frac{1}{2} |\nabla \varphi|^2(x, \eta(x)) \, dx + o(\varepsilon).
\]
Now consider the difference of the two terms in (2.27)

\[
\int_0^{2\pi} \int_{-\frac{\eta}{2}}^{\eta} \frac{1}{2} |\nabla \varphi_{\varepsilon}|^2 dy \, dx - \int_0^{2\pi} \int_{-\frac{\eta}{2}}^{\eta} \frac{1}{2} |\nabla \varphi|^2 dy \, dx = \int_0^{2\pi} \int_{-\frac{\eta}{2}}^{\eta} \frac{1}{2} |\nabla \varphi_{\varepsilon}|^2 - \frac{1}{2} |\nabla \varphi|^2 dy \, dx.
\]

Using the Taylor expansion of \( \nabla \varphi_{\varepsilon} \) (2.31) we have

\[
|\nabla \varphi_{\varepsilon}|^2 = \nabla (\varphi + \varepsilon \varphi^{(1)}) \cdot \nabla (\varphi + \varepsilon \varphi^{(1)})
= \nabla \varphi \cdot \nabla \varphi + 2\varepsilon \nabla \varphi \cdot \nabla \varphi^{(1)} + o(\varepsilon)
\]

which implies

\[
\frac{1}{2} |\nabla \varphi_{\varepsilon}|^2 - \frac{1}{2} |\nabla \varphi|^2 = \varepsilon \nabla \varphi \cdot \nabla \varphi^{(1)} + o(\varepsilon).
\]

Thus, by applying Green’s first identity. The second term of (2.32) vanishes since \( \Delta \varphi = 0 \) in the fluid domain. In the first term, the side boundaries vanish because of the periodic boundary conditions and the bottom boundary contribution vanishes since \( \nabla \varphi \cdot n = 0 \) for \( y = -h \).

Thus, only the terms with the free surface boundary \( y = \eta(x) \) remain. Therefore,

\[
\int_0^{2\pi} \int_{-\frac{\eta}{2}}^{\eta} \frac{1}{2} |\nabla \varphi_{\varepsilon}|^2 - \frac{1}{2} |\nabla \varphi|^2 dy \, dx = \varepsilon \int_0^{2\pi} \int_{-\frac{\eta}{2}}^{\eta} \nabla \varphi \cdot \nabla \varphi^{(1)} dy \, dx + o(\varepsilon),
\]

(2.32)

Since \( \xi \) is fixed then for every \( \varepsilon \)

\[
\varphi_{\varepsilon}(x, \eta(x) + \varepsilon \rho(x)) = \xi(x).
\]

(2.33)
Substituting in the Taylor expansion of $\varphi_\varepsilon$,

$$\varphi(x, \eta(x) + \varepsilon \rho(x)) + \varepsilon \varphi^{(1)}(x, \eta(x) + \varepsilon \rho(x)) + o(\varepsilon) = \xi(x), \quad (2.34)$$

and additionally using

$$\varphi(x, \eta(x) + \varepsilon \rho(x)) = \varphi(x, \eta(x)) + \varepsilon \frac{\partial \varphi}{\partial y}(x, \eta(x)) \rho(x) + o(\varepsilon),$$

$$\varphi^{(1)}(x, \eta(x) + \varepsilon \rho(x)) = \varphi^{(1)}(x, \eta(x)) + \varepsilon \frac{\partial \varphi^{(1)}}{\partial y}(x, \eta(x)) \rho(x) + o(\varepsilon),$$

we obtain

$$\varphi(x, \eta(x)) + \varepsilon \frac{\partial \varphi}{\partial y}(x, \eta(x)) \rho(x) + \varepsilon \varphi^{(1)}(x, \eta(x)) + o(\varepsilon) = \xi(x). \quad (2.35)$$

Letting $\varepsilon = 0$, in (2.34) yields

$$\phi(x, \eta(x)) = \xi(x), \quad (2.36)$$

in agreement with the boundary condition at zero order. Using this, equation (2.35) becomes

$$\varepsilon \frac{\partial \varphi}{\partial y}(x, \eta(x)) \rho(x) + \varepsilon \varphi^{(1)}(x, \eta(x)) + o(\varepsilon) = 0. \quad (2.37)$$

Therefore, dividing by $\varepsilon$ and letting $\varepsilon \to 0$, we obtain

$$\varphi^{(1)}(x, \eta(x)) = - \rho(x) \frac{\partial \varphi}{\partial y}(x, \eta(x)). \quad (2.38)$$

Thus,

$$\int_0^{2\pi} \varepsilon \varphi^{(1)}(\nabla \varphi \cdot \vec{n}) \, dx = - \int_0^{2\pi} \varepsilon \rho(x) \frac{\partial \varphi}{\partial y}(x, \eta) \left( \partial_y \varphi - \partial_x \eta \partial_y \varphi \right) \, dx. \quad (2.39)$$
Combining (2.5) and (2.39) we get that

\[ \frac{\delta K}{\delta \eta} = \frac{1}{2} |\nabla \varphi|^2 - \partial_y \varphi \left( \partial_y \varphi - \partial_x \eta \partial_y \varphi \right). \]  

(2.40)

Therefore,

\[ -\frac{\delta H}{\delta \eta} = -\frac{\delta K}{\delta \eta} - \frac{\delta P}{\delta \eta} \]

(2.41)

\[ = \partial_y \varphi \left( \partial_y \varphi - \partial_x \eta \partial_y \varphi \right) - \frac{1}{2} |\nabla \varphi|^2 - g \eta \]

(2.42)

\[ = \partial_t \xi. \]

(2.43)

Thus, the second boundary condition of the water wave problem is shown to have the canonical Hamiltonian form (2.22).

2.6 Taylor Expansion of the Dirichlet-Neumann operator

Theorem [3]: The Dirichlet-Neumann operator \( G(\eta) \) is a bounded mapping \( H^1(0, 2\pi) \to L^2(0, 2\pi) \) which is analytic in \( \eta \) with values in the class of operators from \( H^1 \) to \( L^2 \), at least on a set of \( \eta \in C[0, 2\pi] \) such that for some \( R > 0 \), \( |\eta(x)| \leq R \), and \( \frac{|\eta(x) - \eta(y)|}{|x - y|} \leq R \), \( x, y \in [0, 2\pi] \).

The proof of this theorem follows from general results in harmonic analysis by Coifman and Meyer [2] and is beyond the scope of this thesis. Nevertheless, the analyticity of operator \( G(\eta) \) justifies the representation of \( G(\eta) \) in terms of its Taylor series

\[ G(\eta) = \sum_{j \geq 0} G_j(\eta), \]

(2.44)

where each \( G_j(\eta) \) is homogeneous of degree \( j \) in \( \eta \). Substituting the Taylor expansion of \( G(\eta) \) into the Hamiltonian (2.13) leads to
\[ H = \int_0^{2\pi} \frac{1}{2} \xi G(\eta)\xi + \frac{1}{2} g\eta^2 \, dx \]

\[ = \int_0^{2\pi} \left( \sum_{j \geq 0} G_j(\eta)\xi \right) \xi + \frac{1}{2} g\eta^2 \, dx \]

\[ = \int_0^{2\pi} \frac{1}{2} \xi G_0\xi + \frac{1}{2} g\eta^2 \, dx + \sum_{j \geq 3} \int_0^{2\pi} \frac{1}{2} \xi G_{j-2}(\eta)\xi \, dx \]

\[ = H_2(\eta, \xi) + \sum_{j \geq 3} H_j(\eta, \xi). \]

Thus \( H \) can be expressed as the sum of the quadratic term, \( H_2(\eta, \xi) \), and the higher order terms, \( H_j \) for \( j \geq 3 \). For the higher order terms the non-linear terms of \( G_{j-2} \) can be computed by a recursive formula. We first write out the first few terms of \( G_k \) explicitly. They are

\[
G_0 = D \tanh(hD),
\]

\[
G_1(\eta) = D\eta D - G_0\eta G_0,
\]

\[
G_2(\eta) = -\frac{1}{2} \left( G_0\eta^2 D^2 + D^2\eta^2 G_0 - 2G_0\eta G_0\eta G_0 \right),
\]

\[
G_3(\eta) = \frac{1}{6} D\eta^3 D^3 - \frac{1}{6} G_0\eta^3 D^2 G_0 - \frac{1}{2} G_1\eta^2 D^2 - G_2\eta G_0,
\]

where \( D = -i\partial_x \).

In this paper we will explicitly show how \( G_0 \) and \( G_1 \) are computed. The higher-order terms can then be calculated using the same method in an iterative manner.

Consider the Taylor series (2.44). The operator \( G_0 \) corresponds to a flat surface defined by \( \eta(x) = 0, \ x \in [0, 2\pi] \). In this case, we can solve for \( G_0 \) by separation of variables. Assume that the solution takes the form

\[
\varphi(x, y) = X(x)Y(y).
\]

\[38\]
Laplace’s equation $\Delta \varphi = 0$ implies

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0$$

or equivalently

$$- \frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{Y} \frac{d^2 Y}{dy^2}.$$ 

Since this solution must hold for all choices of $x$ and $y$,

$$-1 \frac{d^2 X}{X \ dx^2} = \frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda$$

for some separation constant $\lambda \in \mathbb{R}$. Thus, we obtain the following system of differential equations

$$\frac{d^2 X}{dx^2} + \lambda X = 0,$$

$$\frac{d^2 Y}{dy^2} - \lambda Y = 0.$$

Applying the periodic boundary conditions:

$$X(0) = X(2\pi)$$

$$X'(0) = X'(2\pi)$$
implies that solutions only exist when $\lambda > 0$. Therefore, let $\lambda = k^2$ for some $k \in \mathbb{R}$. So we have the general form of the solution

$$X(x) = c_1 \cos(kx) + c_2 \sin(kx).$$

Once again applying the periodic condition yields

$$c_1 = c_1 \cos(2\pi k),$$

which implies that $\cos(2\pi k) = 1$ which is true for $k = 0, 1, 2, \ldots$. The general form of $Y$ can be taken as

$$Y(y) = \tilde{c}_1 \cosh(k(y + h)) + \tilde{c}_2 \sinh(k(y + h)),$$

since $Y$ is a solution implies that $Y$ shifted by $-h$ is also a solution.

Differentiating and evaluating at $y = -h$,

$$Y(y) = \tilde{c}_1 \cosh(k(y)).$$

Therefore, separated solutions are obtained in the form

$$\varphi_k(x, y) = a_k \cos(kx) \cosh(k(y + h)) + b_k \sin(kx) \cosh(k(y + h)),$$

or, using complex form,

$$\varphi_k(x, y) = c_1 \exp(ikx) \cosh(k(y + h)), \quad k \in \mathbb{Z}.$$  

Applying the top boundary condition $\varphi(x, 0) = \xi(x)$,

$$\xi_k(x) = c_1 \exp(ikx) \cosh(kh).$$
Therefore, if \( \xi \) has the form
\[
\xi_k(x) = \exp(ikx)
\]
then
\[
\varphi_k(x, y) = \exp(ikx) \frac{\cosh(k(y + h))}{\cosh(hk)}.
\]

Since any \( \xi \) can be represented as a series in \( \exp(ikx) \)
\[
\xi(x) = \sum_{k=-\infty}^{\infty} c_k \exp(ikx),
\]
we obtain
\[
\varphi(x, y) = \sum_{k=-\infty}^{\infty} c_k \exp(ikx) \frac{\cosh(k(y + h))}{\cosh(hk)}.
\]

Computing the normal derivative and evaluating on the surface we obtain
\[
G_0 \xi(x) = \left. \frac{\partial \varphi}{\partial y} \right|_{y=0} = \sum_{k=-\infty}^{\infty} c_k \exp(ikx) k \sinh(k(y + h)) \frac{\cosh(hk)}{\cosh(hk)} \bigg|_{y=0} = \sum_{k=-\infty}^{\infty} c_k k \tanh(hk) \exp(ikx).
\]

Therefore, \( G_0 = D \tanh(hD) \) where \( D = -i\partial_x \).

To compute higher-order terms of \( G(\eta) \) we make use of the fact that any \( \varphi \) harmonic in \( \Gamma \) and satisfying the Neumann boundary conditions at the bottom and periodic conditions on the sides can be represented by a series with terms corresponding to \( \varphi(x, 0) = \exp(ikx) \), for \( k \in \mathbb{Z} \). Therefore, we study the general form,
\[
\varphi_k(x, y) = \exp(ikx) \cosh(k(y + h)), \quad k \in \mathbb{Z}.
\]
Computing the partial derivatives with respect to $x$ and $y$ leads to

$$
\nabla \varphi_k(x, y) = (ik \exp(ikx) \cosh(ky + h), k \exp(ikx) \sinh(ky + h))).
$$

From the expression for $\partial_y \varphi_k$ we see that $\varphi_k$ satisfies the bottom boundary condition $\frac{\partial \varphi_k}{\partial y} \bigg|_{(x,-h)} = 0$, and clearly $\varphi$ is $2\pi$-periodic in $x$. Substituting $\varphi_k(x, y)$ into

$$
G(\eta) \varphi_k(x, \eta) = \partial_y \varphi_k - \partial_x \eta \partial_x \varphi_k
$$

yields

$$
G(\eta) \varphi_k(x, \eta) = k \exp(ikx) [\sinh(k(\eta(x) + h)) - i\eta x \cosh(k(\eta(x) + h))],
$$

from where we see that the Taylor expansion of $G(\eta)$ must satisfy

$$
\left( \sum_{j=0}^{\infty} G_j(\eta) \right) \exp(ikx) \cosh(k(\eta(x) + h)) = k \exp(ikx) [\sinh(k(\eta(x) + h)) - i\eta x \cosh(k(\eta(x) + h))].
$$

(2.45)

This relation allows us to compute expressions for $G_k(\eta)$ using the Taylor expansions of $\cosh(k(\eta + h))$ and $\sinh(k(\eta + h))$ about $\eta = 0$

$$
\cosh(k(\eta + h)) = \cosh(kh) + k \sinh(kh) \eta + \frac{k^2}{2!} \cosh(kh) \eta^2 + ...
$$

$$
= \sum_{j \text{ even}} \frac{(k\eta)^j}{j!} \cosh(kh) + \sum_{j \text{ odd}} \frac{(k\eta)^j}{j!} \sinh(kh),
$$

and

$$
\sinh(k(\eta + h)) = \sinh(kh) + k \cosh(kh) \eta + \frac{k^2}{2!} \sinh(kh) \eta^2 + ...
$$

$$
= \sum_{j \text{ even}} \frac{(k\eta)^j}{j!} \sinh(kh) + \sum_{j \text{ odd}} \frac{(k\eta)^j}{j!} \cosh(kh)
$$

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Substituting the above Taylor expansions into (2.45) leads to

\[
\left( \sum_{j=0}^{\infty} G_j(\eta) \right) \exp(ikx) \left( \sum_{j \text{ even}} \frac{(k\eta)^j}{j!} \cosh(hk) + \sum_{j \text{ odd}} \frac{(k\eta)^j}{j!} \sinh(hk) \right) = 
\]

\[
k \exp(ikx) \left( \sum_{j \text{ even}} \frac{(k\eta)^j}{j!} \sinh(hk) + \sum_{j \text{ odd}} \frac{(k\eta)^j}{j!} \cosh(hk) \right) - 
\]

\[
i\eta x k \exp(ikx) \left( \sum_{j \text{ even}} \frac{(k\eta)^j}{j!} \cosh(hk) + \sum_{j \text{ odd}} \frac{(k\eta)^j}{j!} \sinh(hk) \right) .
\]

(2.47)

Thus we can recursively solve for \(G_j(\eta)\) by equating the the same powers of \(\eta\) in the above expression beginning with \(G_0\). Equating the constant terms in the expression:

\[
G_0 \exp(ikx) \cosh(hk) = k \sinh(hk) \exp(ikx)
\]

⇔ \(G_0 \exp(ikx) = k \tanh(hk) \exp(ikx)\). 

Therefore \(G_0 = D \tanh(hD)\).

For the derivation of \(G_1(\eta)\) we equate all the first order terms of (2.47):

\[
G_1(\eta) \cosh(hk) \exp(ikx) + G_0 \sinh(hk)(k\eta) \exp(ikx) = k \exp(ikx) \cosh(hk)(k\eta) - i\eta x k \exp(ikx) \cosh(hk).
\]

(2.48)
Dividing by $\cosh(hk)$,

$$G_1(\eta) \exp(ikx) = k^2\eta \exp(ikx) - i\eta_x k \exp(ikx)$$
$$- G_0 \eta k \tanh(hk) \exp(ikx)$$
$$= \eta(- i \partial_x k \exp(ikx)) - i \partial_x \eta k \exp(ikx)$$
$$- G_0 \eta G_0 \exp(ikx)$$
$$= - i \partial_x (\eta k \exp(ikx)) - G_0 \eta G_0 \exp(ikx)$$
$$= - i \partial_x (\eta(- i \partial_x \exp(ikx))) - G_0 \eta G_0 \exp(ikx)$$
$$= D\eta D \exp(ikx) - G_0 \eta G_0 \exp(ikx)$$

The higher terms in the expansion can be derived in the same manner by equating terms of equal order. Furthermore, there exists a recursion formula for higher-order terms [5, p. 75]. For $j > 0$ even,

$$G_j(\eta) = \frac{1}{j!} \left( \eta^j D^{j+1} \tanh(hD) - i(\eta^j)_x D^j \tanh(hD) \right)$$
$$- \sum_{l<j \text{ even}} G_l(\eta) \frac{1}{(j-l)!} \eta^{j-l} D^{j-l}$$
$$- \sum_{l<j \text{ odd}} G_l(\eta) \frac{1}{(j-l)!} \eta^{j-l} D^{j-l} \tanh(hD)$$

(2.50)
and for $j$ odd,

$$G_j(\eta) = \frac{1}{j!} \left( \eta^j D^{j+1} - i(\eta^j)_x D^j \right) - \sum_{l < j \text{ odd}} G_l(\eta) \frac{1}{(j-l)!} \eta^{j-l} D^{j-l} - \sum_{l < j \text{ even}} G_l(\eta) \frac{1}{(j-l)!} \eta^{j-l} D^{j-l} \tanh(hD).$$

2.7 Linear Theory

If we go back to the top boundary conditions, (2.19) and (2.20), it is clear that the constant zero functions $\eta = 0$ and $\xi = 0$ are equilibrium solutions. Equations linearized about zero may be expressed in terms of the quadratic term of the Hamiltonian

$$H_2(\eta, \xi) = \int_0^{2\pi} \frac{1}{2}(\xi G_0 \xi + g\eta^2) \, dx. \quad (2.51)$$

We now apply the Fourier transform to $H_2$ to simplify it and express it as a function of the Fourier coefficients of $\eta$ and $\xi$. Given a function $f(x)$, $x \in (0, 2\pi)$ define the Fourier coefficients,

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) \exp(-ikx) \, dx, \quad (2.52)$$
corresponding to the Fourier series expansion (“the inverse Fourier transformation”)

$$f(x) = \sum_{k=-\infty}^{\infty} f_k \exp(ikx). \quad (2.53)$$

Theorem (Plancherel’s Identity): Let $f, g$ be square integrable functions on $(0, 2\pi)$. Then

$$\int_0^{2\pi} f(x) \overline{g(x)} \, dx = 2\pi \sum_{k=-\infty}^{\infty} f_k \overline{g_k}. \quad (2.54)$$
In particular, if \( f = g \) then

\[
\int_0^{2\pi} |f(x)|^2 \, dx = 2\pi \sum_{k=-\infty}^{\infty} |f_k|^2,
\] (2.55)

known as Parseval’s identity.

Proof: The proof involves substituting in the transformations of \( f \) and \( g \):

\[
\int_0^{2\pi} f(x)\overline{g(x)} \, dx = \frac{1}{(2\pi)^2} \int_0^{2\pi} \left( \sum_{k=-\infty}^{\infty} f_k \exp(ikx) \right) \left( \sum_{l=-\infty}^{\infty} g_l \exp(ilx) \right) \, dx
\]

\[
= \frac{1}{(2\pi)^2} \sum_{k,l} f_k g_l \int_0^{2\pi} \exp(i(k-l)x) \, dx. \tag{2.56}
\]

The function \( \exp(i(k-l)x) \) satisfies

\[
\int_0^{2\pi} \exp(i(k-l)x) \, dx = \begin{cases} 
2\pi, & k = l \\
\left[ \frac{\exp(i(k-l)x)}{i(k-l)} \right]_{x=0}^{x=2\pi}, & k \neq l
\end{cases}
\]

thus the only nonzero terms of the sum (2.56) are those with \( k = l \), and we obtain

\[
\int_0^{2\pi} f(x)\overline{g(x)} \, dx = 2\pi \sum_{k=-\infty}^{\infty} f_k \overline{g_k}. \tag{2.57}
\]

\[ \square \]

Applying Plancherel’s Identity to \( H_2 \) leads to

\[
H_2(\eta, \xi) = \int_0^{2\pi} \frac{1}{2} \xi G_0 \xi + \frac{1}{2} g \eta^2 \, dx
\] (2.58)

\[
= 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{2} k \tanh(hk) |\xi_k|^2 + \frac{1}{2} g |\eta_k|^2. \tag{2.59}
\]
Therefore,
\[
\dot{\eta}_k = \frac{1}{2\pi} \frac{\delta H}{\delta \eta_k} = k \tanh(hk) \xi_k
\]
(2.60)
\[
\dot{\xi}_k = -\frac{1}{2\pi} \frac{\delta H}{\delta \xi_k} = -g \eta_k
\]
(2.61)
where the dot denotes the derivative with respect to $t$.

Proposition: The water wave problem linearized about equilibrium is a harmonic oscillator of infinitely many degrees of freedom, with frequencies
\[
\omega_k = \sqrt{gk \tanh(hk)}, \quad k = 1, 2, ....
\]
(2.62)

In the case of deep water, $h = \infty$, the frequencies are
\[
\omega_k = \sqrt{g|k|}, \quad k = 1, 2, ....
\]
(2.63)

Proof: Taking the derivative with respect to $t$ of (2.60) we obtain
\[
\ddot{\eta}_k = k \tanh(hk) \dot{\xi}_k.
\]

Now substitute in (2.61) to get a second order differential equation
\[
\ddot{\eta}_k + gk \tanh(hk) \eta_k = 0.
\]

Thus, the solution $\eta_k(t)$ can be obtained in the form
\[
\eta_k(t) = c_1 \cos(\omega_k t) + c_2 \sin(\omega_k t),
\]
where $\omega_k$ is given by (2.63). Since $\xi_k(t) = \frac{g}{\omega_k^2} \dot{\eta}_k$, we have

$$\dot{\xi}_k(t) = -\frac{c_1 g}{\omega_k} \sin(\omega_k t) + \frac{c_2 g}{\omega_k} \cos(\omega_k t).$$

The values of the constants $c_1, c_2$ can be expressed through the initial data $\eta_k(0), \xi_k(0)$, thus giving the solution of the form

$$\dot{\eta}_k(t) = \eta_k(0) \cos(\omega_k t) + \frac{\omega_k \xi_k(0)}{g} \sin(\omega_k t)$$

$$\dot{\xi}_k(t) = -\frac{g}{\omega_k} \eta_k(0) \sin(\omega_k t) + \xi_k(0) \cos(\omega_k t).$$

The asymptotic behavior of $\tanh(hk)$ gives the result for infinite depth.

2.8 Higher Order Terms

We now expand on the linear model established in the previous section to include both the quadratic and the cubic terms in the Hamiltonian function. Let $H = H_2 + H_3$:

$$H = \int_0^{2\pi} \frac{1}{2} \xi G_0 \xi + \frac{1}{2} g \eta^2 dx + \int_0^{2\pi} \frac{1}{2} \xi G_1(\eta) \xi dx.$$

We use the same approach based on Fourier transform as in the previous section in order to express $H$ as function of the Fourier coefficients. Thus what remains to be shown is the transformation of $H_3$.

Recall from Section 2.5 that

$$G_1(\eta) \xi = D\eta D\xi - G_0 \eta G_0 \xi.$$  (2.64)
Therefore

\[ H_3 = \int_0^{2\pi} \frac{1}{2} \xi G_1(\eta) \xi \, dx \quad (2.65) \]

\[ = \frac{1}{2} \int_0^{2\pi} \xi D\eta D\xi \, dx - \frac{1}{2} \int_0^{2\pi} \xi G_0 \eta G_0 \xi \, dx. \quad (2.66) \]

Before proceeding any further we offer a word of caution in interpreting (2.66). It is understood that the expressions are compositions of operators and not products. In other words

\[ D\eta D\xi = D(\eta(D\xi)). \]

We begin with the transform of the first term. Let \((\eta D\xi)_k\) represent the sequence of Fourier coefficients (the finite interval “Fourier transform”) of \(\eta D\xi\), thus

\[ (\eta D\xi)_k = 2\pi \sum_{l=-\infty}^{\infty} \eta_l (D\xi)_{k-l} = 2\pi \sum_{l=-\infty}^{\infty} (k-l)\eta_l \xi_{k-l}, \]

recalling that \(D = -i\partial_x\) thus \((D(\xi))_k = k\xi_k\). Therefore,

\[ \int_0^{2\pi} \xi D\eta D\xi \, dx = 2\pi \sum_{k=-\infty}^{\infty} \xi_k (D\eta D\xi)_k = 2\pi \sum_{k,l=-\infty}^{\infty} k(k-l) \xi_k \xi_{k-l} \xi_{k-l} \xi_{k-l}. \quad (2.67) \]

Computing similarly the second term in (2.66) we obtain

\[ \int_0^{2\pi} \xi G_0 \eta G_0 \, dx = 2\pi \sum_{k,l=-\infty}^{\infty} \tanh(hk)(k-l) \tanh(h(k-l)) \xi_k \xi_{k-l} \xi_{k-l} \xi_{k-l}. \quad (2.68) \]

Thus combining (2.67) and (2.68) we arrive at the expression of \(H_3\) using the transformed variables:

\[ H_3 = 2\pi \sum_{k,l=-\infty}^{\infty} \frac{1}{2} [k(k-l) - k \tanh(hk)(k-l) \tanh(h(k-l))] \xi_k \xi_{k-l} \xi_{k-l} \xi_{k-l}. \quad (2.69) \]
The derivatives of $H_3$ are expressed most easily using the notion of discrete convolution of sequences.

Definition: Let $a_k$ and $b_k$ be an infinite sequences for $k \in \mathbb{Z}$, then their convolution, denoted $a_k * b_k$ is defined as

$$a_k * b_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}, \quad k \in \mathbb{Z}. \quad (2.70)$$

Computing the derivative of $H_3$ with respect to $\eta_l$, we obtain

$$\frac{1}{2\pi} \frac{\partial H_3}{\partial \eta_l} = \frac{1}{2} \sum_{k=-\infty}^{\infty} (k(k - l) - k \tanh(hk)(k - l) \tanh(h(k - l))) \xi_k \xi_{k-l}$$

$$= -\frac{1}{2} \sum_{k=-\infty}^{\infty} k(l - k) \xi_k \xi_{l-k}$$

$$- \frac{1}{2} \sum_{k=-\infty}^{\infty} k \tanh(hk)(l - k) \tanh(h(l - k)) \xi_k \xi_{l-k}$$

$$= -\frac{1}{2} \sum_{k=-\infty}^{\infty} k \xi_k (l - k) \xi_{l-k} \quad (2.71)$$

$$- \frac{1}{2} \sum_{k=-\infty}^{\infty} k \tanh(hk) \xi_k (l - k) \tanh(h(l - k)) \xi_{l-k}. $$

The convolution notation allows us to express

$$\frac{1}{2\pi} \frac{\partial H_3}{\partial \eta_l} = -\frac{1}{2} [(k\xi_k) * (k\xi_k) + (k \tanh(hk)\xi_k) * (k \tanh(hk)\xi_k)]. \quad (2.72)$$

To compute the derivative with respect to $\xi_m$ we notice that all the terms of (2.69) will
have zero contribution except when \( m = k - l \):

\[
\frac{1}{2\pi} \frac{\partial H_3}{\partial \xi_m} = \frac{1}{2} \sum_{l=-\infty}^{\infty} (m + l) m \xi_{m+l} \eta_l
\]

\[
- \frac{1}{2} \sum_{l=-\infty}^{\infty} (m + l) m \tanh(h(m + l)) \tanh(hm) \xi_{m+l} \eta_l
\]

\[
= \frac{1}{2} m \sum_{l=-\infty}^{\infty} (m - l) \xi_{m-l} \eta_l
\]

\[
- \frac{1}{2} m \tanh(hm) \sum_{l=-\infty}^{\infty} (m - l) \tanh(h(m - l)) \xi_{m-l} \eta_l.
\]

Expressing the above expression as a convolution yields

\[
\frac{1}{2\pi} \frac{\partial H_3}{\partial \xi_k} = \frac{1}{2} k(k \xi_k) * (\eta_k) - \frac{1}{2} k \tanh(hk)(k \tanh(hk) \xi_k * (\eta_k)). \tag{2.73}
\]

Therefore, combining the derivative of the quadratic and cubic terms of the Hamiltonian:

\[
\frac{d\eta_k}{dt} = \frac{1}{2\pi} \frac{\partial}{\partial \xi_k} (H_2 + H_3)
\]

\[
= k \tanh(hk) \xi_k + k(k \xi_k) * (\eta_k)
\]

\[
- k \tanh(hk))(k \tanh(hk) \xi_k) * (\eta_k), \tag{2.74}
\]

\[
\frac{d\xi_k}{dt} = -\frac{1}{2\pi} \frac{\partial}{\partial \eta_k} (H_2 + H_3)
\]

\[
= -g \eta_k + \frac{1}{2}(k \xi_k) * (k \xi_k) \tag{2.75}
\]

\[
+ \frac{1}{2}(k \tanh(hk) \xi_k) * (k \tanh(hk) \xi_k).
\]
In this section we will discuss how the linear and higher-order theory for the Hamiltonian can be implemented in MATLAB using built-in Fast Fourier algorithm and Runge-Kutta fourth-order solver.

3.1 Discrete Fourier Transform

Thus far we have managed to transform our water wave problem into a relatively simple system of differential equations in terms of each set of Fourier coefficients, $\eta_k$ and $\xi_k$. Therefore, what we now require is a computationally efficient method of computing Fourier coefficients (finite interval "Fourier transform") and the corresponding inverse Fourier transforms.

Given an $L$ periodic function $f(x)$ for $x \in [0, L]$, consider its Fourier coefficients

$$c_k = \frac{1}{L} \int_0^L f(x) \exp \left( -2\pi i \frac{kx}{L} \right) \, dx, \quad k \in \mathbb{Z}. \quad (3.1)$$

For a finite number of Fourier coefficients, it is certainly possible to approximate each Fourier coefficient directly using MATLAB’s integral function. However, such a straightforward method, despite its relative ease of programming, is horribly inefficient especially when the number of Fourier coefficients gets large. So instead we turn to the discrete Fourier transform [11].

Given a finite sequence $(f_0, f_1, f_2, \ldots, f_{N-1})$, the discrete Fourier transform is defined as

$$F_k = \sum_{n=0}^{N-1} f_n \exp \left( -2\pi i \frac{kn}{N} \right), \quad k = 0, 1, \ldots, N - 1 \quad (3.2)$$

(in fact this definition can be extended to $k \in \mathbb{Z}$; the result is a sequence which is $N$-
periodic). Now let us compare this with the expression for the “left-endpoint” Riemann sum approximation of (3.1) by equally dividing the interval $[0, L]$ into $N$ subintervals, and taking $x_n = \frac{nL}{N}$:

$$
\frac{1}{L} \int_0^L f(x) \exp \left( -2\pi i \frac{kx}{L} \right) \, dx \approx \frac{1}{L} \sum_{n=0}^{N-1} f(x_n) \exp \left( -2\pi i \frac{kn}{N} \right) \frac{L}{N} \quad (3.3)
$$

$$
= \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \exp \left( -2\pi i \frac{kn}{N} \right). \quad (3.4)
$$

Thus, we obtain the relation between the Fourier coefficients and the discrete Fourier transform of $f(x)$:

$$
c_k = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n) \exp \left( -2\pi i \frac{kn}{N} \right), \quad (3.5)
$$

which means that for $k$ fixed, and for $N$ sufficiently large the Discrete Fourier transform multiplied by a factor $1/N$ will be a close approximation of the Fourier coefficients.

For $N$ fixed, the Discrete Fourier transform is a linear operator on $\mathbb{C}^N$. The inverse discrete Fourier transform is given by the following expression (see for example [11]):

$$
f_k = \sum_{n=0}^{N-1} F_n \exp \left( 2\pi i \frac{kn}{N} \right). \quad (3.6)
$$

The Discrete Fourier transform and its inverse can be computed using the native MATLAB functions $\texttt{fft()}$ and $\texttt{ifft()}$ respectively. Not only do these two functions save us the time of writing our own algorithms but they utilize the fast Fourier algorithm, which as its name suggests is “fast.” Given a vector of length $N$, a direct computation of discrete Fourier transform requires $2N^2$ floating point operations, not including the computation of the complex roots of unity. The fast Fourier algorithm utilized by MATLAB requires $3N \log_2 N$ floating point operations. If $N = 1024$ then the direct method for computing the transform will take at least 2.1 million operations. At the same time, the fast Fourier
algorithm takes 21,293 operations so it is 98 times more efficient. Figure 3.1 plots the number of floating point operations necessary using both methods.

![Fast Fourier Algorithm Efficiency](image)

Figure 3.1: Theoretical efficiency of the Fast Fourier algorithm compared with direct computation of DFT.

Lastly, but of equal importance is how to correctly interpret MATLAB’s `fft()` output. Passing a vector through the `fft` function returns the Discrete Fourier transform vector $(F_0, F_1, ..., F_{N-1})$. However, once multiplied by a factor of $1/N$ the values from $N/2$ to $N-1$ are not the approximation for $c_{N/2}$ to $c_{N-1}$. They are the approximation for $c_{N/2-N}$ to $c_{-1}$. Or in other words, the resultant vectors are approximations for the Fourier coefficients

$$(c_0, c_1, ..., c_{N/2-1}, c_{-N/2}, ..., c_{-1}).$$

Since for a real-valued sequence the Fourier coefficients are complex conjugates of their negative index counterparts, i.e. $c_k = \overline{c_{-k}}$ the `fft()` function in MATLAB gives us $N/2 + 1$ “useful” Fourier coefficients. The zero index and the $-N/2$ index term appear only once, thus we have approximations for $c_n$ for $n = 0, \pm 1, \pm 2, ..., \pm N/2 - 1, -N/2$. The benefit of this is that should we desire, we could effectively cut by about half the num-
ber of computations required to solve the system of differential equations for the Fourier coefficients.

We now illustrate the above technique with an elementary periodic function from 0 to $2\pi$. Let $N = 8$ and $f(x) = \sin(x)$. The values of $f$ computed at the eight values of $x = 2\pi k / 8$ for $k = 0, 1, \ldots, 7$ are $\{0, 0.7071, 1, 0, -0.7071, -1, -0.7071\}$. Taking the \texttt{fft()} of this row yields $\{0, 4i, 0, 0, 0, 0, -4i\}$. Thus, our Fourier coefficients are $1/8$ times these values: we have $c_1 = i/2$ and $c_{-1} = -i/2$, while the remaining coefficients are zero. In fact these are precisely the coefficients we expected to get, since

$$\sin(x) = \sum_{k=-4}^{3} c_k \exp(ikx)$$

$$= c_1 \exp(ix) + c_{-1} \exp(-ix)$$

$$= \frac{\exp(ix) - \exp(-ix)}{2i}.$$  

3.2 The Fourth Order Runge-Kutta Method

From the linear theory we established the system of ordinary differential equations

$$\dot{\eta}_k = k \tanh(hk)\xi_k \quad \text{and} \quad \dot{\xi}_k = -g\eta_k.$$  

Although, this system can be solved exactly, we use MATLAB’s \texttt{ode45()} solver which implements a fourth-order Runge-Kutta method. This is done for two reasons: in order establish a foundation that can be built upon for coding higher order non-linear terms and to compare the accuracy of the \texttt{ode45} solution to the exact solution which can be calculated separately.

We implement the \texttt{ode45()} solution in the following way. We first begin by creating a MATLAB function that encompasses our system of differential equations for the Fourier coefficients. Explicitly, our system can be described in matrix form as:
\[
\begin{pmatrix}
\dot{\eta}_k \\
\dot{\xi}_k
\end{pmatrix} =
\begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
\begin{pmatrix}
\eta_k \\
\xi_k
\end{pmatrix}
\]  
(3.8)

where \( A \) is a diagonal matrix with nonzero elements \( a_{(j,j)} = k_j \tanh(hk_j) \), where \( k_j \) indicates the proper \( k \)-th index as described in the previous section and \( B \) is the diagonal matrix with all diagonal entries equal to \(-g\).

If \( \eta_0 \) and \( \xi_0 \) are two functions that initially define the fluid surface and velocity potential, we first take the discrete Fourier transform of each. We then adjoin the two rows of computed coefficients into a single row where the first half of the entries represents \( \eta_k \) and the second half represents \( \xi_k \). This row, the function describing the right-hand side of the system, and a terminal time value are then passed through \texttt{ode45()} function.

What we get back from \texttt{ode45()} solver is a column vector \( T \) of length equal to the total number of time steps that the \texttt{ode45()} function generated and a matrix \( X \) where each row represents the Fourier coefficients of \( \eta \) and \( \xi \) at time \( t \) corresponding to the row entry in \( T \). We then extract the \( \eta_k \) and \( \xi_k \) values out of \( X \) and take the inverse Fourier transform of the coefficients at each time step and multiply by \( N \) to get real-valued data. Finally, we can plot the two-dimensional data of \( \eta \) and \( \xi \) versus time as a mesh plot in MATLAB.

For the higher-order model an additional column vector is added to the right-hand side of (3.8) which is computed using the discrete convolution operations. In doing so, we apply the Fourier Convolution Theorem in order to compute the convolution of two sequences in the most computationally effective manner known to us.

3.3 Numerical Results

The goal of the numerical simulation in this section is to show that the Hamiltonian formulation for the water wave problem is robust and can be applied to different time-
dependent models. The examples we have chosen to simulate are the modulated wave packet and a second-order approximation to the Stokes’ wave also used in [5].

Example 1: Let the initial top surface of the water wave be represented by a slowly modulated wave packet function

$$\eta_0(x) = 0.01 \exp\left(-(4/3)(x - \pi)^2\right) \cos(10x),$$

(3.9)

and we take the initial velocity potential at the surface to be zero and \(h = 1\). Figure 3.2 shows the evolution of the modulated wave packet for \(t \in [0, 6]\), while Figure 3.3 shows the solution up to \(t = 100\).

Figure 3.2: Solution \(\eta(x, t)\) corresponding to \(\eta_0(x) = 0.01 \exp\left(-(4/3)(x - \pi)^2\right) \cos(10x), \xi_0(x) = 0, h = 1,\) for \(t \in [0, 6]\).

We can also analyze the frequency of sinusoidal waves comprising the modulated wave packets by examining the real and imaginary components of \(\eta_k\). Figure 3.4 reveals that the real part of \(\eta_k\) are all zero except for \(\eta_6\) to \(\eta_{14}\) and by symmetry \(\eta_{-14}\) to \(\eta_{-6}\). Thus the frequency of the waves which is given by \(\omega(k) = \sqrt{gk \tanh(hk)}\) implies that modulated wave packet is comprised of waves of frequency from 1.22 to 1.86 hz. On the other hand,
Figure 3.3: Solution $\eta(x, t)$ corresponding to $\eta_0(x) = 0.01 \exp\left(-\frac{4}{3}(x - \pi)^2\right) \cos(10x)$, $\xi = 0$, $h = 1$, for $t \in [0, 100]$.

Figure 3.4: Real part of $\eta_k$ at $t = 0$. 
the imaginary part of $\eta_k$ are negligibly small (they have values on the order of numerical noise).

We now compare the computed ODE45 solution to the exact solution for the linear model. We have that

$$\dot{\eta}_k = k \tanh(hk)\xi_k \quad \text{and} \quad \dot{\xi}_k = -g\eta.$$ 

We refer to the proof of the proposition in Section 2.7 for the exact solution to the linear model.

![Log scale plot showing the magnitude of the exact solution for the Fourier coefficients.](image)

**Figure 3.5: Magnitude of the exact solution.**

Figure 3.5 shows the magnitude of the exact solution for the Fourier coefficients on a $\log_{10}$ scale. The plot shows that all the Fourier coefficients except -14 to -6 and 6 to 14 are all negligible. This coincides with the data on Figure 3.4. Figure 3.6 shows the absolute error of the ODE45 solution compared to the exact solution. We can see a rather pronounced absolute error on the two extremes of the $x$-axis. These values should have a magnitude of $10^{-11}$ thus an error of the magnitude of $10^{-5}$ is rather severe. This can be explained by ODE45’s default tolerance settings of an absolute and relative error tolerances of $10^{-3}$ and $10^{-6}$ respectively. Thus, an absolute tolerance of $10^{-3}$ is insufficient given the magnitude of the numbers involved. Increasing the tolerance to $10^{-9}$ for both settings gives the results shown in Figure 3.7. Here we can see that the large pronounced error
Figure 3.6: Absolute error from default ODE45 tolerance.

Figure 3.7: Absolute error with absolute and relative error set to 1e-9.
at the extremes of the $x$-axis has vanished. The absolute errors for the non-zero Fourier coefficients are to be expected and we can see that it increase linearly as time progresses. However, after 12 time units the error is of magnitude $10^{-8}$ which is still insignificant.

Figure 3.8: Higher-order model to $t=6$.

Figure 3.9: Difference between higher-order model and linear model.

Figure 3.8 shows the time evolution of the wave using the higher-order model. Plotting the difference between the output of the higher-order model and the linear model shows the
contribution of the cubic term of the Hamiltonian in Figure 3.9. Examining the magnitude of the scale indicates that the contribution of the cubic term is approximately 10%.

Example 2: As our next example we take the approximate traveling wave solution given by truncated expansion of the Stokes wave. The term Stokes’ waves refers to non-linear waves on a surface of inviscid fluid. The theory of non-linear traveling waves started with Stokes’ first publication in 1847 [9]. Stokes discovered that periodic wavetrains are possible in non-linear systems where the amplitude of the wave plays a vital role in the dispersion of the wave. His work showed that changes in the amplitude not only changed the behavior of the waves but could also introduce new phenomena, see [6].

Stokes approached the problem of finding exact traveling wave solutions by means of an asymptotic expansion based on Fourier series and using a small parameter $a$, which is related to the amplitude of the wave. Stokes’ solution is a “traveling wave” which means precisely that the dependence on $x$ and $t$ is through a combination $\theta = kx - \omega t$. Here, $k$ is the wave number (roughly, the ”number of peaks per unit length”) and $\omega$ is the time frequency (speed of the wave) to be determined. In addition, the term dispersive waves means that waves of different wave numbers propagate with different speeds; typically there exists a relation between $\omega$ and $k$, called the dispersion relation.

Given the same set of governing equations as the water wave problem, we look for a solution of the form $\eta(\theta)$ and $\varphi(\theta, y)$ where $\eta$ and $\varphi$ are periodic in $\theta$. Choosing the origin $y = 0$, so that $\eta$ has zero mean value and expanding $\eta$ yields:

$$\eta = a \cos(\theta) + \mu_2 a^2 \cos(2\theta) + ..., \quad (3.10)$$

where $\mu_2$ is a coefficient to be determined. With the choice that mean value of $\eta$ is zero, the second free-surface condition implies that the mean value of $\varphi_t$ cannot be zero. Thus, $\varphi$ must have at least a term of $t$ in its expansion [12, p.474]. The expansion of $\varphi$ is taken to
be:

\[ \varphi = \nu_0 a^2 t + \nu_1 a \cosh k(y + h) \sin \theta + \nu_2 a^2 \cosh 2k(y + h) \sin 2\theta + \ldots, \]  

(3.11)

where \( \nu_j \) for \( j = 0, 1, \ldots \) are to be determined. In the case of third order expansion, which we will not cover here, the frequency \( \omega \) must also be expanded as

\[ \omega = \omega_0(k) + a^2 \omega_2(k) + \ldots, \]  

(3.12)

in order to avoid secular terms.

Since we need to differentiate \( \eta \) and \( \varphi \) in order to solve for the unknown coefficients, we list the results here for convenience;

\[
\begin{align*}
\eta_x &= -ak \sin \theta - \mu_2 a^2(2k) \sin 2\theta - \ldots \\
\eta_t &= \omega a \sin \theta + \mu_2 a^2(2\omega) \sin(2\theta) + \ldots \\
\varphi_t &= \nu_0 a^2 - \nu_1 a \omega \cosh k(y + h) \cos \theta - \nu_2 a^2(2\omega) \cosh 2k(y + h) \cos 2\theta + \ldots, \\
\varphi_y &= \nu_1 a k \sinh k(y + h) \sin \theta + \nu_2 a^2(2k) \sinh 2k(y + h) \sin 2\theta + \ldots, \\
\varphi_x &= \nu_1 a k \cosh k(y + h) \cos \theta + \nu_1 a^2(2k) \cosh 2k(y + h) \sin 2\theta + \ldots.
\end{align*}
\]

In addition, we restate the two free-surface boundary condition for convenience, they are:

\[
\begin{align*}
\eta_t + \varphi_x \eta_x - \varphi_y &= 0, \quad y = \eta \\
\varphi_t + \frac{1}{2} \varphi_x^2 + \frac{1}{2} \varphi_y^2 + g\eta &= 0, \quad y = \eta.
\end{align*}
\]

Substituting the expansion of \( \eta \) and \( \varphi \) into the free-surface condition and equating terms of equal powers of \( a \) allows us to approximate the solution to any given order.

Let us first consider the terms of first-order \( a \). Substitution into the first free-surface
boundary condition and isolating the first-order term yields:

\[ \omega_0 a \sin \theta - \nu_1 a k \sinh k(y + h) \sin \theta = 0, \]

which implies on \( y = 0 \)

\[ \nu_1 = \frac{\omega_0}{k \sinh kh}. \]  \hfill (3.13)

Substitution into the second free-surface condition, gathering terms of the first order in \( a \) and evaluating on \( y = 0 \) yields:

\[ \nu_1 \omega_0 \cosh kh \cos \theta - g a \cos(\theta) = 0. \]

Therefore, substituting for \( \nu_1 \) from the above:

\[ \omega_0^2 = g k \tanh kh. \]  \hfill (3.14)

For the second-order approximation, we need to evaluate free-surface condition on \( y = \eta \). Thus, it is necessary to expand \( \sinh k(y + h) \) and \( \cosh k(y + h) \). The expansions are:

\[ \sinh k(y + h) = \sinh(kh) + k \cosh(kh) \eta + \frac{1}{2} k^2 \sinh(kh) \eta^2 + ..., \]

\[ \cosh k(y + h) = \cosh(kh) + k \sinh(kh) \eta + \frac{1}{2} k^2 \cosh(kh) \eta^2 + .... \]

Substituting in (3.10) we obtain

\[ \sinh k(y + h) = \sinh(kh) + ka \cosh(kh) \cos \theta + ka^2 \mu_2 \cosh(kh) \cos 2\theta \]
\[ + \frac{1}{2} k^2 a \sinh(kh) \cos \theta + \frac{1}{2} k^2 a^2 \mu_2 \sinh(kh) \cos^2 2\theta + ...., \]

\[ \cosh k(y + h) = \cosh(kh) + ka \sinh(kh) \cos \theta + ka^2 \mu_2 \sinh(kh) \cos 2\theta \]
\[ + \frac{1}{2} k^2 a \cosh(kh) \cos \theta + \frac{1}{2} k^2 a^2 \mu_2 \sinh(kh) \cos^2 2\theta + .... \]
Next, we use the second-order expansion of \( \eta \) and \( \varphi \) and evaluate the free-surface boundary conditions. After eliminating terms of higher-order and discarding linear terms that do not involve expansion of \( \sinh \) or \( \cosh \), we obtain

\[
\eta^{(2)}_t + \varphi^{(1)}_x \eta^{(1)}_x - \varphi^{(1)}_y - \varphi^{(2)}_y = 0, \quad y = \eta^{(1)} + \eta^{(2)},
\]

\[
\nu_0 a^2 + \varphi^{(1)}_i + \varphi^{(2)}_i + \frac{1}{2} \left( \varphi^{(1)}_x \right)^2 + \frac{1}{2} \left( \varphi^{(1)}_y \right)^2 + g \eta^{(2)} = 0, \quad y = \eta^{(1)} + \eta^{(2)}.
\]

Explicitly expressing the first,

\[
\mu_2 a^2 (2\omega_0) \sin 2\theta = a^2 k \omega_0 \frac{\cosh k (y + h) \sin 2\theta}{\sinh kh} - a \omega_0 \frac{\sinh k (y + h)}{\sinh kh} \sin \theta - \nu_2 a^2 (2k \sinh(2k(y + h))) \sin 2\theta = 0,
\]

where we evaluate on \( y = \eta^{(1)} + \eta^{(2)} \) using the expansion of \( \cosh \) and \( \sinh \) while maintaining an order of \( a^2 \). In the second term we substitute \( \cosh kh \) to get the correct second order term and obtain

\[
\frac{1}{2} a^2 k \omega_0 \frac{\cosh kh}{\sinh kh} \sin 2\theta = \frac{1}{2} a^2 \omega_0 k \coth kh \sin 2\theta.
\]

The third term requires the \( ka \cosh(kh) \cos \theta \) of the sinh expansion to get the second-order term, thus we also get

\[
a \omega_0 \frac{ka \cosh kh \cos \theta}{\sinh kh} \sin \theta = \frac{1}{2} a^2 \omega_0 k \coth kh \sin 2\theta.
\]

For the third term we first use the identity \( \sinh 2\theta = 2 \sinh \theta \cosh \theta \), then we use the expansion terms \( \cosh kh \) and \( \sinh kh \) to obtain

\[
4\nu_2 ka^2 \cosh(kh) \sinh(kh) \sin 2\theta.
\]
Therefore, we get

\[ \mu_2 \omega_0 - 2 \nu_2 k \cosh kh \sinh kh = \frac{1}{2} \omega_0 k \coth kh. \]  \hspace{1cm} (3.15)

Now consider the second free-surface boundary condition, which with a little rearranging can be expressed as

\[ \eta = -\frac{1}{g} \left( \varphi_t + \frac{1}{2} (\varphi_x)^2 + \frac{1}{2} (\varphi_y)^2 \right), \]

which implies

\[ \eta_t = -\frac{1}{g} (\varphi_{tt} + \varphi_x \varphi_{xt} + \varphi_y \varphi_{yt}). \]

Thus, the \( a^2 \)-order terms of the expansion of \( \eta \) may be expressed as

\[ \eta^{(2)} = -\frac{1}{g} \left( -\omega_0^3 a^2 \sin 2\theta - 4 \nu_2 \omega_0^2 a^2 \cosh 2kh \sin 2\theta + \frac{1}{2} \omega_0^3 a^2 \coth^2 kh \sin 2\theta - \frac{1}{2} \omega_0^3 a^2 \sin 2\theta \right). \]

Substituting into the first free-surface boundary condition yields

\[ \frac{1}{g} \left( \omega_0^3 + 4 \nu_2 \omega_0^2 \cosh 2kh - \frac{1}{2} \omega_0^3 \coth^2 kh + \frac{1}{2} \omega_0^3 \right) - \omega_0 k \coth kh - 4 \nu_2 k \cosh kh \sinh kh = 0. \]

After simplification and using \( \omega_0^2 = gk \tanh hk \), we obtain

\[ \omega_0 k \tanh hk + 4 \nu_2 k \left( \frac{\sinh^3 kh}{\cosh kh} + \sinh kh \cosh kh \right) \]

\[ - \frac{\omega_0^3}{2g \sinh^2 kh} - \omega_0 k \coth kh - 4 \nu_2 k \cosh kh \sinh kh = 0, \]

\[ 4 \nu_2 k \frac{\sinh^3 kh}{\cosh kh} = \frac{\omega_0^3}{2g \sinh^2 kh} + \frac{\omega_0 k}{\sinh kh \cosh kh}, \]
\[ \nu_2 = \frac{\omega_0^3 \cosh kh}{8gk \sinh^3 hk} + \frac{\omega_0}{4 \sinh^4 kh}, \]

\[ \nu_2 = \frac{\omega_0^3 \cosh kh + 2\omega_0 gk \sinh kh}{8gk \sinh^3 hk}, \]

\[ \nu_2 = \frac{\omega_0^3 + 2\omega_0 gk \tanh kh}{8 \sinh^4 hk} \cdot \frac{1}{gk \tanh kh}, \]

\[ \nu_2 = \frac{\omega_0^3 + 2\omega_0^3}{8 \sinh^4 hk} \frac{1}{\omega_0^2}, \]

\[ \nu_2 = \frac{3\omega_0^3}{8 \sinh^4 hk} \frac{1}{\omega_0^2}. \]

Therefore, we obtain
\[ \nu_2 = \frac{3}{8} \frac{\omega_0}{\sinh^4 kh}. \] \hspace{1cm} (3.16)

Substituting back into (3.15) yields
\[ \mu_2 = \frac{1}{2} k \coth kh \left(1 + \frac{3}{2 \sinh^2 kh}\right). \] \hspace{1cm} (3.17)

Finally, finding \( \nu_0 \) is a straightforward computation using the second free-surface boundary conditions once \( \nu_1, \nu_2, \) and \( \mu_2 \) are known. Thus,
\[ \nu_0 = \frac{-gk}{2 \sinh 2kh}. \] \hspace{1cm} (3.18)

We now use the second-order Stokes’ wavetrain with both the linear and higher order model we have developed. Figures 3.10 and 3.11 shows the time evolution of the second
Figure 3.10: Linear Model of the Stokes wavetrain with $k = 5$, $a = 0.02$, $h = 1$, for $t \in [0, 6]$.

Figure 3.11: Higher-order model of the Stokes wavetrain with $k = 5$, $a = 0.02$, $h = 1$, for $t \in [0, 6]$. 
order approximation of the Stokes’ wavetrain with $N = 128$, $a = 0.02$, $k = 5$ and $h = 1$. The linear model shows that the wavetrain is stable with some fluctuations in the wave height as time progresses. Adding the cubic term of the Hamiltonian to the linear model helps to stabilize the fluctuations of the wave height, see Figure 3.11.

Stokes’ theory states that the amplitude of the waves plays a vital importance in the behavior and stability of the wave. In addition, Stokes’ solution can display non-realistic behavior when the depth $h$ is much smaller than the wave length, that is when Stokes’ solution is used to model a shallow water wave. We illustrate this by setting wave amplitude to 0.02, number of waves to 1 so that the wave length is $2\pi$, and water depth to a mere 0.2. The time evolution of this shallow wave train shows that the wave height erroneously increases to a height of 0.12, see figure 3.12.

![Higher-Order Model of Stokes’ Wavetrain](image)

Figure 3.12: Higher-order model of the Stokes wavetrain with $k = 1$, $a = 0.02$, $h = 0.2$, for $t \in [0, 6]$

Furthermore, we witnessed that the higher-order model that uses the quadratic and cubic terms of the Hamiltonian is stable for wave of small amplitude but becomes unstable for higher values. The value at which the model becomes unstable is smaller as the slope of
Figure 3.13: Higher-order model of the Stokes wavetrain with $k = 5$, $a = 0.04$, $h = 2$, for $t \in [0, 6]$

Figure 3.14: Magnitude of the Fourier coefficients $\eta_k$; $a = 0.04$, $h = 0.2$, for $t \in [0, 6]$
the wave increases. In Figure 3.13 you can see the onset of the instability as the wave height increases at the end of the time interval show in in the plot. Examining the Fourier coefficients in Figure 3.14 shows that the higher-order Fourier coefficients starts to increase in magnitude. This behavior increases exponentially and the plot values were taken so that only the onset of the increase can be seen as to avoid an axis scale that becomes meaningless.
References


Appendix A

Euler’s Identity

Given a mapping \((X, t) \mapsto \vec{x}(X, t)\) where \(X = (X, Y, Z)\), \(\vec{x} = (x, y, z)\), velocity vector \(\vec{u} = D\vec{x}/Dt\), where \(\vec{u}(\vec{x}, t) = (u_1, u_2, u_3)\) and \(J(t)\) the Jacobian of the transformation \(X \mapsto \vec{x}(X, t)\), we have

\[
\frac{dJ}{dt} = J(\nabla \cdot u).
\]

Proof:

\[
\frac{dJ}{dt} = \frac{D}{Dt} \begin{pmatrix}
\frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\
\frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\
\frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z}
\end{pmatrix}
\]

\[
= \left( \frac{D}{Dt} \begin{pmatrix}
\frac{\partial x}{\partial X} \\
\frac{\partial y}{\partial X} \\
\frac{\partial z}{\partial X}
\end{pmatrix} \right) \cdot \left( \frac{D}{Dt} \begin{pmatrix}
\frac{\partial x}{\partial Y} \\
\frac{\partial y}{\partial Y} \\
\frac{\partial z}{\partial Y}
\end{pmatrix} \right) + \left( \frac{D}{Dt} \begin{pmatrix}
\frac{\partial x}{\partial Z} \\
\frac{\partial y}{\partial Z} \\
\frac{\partial z}{\partial Z}
\end{pmatrix} \right) \cdot \left( \frac{D}{Dt} \begin{pmatrix}
\frac{\partial x}{\partial Z} \\
\frac{\partial y}{\partial Z} \\
\frac{\partial z}{\partial Z}
\end{pmatrix} \right)
\]

Recall that the derivative \(D/Dt\) is evaluated with a fixed initial position \(\vec{X}\) therefore it commutes with the partial derivatives with respect to \(X\), \(Y\), and \(Z\), and \(D\vec{x}/Dt = \vec{u}\). Thus,

\[
\frac{DJ}{Dt} = \left( \frac{\partial u_1}{\partial X} \right) \begin{pmatrix}
\frac{\partial x}{\partial X} \\
\frac{\partial y}{\partial X} \\
\frac{\partial z}{\partial X}
\end{pmatrix} + \left( \frac{\partial u_2}{\partial X} \right) \begin{pmatrix}
\frac{\partial x}{\partial Y} \\
\frac{\partial y}{\partial Y} \\
\frac{\partial z}{\partial Y}
\end{pmatrix} + \left( \frac{\partial u_3}{\partial X} \right) \begin{pmatrix}
\frac{\partial x}{\partial Z} \\
\frac{\partial y}{\partial Z} \\
\frac{\partial z}{\partial Z}
\end{pmatrix}.
\] (A.1)
Let $d_1$, $d_2$, and $d_3$ be the determinants of the three matrices on the right-hand side. Consider the first matrix on the right-hand side and apply the chain rule to each of the terms in the first row:

\[
\begin{align*}
\frac{\partial u_1}{\partial X} &= \frac{\partial u_1}{\partial x} \frac{\partial x}{\partial X} + \frac{\partial u_1}{\partial y} \frac{\partial y}{\partial X} + \frac{\partial u_1}{\partial z} \frac{\partial z}{\partial X}, \\
\frac{\partial u_1}{\partial Y} &= \frac{\partial u_1}{\partial x} \frac{\partial x}{\partial Y} + \frac{\partial u_1}{\partial y} \frac{\partial y}{\partial Y} + \frac{\partial u_1}{\partial z} \frac{\partial z}{\partial Y}, \\
\frac{\partial u_1}{\partial Z} &= \frac{\partial u_1}{\partial x} \frac{\partial x}{\partial Z} + \frac{\partial u_1}{\partial y} \frac{\partial y}{\partial Z} + \frac{\partial u_1}{\partial z} \frac{\partial z}{\partial Z}. 
\end{align*}
\]

Thus, substitution leads to:

\[
d_1 = \frac{\partial u_1}{\partial x}(\begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{pmatrix}) + \frac{\partial u_1}{\partial y}(\begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{pmatrix}) + \frac{\partial u_1}{\partial z}(\begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial z}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial z}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{pmatrix}).
\]

The determinant of the first matrix on the right-hand is $J$, while the determinate of other two matrices are zero since they both contain a duplicate row. Thus,

\[
d_1 = \frac{\partial u_1}{\partial x}J.
\]

Similarly, $d_2 = \frac{\partial u_2}{\partial y}J$ and $d_3 = \frac{\partial u_3}{\partial z}J$. Therefore,

\[
\frac{DJ}{Dt} = \frac{\partial u_1}{\partial x}J + \frac{\partial u_2}{\partial y}J + \frac{\partial u_3}{\partial z}J \\
= J (\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}) \\
= J(\nabla \cdot \vec{u}).
\]
Appendix B

Conservative Vector Fields

Theorem: Let \( \mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} \) be a differentiable vector field defined on \( D \subset \mathbb{R}^3 \); except possibly for a finite number of points. Then the following conditions on \( \mathbf{F} \) are equivalent:

1. For any simple closed curve \( C \), \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \).

2. Path Independence: For any two simple curves, \( C_1 \) and \( C_2 \), with common end points,
\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.
\]

3. Existence of potential function: There exists a potential function \( f \) such that
\[
\mathbf{F} = \nabla f.
\]

4. \( \nabla \times \mathbf{F} = 0 \).

If \( \mathbf{F} \) satisfied any of the above conditions then \( \mathbf{F} \) is said to be a conservative vector field.

Proof: \((1) \implies (2)\): Let \( C_1 \) and \( C_2 \) be simple curves with common end points, then \( C_1 \cup -C_2 \), where \( -C_2 \) indicates \( C_2 \) traversed in the reverse direction, is a simple closed curve. Thus,
\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r}
= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0.
\]
(2) $\implies$ (3): Assume $C$ is an oriented simple curve in $D$ connecting a fixed point $(a, b, c)$ in $D$ to any arbitrary point $(x, y, z)$ in $D$. Let $f(x, y, z) = \oint_C \vec{F} \cdot d\vec{r}$. By (2) $f$ is independent of $C$. Thus, only the terminal point $(x, y, z)$ matters. Recall the definition of the partial derivative with respect to $x$:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h},$$

and let $C$ be any path from $(a, b, c)$ to $(x, y, z)$ and let $C_1$ be the straight line path from $(x, y, z)$ to $(x + h, y, z)$. Then

$$f(x + h, y, z) - f(x, y, z) = \int_{C \cup C_1} \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \vec{F} \cdot d\vec{r} + \int_{C_1} \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r}.$$

Now, parametrize $C_1$ by $\varphi(t) = (x + t, y, z)$, $0 \leq t \leq h$, then $\varphi'(t) = (1, 0, 0)$ and

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^h (Q(x + h, y, z), P(x + h, y, z), R(x + h, y, z)) \cdot (1, 0, 0) \, dt$$

$$= \int_0^h P(x + h, y, z) \, dt.$$

Hence,

$$\frac{\partial f}{\partial x}(x, y, z) = \lim_{h \to 0} \frac{\int_0^h P(x + h, y, z) \, dt}{h}$$

$$= \lim_{h \to 0} P(x + h, y, z)$$

$$= P(x, y, x).$$

Similarly, we get that $\partial_y f(x, y, z) = Q(x, y, z)$ and $\partial_z f(x, y, z) = R(x, y, z)$. Therefore,
we have
\[ \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k} = \mathbf{F}. \]

(3) \implies (4): \( \nabla \times \mathbf{F} = \nabla \times (\nabla f) = 0 \) since curl grad =0.

(4) \implies (1): Let \( C \) be a closed curve in \( D \) and let \( D_1 \) be a simply connected surface contained in \( D \) such that is \( C \) is the boundary of \( D_1 \). Invoking Stokes’ theorem yields:

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{D_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = 0. \]
Appendix C

MATLAB Codes

This appendix contains the MATLAB codes used in the numerical simulations.

The following function “yprimecomplex.m” sets up the ODE system for the linear model necessary for the implementation of the ODE45 solver.

```matlab
function dydt = yprimecomplex(t,y)
%function the describes the system of differential equations
%y = (eta_k's, xi_k's) as input row vector of length 2*N

g=9.80665; %standard accel of gravity
n=length(y)/2;

h=1;

%create index row for coefficients
%0,1,2,3,...,n/2-1,-n/2,....,-1
%example n=8: w=0,1,2,3,-4,-3,-2,-1
w=[0:n/2-1,-(n/2):-1];

%create n x n diagonal with k*tanh(k*h) element for a_k,k
A=eye(n);
for k=1:n
    m=w(k);
    A(k,k)=m*tanh(h*m);
end

B=-g*eye(n); %diagonal with -g on b_k,k
C=zeros(2*n,2*n);
%A into top right
C(1:n,n+1:2*n) = A;
%B into lower left
C(n+1:2*n,1:n) = B;
```
dydt=C*y;
end

Linear Model code used in the first modulated wave packet example.

function [ eta1, xi1,T] = ModulatedWavePacket( N )
%Simulation of the Modulated WavePacket with zero initial
%velocity potential at the surface

tend=6; %end time
h=1; %water depth
g=9.80665; %gravity

w=[0:N-1];
w=(2*pi*w)/N;

%eta function defining top surface
eta = 0.01*exp(-(4/3)*(w-pi).^2).*cos(10*w);

%xi potential velocity; taken to be initially zero
xi = zeros(1, N);

%take the fast fourier transform and divide by 1/N
Eta = fft(eta)/N;
Xi = fft(xi)/N;
%reorder

C=[Eta, Xi];
options = odeset('RelTol',1e-9,'AbsTol',1e-9);
[T, X] = ode45(@yprimecomplex, [0,tend], C,options);
[x1, x2]=size(X);
eta1=zeros(x1,N);
for k=1:x1
    Y1=X(k,1:N);
    y1=ifft(Y1);
    y1=N*real(y1);
    eta1(k,:)=y1;
end

xi1=zeros(x1,N);
for k=1:x1
    Y2=X(k,N+1:2*N);
    y2=ifft(Y1);
    y2=N*real(y2);
    xi1(k,:)=y2;
end

Linear model code used in the second Stokes wavetrain example.

function [ eta1, xi1,T ] = Stokes( N )
%second order approx to stoke's wavetrain
tend=6; %end time
h=1; %water depth
g=9.80665; %gravity
j=5;
w=[0:N-1];
w=(2*pi*w)/N;
a=0.02;
mu=0.5*j*coth(h+j)*(1+3/(2*(sinh(j*h))^2));
omega=sqrt(g*j*tanh(j*h));
u1=(omega)/(j*sinh(h*j));
u2=(3*omega)/(8*(sinh(j*h))^4);
%eta function defining top surface
eta = a*cos(j*w)+mu*a^2*cos(2*j*w);

%xi potential velocity; taken to be initially zero
xi = nu1*a*cosh(j*(eta+h)).*sin(j*w)+nu2*a^2*cosh(2*j*(eta+h)).*sin(2*j*w);

%take the fast fourier transform and divide by 1/N
Eta = fft(eta)/N;
Xi = fft(xi)/N;
C=[Eta, Xi];

options = odeset('RelTol',1e-9,'AbsTol',1e-9);
[T, X] = ode45(@yprimecomplex, [0,tend], C,options);

[x1, x2]=size(X);
etal=zeros(x1,N);
for k=1:x1
    Y1=X(k,1:N);
y1=ifft(Y1);
y1=N*real(y1);
etal(k,:)=y1;
end

xi1=zeros(x1,N);
for k=1:x1
    Y2=X(k,N+1:2*N);
y2=ifft(Y1);
y2=N*real(y2);
xi1(k,:)=y2;
end

ODE45 Function file for setting up the ODE for the higher order mode

function dydt = higherorder(t,y)
%function the describes the system of differential equations
%y = (eta_k's, xi_k's) as input row vector of length 2*N

\[
g = 9.80665; \quad \text{%standard accel of gravity}
\]

\[
n = \text{length}(y)/2;
\]

\[
h = 1.0;
\]

%extract Eta and Xi from vector y

\[
\text{Eta} = y(1:n); \quad \text{%length n}
\]

\[
\text{Xi} = y(n+1:2*n); \quad \text{%length n}
\]

\[
L = [-n/2:n/2-1]; \quad \text{%length n}
\]

\[
Lt = \text{tanh}(h*L); \quad \text{%length n}
\]

\[
\text{Xi} = \text{transpose}(\text{Xi});
\]

\[
\text{Eta} = \text{transpose}(\text{Eta});
\]

\[
\text{Xi2} = L.*\text{Xi};
\]

\[
\text{XiConv1} = \text{convolution1}(\text{Xi2}, \text{Xi2});
\]

\[
\text{Xi3} = Lt.*\text{Xi2};
\]

\[
\text{XiConv2} = \text{convolution1}(\text{Xi3}, \text{Xi3});
\]

\[
\text{EtaConv1} = \text{convolution1}(\text{Eta}, \text{Xi2}); % \text{Xi1} = k \xi_k
\]

\[
\text{EtaConv1} = L.*\text{EtaConv1};
\]

\[
\text{EtaConv2} = \text{convolution1}(\text{Eta}, \text{Xi3}); % \text{Xi3} = k \text{tanh}(hk) \xi_k
\]

\[
\text{EtaConv2} = L.*Lt.*\text{EtaConv2};
\]

\[
dXi = (1/2)\ast(\text{EtaConv1} - \text{EtaConv2});
\]

\[
dEta = (1/2)\ast(\text{XiConv2} + \text{XiConv1});
\]

\[
D = [dXi, dEta];
\]

\[
D = \text{transpose}(D);
\]

\[
A = \text{eye}(n);
\]

\[
\text{for } k = 1:n
\]

\[
m = L(k);
\]

\[
A(k,k) = m \ast \text{tanh}(h*m);
\]

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end

B=-g*eye(n); %diagonal with -g on b_k,k
C=zeros(2*n,2*n);
%A into top right
C(1:n,n+1:2*n) = A;
%B into lower left
C(n+1:2*n,1:n) = B;
dydt=C*y+D;1
end

Function that manually computes the convolution

function [ D ] = convolution( A,B )
%convolution of two sequences
N=length(A);
N2=N/2;
L=[-N2:N2-1];
B=transpose(fliplr(B));
C=[];
for k=1:N
    b=L(k)+1;
    C(:,k)=circshift(B,b);
end
D=A*C;
end

Convolution function that takes advantage of the Fourier convolution theorem.

function [ C ] = convolution1( A,B )
%UNTITLED2 Summary of this function goes here
function [ etal, xi1,T,Etak,Xik ] = ModulatedWavePacketHO( N )

%Simulation of the Modulated Wave Packet with zero initial
%velocity potential at the surface

%eta function defining top surface
eta = 0.01*exp(-(4/3)*(w-pi).^2).*cos(10*w);

%xi potential velocity; taken to be initially zero
xi = zeros(1, N);

%take the fast fourier transform and divide by 1/N
Eta = fft(eta)/N;
Xi = fft(xi)/N;
Eta=[Eta(N/2+1:N),Eta(1:N/2)];
Xi=[Xi(N/2+1:N),Xi(1:N/2)];
C=[Eta, Xi];

options = odeset('RelTol',1e-9,'AbsTol',1e-9);

[T, X] = ode45(@higherorder, [0,tend], C,options);

[x1, x2]=size(X);

A=X(:,1:x2/4);
B=X(:,x2/4+1:x2/2);
C=X(:,x2/2+1:3*x2/4);
D=X(:,3*x2/4+1:x2);

X=[B,A,D,C];

Etak=[A,B];

Xik=[C,D];

eta1=zeros(x1,N);
for k=1:x1
    Y1=X(k,1:N);
    y1=ifft(Y1);
    eta1(k,:)=y1;
end

xi1=zeros(x1,N);
for k=1:x1
    Y2=X(k,N+1:2*N);
    y2=ifft(Y1);
    xi1(k,:)=y2;
end