ODE Methods for Eigenvalue Problems in Riemannian Geometry

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by

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Dedications

In memory of my father Hassan. To my mother Amal and sister Ryann.
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Abstract

ODE Methods for Eigenvalue Problems in Riemannian Geometry

by

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We study Sturm Liouville problems in relation to eigenvalue problems in Riemannian geometry and prove some standard comparison theorems for eigenvalues in the case of spherically symmetric domains in warped products. Our main goal is to investigate fourth order Sturm Liouville operators and the Bilaplacian. We characterize the eigenfunctions of the clamped plate problem on discs, and finally prove a generalization of Szego’s lower bound of the first eigenvalue to positively curved warped products.
Chapter 1

Comparison Theorems for Second Order ODE's

1.1 The Sturm Comparison Theorem

We wish to examine the behavior of linear second order ODE's

\[(pu)' + qu = 0.\]  \hspace{1cm} (1.1)

**Definition 1.1.1** A second order ODE is oscillatory on \((a, \infty)\) if every solution on \((a, \infty)\) has infinitely many zeroes.

We know for \(p(t) = 1, q = k^2\), the above equation has solutions of the form of

\[u = c_1 \cos (kt) + c_2 \sin (kt).\]

Thus, the equation is oscillatory because every solution has infinitely many zeros. By finding the location of the zeros, we can rigorously analyze how much the solution oscillates. First, we will need a couple of definitons that will help us examine this behavior for a second order ODE.

**Definition 1.1.2** Suppose \(u\) is a solution of Eq. (1.1) with \(u(0) = 0\). Then Eq. (1.1) has a conjugate point at \(R\) if \(u(R) = 0\).

**Definition 1.1.3** Suppose \(u\) is a solution of Eq. (1.1) with \(u'(0) = 0\). Then Eq. (1.1) has a focal point at \(R\) if \(u(R) = 0\).

**Example 1** For \(q = k^2\) we see that the first conjugate point is the first zero of \(\sin (kt)\), which is \(\frac{\pi}{k}\). In addition, we also see that the first focal point is the first zero of \(\cos (kt)\), which is \(\frac{\pi}{2k}\).

We now have find a way to examine the oscillatory behavior of Eq. (1.1). The Sturm Comparison Theorem allows us to find conjugate points of second order differential equations. Showing that an ODE is oscillatory is then equivalent to showing the existence of infinitely many conjugate points. Finding the location of the conjugate points allows us to examine how rapidly the solution oscillates. We will now prove the Sturm Comparison Theorem for conjugate points.

**Theorem 1.1.4 (Sturm Comparison)** Let \(u\) and \(v\) be solutions to

\[u'' + qu = 0,\]  \hspace{1cm} (1.2)
\[ v'' + \bar{q}v = 0. \] (1.3)

Let 0, R be consecutive zeros of \( v \). That is \( v(0) = v(R) = 0 \) and \( v(t) \neq 0 \) for \( t \in (0, R) \).

Suppose \( q > \bar{q} \).

Then \( u \) will have a zero \( r_0 \) with \( 0 < r_0 < R \). In particular, if \( u(0) = 0 \), \( u \) has a conjugate point before \( v \).

To prove the Sturm Comparison Theorem, we will need some identities using the Wronskian. By definition, \[ W'(t) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}' = (uv' - vu')' \] which by the product rule gives us, \( W'(t) = uv'' - vu'' \). Using Eq.(1.2) and Eq.(1.3), and substituting into \( uv'' - vu'' \),

\[ W'(t) = (q - \bar{q})uv. \]

Therefore, integrating over \([0, R]\), we obtain the following:

\[
\int_{0}^{R} (uv' - u'v)' dt = \int_{0}^{R} (q - \bar{q})uv dt.
\]

So if \( u \) and \( v \) are solutions to Eq.(1.2) and Eq.(1.3) then by the Fundamental Theorem of Calculus,

\[ W(R) - W(0) = \int_{0}^{R} (q - \bar{q})uv dt. \] (1.4)

Now we can prove the Sturm Comparison Theorem.

**Proof.** Assume that \( v(t) > 0 \) on \((0, R)\) and \( u(0) \geq 0 \). Otherwise we can multiply by \(-1\) to get solutions to Eq.(1.2) and Eq.(1.3) such that this holds.

By definition of the Wronskian we have

\[ W(R) - W(0) = (uv' - u'v)(R) - (uv' - u'v)(0) = u(R)v'(R) - u(0)v'(0) \]

where we have used \( v(0) = v(R) = 0 \). Since \( v > 0 \) on \((0, R)\) and \( v(0) = v(R) = 0 \) we have \( v'(0) \geq 0 \) and \( v'(R) \leq 0 \). Therefore, \( u(R)v'(R) - u(0)v'(0) < 0 \) because \( u(0) \geq 0 \).

Hence,

\[
\int_{0}^{R} (q - \bar{q})uv dt < 0.
\]

But since \( q > \bar{q} \) we see from Eq.(1.4) that

\[
\int_{0}^{R} (q - \bar{q})uv dt > 0
\]
which is a contradiction. Hence, $u$ must have a conjugate point before $v$.

The following theorem for focal points can be proved similarly.

**Theorem 1.1.5** Suppose $u, v$ solve Eq.(1.2) and Eq.(1.3) respectively with $q > \bar{q}$. If $R$ is a focal point for $v$, with $v'(0) = v(R) = 0$ and $v$ nonzero on $(0, R)$. Then if $u'(0) = 0$, $u(t_0) = 0$ for some $t_0 < R$. That is, $u$ has a focal point before $v$.

### 1.2 Sturm-Picone Theorem

Despite proving the Sturm Comparison Theorem and now having a way to find conjugate points, restricts us to cases of Eq. (1.1) where $p = 1$. As a result, we would like to find a more general comparison theorem which leads us to the Sturm-Picone Theorem. In order to prove this, we will first need the following fundamental result known as Picone’s Identity.

Recall that the Wronskian was $W(t) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = (uv' - vu')$. For Eq.(1.1) we define the modified Wronskian

$$\tilde{W}(t) = \begin{vmatrix} u & v \\ pu' & \bar{p}v' \end{vmatrix} = (\bar{p}v'u - pu'v).$$

**Theorem 1.2.1 (Picone’s Identity)** Consider the following ODE’s:

\begin{align*}
(pu')' + qu &= 0, \\
(\bar{p}v')' + \bar{q}v &= 0.
\end{align*}

Then,

$$\frac{d}{dt} \left( \frac{v}{u} \tilde{W}(t) \right) = \frac{d}{dt} \left( \frac{v}{u} (\bar{p}v'u - pu'v) \right) = (q - \bar{q})v^2 + (\bar{p} - p)v'^2 + p \left( v' - v \left( \frac{u'}{u} \right) \right)^2.$$

Moreover,

$$\left[ \frac{v}{u} (puv' - \bar{p}u'v) \right]_{t_1}^{t_2} = \int_{t_1}^{t_2} (q - \bar{q})v^2 dt + \int_{t_1}^{t_2} (\bar{p} - p)v'^2 dt + \int_{t_1}^{t_2} \frac{p(v'u - vu')^2}{u^2} dt.$$

In the case of $p = \bar{p} = 1$, we get that

$$\frac{d}{dt} \left[ \frac{v}{u} W(t) \right] = \frac{d}{dt} \left[ \frac{v}{u} (uv' - u'v) \right] = (q - \bar{q})v^2 + p \left( v' - v \left( \frac{u'}{u} \right) \right)^2.$$
Proof.

\[
\frac{d}{dt} \left( \frac{v}{u} \widetilde{W}(t) \right) = \frac{d}{dt} \left( \frac{v}{u} (\bar{p}v' - pu'v) \right)
\]
\[
= \frac{v}{u} (\bar{p}v' - pu')' + \left( \frac{v' - vu'}{u^2} \right) (\bar{p}v' - pu')
\]
\[
= \frac{v}{u} \left[ (\bar{p}v)'u + \bar{p}v'u' - (pu')' - pu'v' \right] + \left( \frac{v' - vu'}{u^2} \right) (\bar{p}v' - pu')
\]
\[
= \frac{v}{u} \left[ (q - \bar{q})uv + (\bar{p} - p)u'v' \right]
\]
\[
+ \bar{p}(v')^2 - \bar{p}u'v' \left( \frac{v}{u} \right) - pu'v' \left( \frac{v}{u} \right) + \bar{p} \left( \frac{v}{u} \right)^2 (u')^2
\]
\[
= (q - \bar{q})v^2 + (\bar{p} - p)u'v' \left( \frac{v}{u} \right)
\]
\[
+ \bar{p}(v')^2 - \bar{p}u'v' \left( \frac{v}{u} \right) - pu'v' \left( \frac{v}{u} \right) + \bar{p} \left( \frac{v}{u} \right)^2 (u')^2
\]
\[
= (q - \bar{q})v^2 + \bar{p}(v')^2 - 2pu'v' \left( \frac{v}{u} \right) + \bar{p} \left( \frac{v}{u} \right)^2 (u')^2
\]
\[
= (q - \bar{q})v^2 + \bar{p}(v')^2 - p(v')^2 + p(v')^2 - 2pu'v' \left( \frac{v}{u} \right) + \bar{p} \left( \frac{v}{u} \right)^2 (u')^2
\]
\[
= (q - \bar{q})v^2 + (\bar{p} - p)(v')^2 + p \left( v' - v \left( \frac{u'}{u} \right) \right)^2.
\]

By the Fundamental Theorem of Calculus,

\[
\left[ \frac{v}{u} (pu' - \bar{p}u') \right]_{t_1}^{t_2} = \int_{t_1}^{t_2} (q - \bar{q})v^2 dt + \int_{t_1}^{t_2} (\bar{p} - p)v^2 dt - \int_{t_1}^{t_2} p \left( \frac{v' - vu'}{u^2} \right)^2 dt.
\]

Now, we can use this to prove the following theorem.

**Theorem 1.2.2 (Sturm-Picone)** Let Eq. (1.5) and Eq. (1.6) be given where

\[ p \leq \bar{p}, \quad q \geq \bar{q}. \]

If \( v \) is a nontrivial solution of Eq. (1.6), on a bounded interval \([0, R]\), such that \( v(0) = v(R) = 0 \) and \( v \) is nonzero on \((0, R)\), then every solution of Eq. (1.5) has at least one zero in \((0, R)\) ([5]).

**Proof.** Let 0 and \( R \) be consecutive zeros of \( v \) and suppose \( u \) does not have a zero within the interval \([0, R]\). We want to show

\[
\left. \frac{v}{u} \widetilde{W}(t) \right|_0^R = 0.
\]

(1.7)
If $u(0) = 0$ then we must evaluate the following limit.

$$\lim_{t \to 0} \left( \frac{v}{u} \tilde{W} \right) = \lim_{t \to 0} \left( \frac{v}{u} (\tilde{p}u' \bar{v} - pu'v) \right) = \lim_{t \to 0} (vpv') = 0.$$  

By L’Hospital’s Rule and using $v(0) = 0$,

$$\lim_{t \to 0} \left( \frac{v}{u} (\tilde{p}u' \bar{v} - pu'v) \right) = \lim_{t \to 0} \left( \frac{p'u'v^2 + pu''v^2 + 2pu'vv'}{u'} \right) = 0.$$  

Therefore,

$$\lim_{t \to 0} \left( \frac{v}{u} (\tilde{p}u' \bar{v} - pu'v) \right) = 0.$$  

If $u$ is not zero on $(0,R)$, then it is clear that

$$\frac{v}{u} \tilde{W} \bigg|_0^R = 0,$$  

because $v(0) = v(R) = 0$. But because $p \leq \bar{p}$ and $q \geq \bar{q}$, if $p \neq \bar{p}$ or $q \neq \bar{q}$ the right hand side of Picone’s Identity

$$\int_0^R (q - \bar{q})v^2 dt + \int_0^R (\bar{p} - p)v'^2 dt + \int_0^R p \frac{(v' - vu')^2}{u^2} dt$$

is positive. By the Picone Identity this contradicts Eq.(1.7).

\[\blacksquare\]

**Theorem 1.2.3** Let Eq.(1.5) and Eq.(1.6) be given where $p \leq \bar{p}$, $q \geq \bar{q}$. If $v$ is a nontrivial solution of Eq.(1.6) such that $v'(0) = v(R) = 0$ and $v$ is nonzero on $(0,R)$. Then if $u'(0) = 0$, $u$ has a focal point before $v$.

### 1.3 The Liouville Transformation

The following is another way to handle $(pu')' + qy = 0$ for $p \neq 1$. First we wish to take a second order ODE $y'' + Py' + Qy = 0$ and put it into the Liouville form, which is $Y'' + \Phi Y = 0$. This will be done by a change of variables. Take $y = zY$ and substitute this into $y'' + Py' + Qy = 0$ to get

$$(zY)'' + P(zY)' + QzY = 0.$$  

We wish to solve for $z$ such that the middle terms vanish. If we expand this by the Product Rule, we get

$$z''Y + 2z'Y' + zY'' + P(z'Y + zY') + QzY = 0.$$
By factoring and rearrangement of terms, we then get
\[ zY'' + (2z' + Pz)Y' + (z'' + Pz' + Qz)Y = 0. \]

Examining the middle term: \( 2z' + Pz = 0 \), we see that we have a first order linear ODE.
\[ \frac{z'}{z} = -\frac{1}{2}P \]

The left side is the logarithmic derivative.
\[ (\ln z)' = -\frac{1}{2}P \]

By integration, we have
\[ \ln z = \int -\frac{1}{2}Pdx + C \]

Hence,
\[ z = Ce^{-\int \frac{1}{2}Pdx} \]

Substituting the above into the coefficient for \( Y \), which was \( z'' + Pz' + Qz \), we obtain the Liouville form \( Y'' + \Phi Y = 0 \) with
\[ \Phi(x) = Q(x) - \frac{1}{4}P(x)^2 - \frac{1}{2}P'(x). \]

Now in the case of \((py')' + qy = 0\) we have
\[ (py')' + qy = py'' + p'y' + qy \]
\[ = y'' + \frac{p'}{p}y' + \frac{q}{p}y. \]

So we take \( P = \frac{p'}{p} \) and \( Q = \frac{q}{p} \). Repeating the same procedure above, we see that
\[ \Phi(x) = Q(x) - \frac{1}{4}P(x)^2 - \frac{1}{2}P'(x) \]
\[ = \frac{q}{p} - \frac{1}{4} \left( \frac{p'}{p} \right)^2 - \frac{1}{2} \left( \frac{p'}{p} \right)' \]
\[ = \frac{q}{p} + \frac{1}{4} \left( \frac{p'}{p} \right)^2 - \frac{1}{2} \frac{p''}{p}. \]

Now we have another comparison theorem for operators of the form
\[ Lu = (pu')' + qu = 0. \]

**Theorem 1.3.1** Consider
\[ (pu')' + qu = 0 \]
and

\[(\bar{p}u')' + \bar{q}v = 0.\]

Let 0, \(R\) be consecutive zeros of \(v\), that is \(v(0) = v(R) = 0\) and \(v(t) \neq 0\) for \(t \in (0, R)\). If \(\Phi \geq \bar{\Phi}\) and \(p, \bar{p} > 0\) on \([0, R]\) then \(u\) will have a zero \(t_0\) with \(0 < t_0 < R\). In particular, if \(u(0) = 0\), \(u\) has a conjugate point before \(v\).

**Proof.** Applying the Liouville transformation to the above system, we get

\[U'' + \Phi U = 0 \tag{1.8}\]

and

\[V'' + \bar{\Phi}V = 0 \tag{1.9}\]

with \(\Phi \geq \bar{\Phi}\). Then since \(u = zU\) for \(z = Ce^{-\int \frac{1}{2}Pdt}\) and \(P\) continuous; in addition, \(p, \bar{p} > 0\). Therefore, \(z(t)\) is nonzero on \([0, R]\), so that \(u\) has a zero if and only if \(U\) has a zero for some \(t_0\) by Theorem 1.1.4.

\[\square\]

### 1.4 Reid’s Comparison Theorem for Focal Points

In applications, we need \(\bar{p} \leq p\) and \(\bar{q} \leq q\). As we saw in the assumptions for the Sturm Picone Theorem, we have the inequalities in the reverse direction on \(p\). In the end, we are required to find a comparison theorem that allows us to analyze the oscillatory properties of a second order ODE with the condition that \(\bar{p} \leq p\) and \(\bar{q} \leq q\). Another drawback to Theorem 1.3.1 is the fact that \(p, \bar{p} > 0\) because in applications a lot of times, \(p = \bar{p} = 0\). Reid’s Comparison Theorem involves different hypotheses that we can use later in our applications. We start with the following lemma which will be used to prove Reid’s Theorem.

**Lemma 1.4.1** If \(u\) solves Eq.(1.5) with \(p, q \geq 0\) and \(u'(0) = 0, u > 0\) on \((0, a)\) then \(u' < 0\) on \((0, a)\).

**Proof.** From Eq.(1.5) we have

\[(pu')' = -qu.\]

By the Fundamental Theorem of Calculus,

\[pu'(r) = \int_0^r -qu(s)ds + pu'(0) \leq 0 + pu'(0) \leq 0\]

on some \(r \in (0, a)\). Hence, \(u' \leq 0\).
Theorem 1.4.2 (Reid’s Theorem) Let \( u \) be the nontrivial solution to Eq.\((1.5)\) and \( v \) be the nontrivial solution to Eq.\((1.6)\) for \( p, \bar{p} \geq 0 \) and \( q, \bar{q} \geq 0 \).

On \([0, R]\), let

1. \( \frac{v'}{p} \leq \frac{\bar{v}'}{\bar{p}} \)
2. \( \frac{\bar{q}}{\bar{p}} \leq \frac{q}{p} \).

If \( u'(0) = v'(0) = v(R) = 0 \) then there exists \( t_0 \in (0, R) \) with \( u(t_0) = 0 \). That is, Eq.\((1.5)\) has a focal point before Eq.\((1.6)\) on \((0, R)\). ([10])

Proof. Suppose that Eq.\((1.5)\) does not have a focal point before Eq.\((1.6)\) on \((0, R)\). Define

\[ Lu = (pu')' + qu \]

then

\[ uLu = u((pu')' + \bar{q}u) = u(\bar{p}'u' + \bar{p}u'' + \bar{q}u). \]

From \((pu')' + qu = 0\)

and the product rule we get that

\[ u'' = \frac{-p'u' - qu}{p}. \]

Now using \( u'' = \frac{-p'u' - qu}{p} \) we obtain

\[ uLu = u\left(\bar{p}'u' + \bar{p}\left(\frac{-p'u' - qu}{p}\right) + \bar{q}u\right) = \left(\bar{p}' - \bar{p}\frac{p'}{p}\right)u' + \left(\bar{q} - \bar{p}\left(\frac{q}{p}\right)\right)u^2. \]

From our assumptions, we have that \( \bar{p}' \geq \bar{p}\left(\frac{v'}{v}\right) \) and \( \bar{q} < \bar{p}\left(\frac{q}{p}\right) \) and by Lemma 1.4.1,

\[ uLu \leq 0. \]

We rewrite Picone’s Identity as

\[ \left(\frac{u}{v} \tilde{W}(t)\right)' = (q - \bar{q})u^2 + (\bar{p} - p)u'^2 + \bar{p}\left(u' - u\left(\frac{v'}{v}\right)\right)^2. \]
Since, $\bar{p} \left( u' - u \left( \frac{u'}{v} \right) \right)^2 > 0$ then,

$$
\int_0^R \left( \frac{u\tilde{W}(t)}{v} \right)' \, dt \geq \int_0^R (q - \bar{q})u^2 + (\bar{p} - p)u'^2 \, dt
$$

$$
= \int_0^R -p(u')^2 + qu^2 \, dt - \int_0^R -\bar{p}(u')^2 + q(u)^2 \, dt.
$$

Using integration by parts we see that

$$
\int_0^R -p(u')^2 + qu^2 \, dt - \int_0^R -\bar{p}(u')^2 + q(u)^2 \, dt
$$

is

$$
= \int_0^R \left( (pu')' + qu \right) ud\xi - \int_0^R \left( (\bar{pu}')' + \bar{q}u \right) ud\xi
$$

$$
= 0 - \int_0^R uLu \, dt \leq 0.
$$

However, using $u'(0) = v'(0) = v(R) = 0$ and the fact that $v'(R) < 0$

$$
\frac{u}{v} (\bar{pu}' - pu') \bigg|_0^R = \left[ pu^2(R) \left( \frac{v'(R)}{v(R)} \right) - pu(R)u'(R) \right] - \left[ \bar{pu}^2(0) \left( \frac{v'(0)}{v(0)} \right) - pu(0)u'(0) \right] < 0
$$

which is a contradiction.
Chapter 2

Second Order Eigenvalue Problems

2.1 Comparison Theorems for Eigenvalues

We now turn our attention towards eigenvalue problems. We first start with the simplest example of a second order eigenvalue problem:

Example 2 Solve

$$u'' + \lambda u = 0$$

on $[0, R]$ with boundary conditions $u(0) = 0, u(R) = 0$. These boundary conditions are called the Dirichlet boundary conditions.

Solving the above ODE, we obtain the following solution:

$$u = \sin \left( \frac{n\pi}{R} \right)$$

and we only obtain nontrivial solutions when $\lambda = \left( \frac{n\pi}{R} \right)^2$. $\lambda_n = \left( \frac{n\pi}{R} \right)^2$ are called the eigenvalues and the solutions $u$ are the eigenfunctions of the Dirichlet problem.

Figure 2.1: Solutions with Dirichlet B.C.’s

Remark 2.1.1 We should note that changing the boundary conditions, yields different solutions for $u$. For instance, $u'(0) = 0$ and $u'(R) = 0$ are known as the Neumann boundary conditions. This gives eigenfunctions $u = c \cos \left( \frac{n\pi}{R} \right)$ with the same eigenvalues.
Figure 2.2: Solutions with Neumann B.C.’S

We now wish to find an upper and lower bounds for our eigenvalues $\lambda_n$ for a given Sturm Liouville eigenvalue problem. To get bounds for infinitely many of the eigenvalues, we need the following two lemmas. First we show that $u$ has at least as many zeros as $v$.

**Lemma 2.1.1** Let $u$ and $v$ be respective solutions to Eq.(1.2) and Eq.(1.3) where $\bar{p} > p$ and $q > \bar{q}$. Suppose both $u$ and $v$ vanish at some point $t_0$ and that $v$ has an infinite number of successive zeros $t_1, t_2, ..., t_n, ...$ that are to the right of $t_0$. Then $u$ has at least as many zeros as $v$ on every closed interval $[t_0, t_n]$. Moreover, if the successive zeros of $u$ to the right of $t_0$ are $a_1, a_2, ..., a_n, ...$ then $a_n < t_n$ for all $n$ ([9]).

**Proof.** By applying the Sturm-Picone Theorem repeatedly, $u$ has at least one zero in each of the open intervals

$$(t_0, t_1), (t_1, t_2), ..., (t_{n-1}, t_n), ....$$

Hence, the result follows. ■

Now we use the Sturm-Picone theorem to get precise control on the zeroes.

**Lemma 2.1.2** Let $p(t), q(t)$ be positive continuous functions on a closed interval $[0, R]$ which satisfies the following inequality

$$0 < k^2 < q(t) < K^2,$$

$$0 < l^2 < p(t) < L^2.$$

Suppose $u$ satisfies Eq.(1.2) on $[0, R]$. If $t_1, t_2$ are successive zeros of $u$ then

$$\left(\frac{l}{K}\right) \pi < t_2 - t_1 < \left(\frac{L}{k}\right) \pi.$$
In addition, if \( u \) vanishes at 0 and \( R \) and at \( n - 1 \) points in the open interval \((0, R)\) then
\[
\left( \frac{k}{L} \right) \frac{R}{\pi} < n < \left( \frac{K}{l} \right) \frac{R}{\pi}
\]
([9]).

**Proof.** We first show \( \left( \frac{l}{K} \right) \frac{R}{\pi} < t_2 - t_1 \left( \frac{l}{K} \right) \frac{R}{\pi} \) by examining the equation of \( Lu'' + k^2 u = 0 \), which can be written as \( u'' + \left( \frac{k}{L} \right)^2 u = 0 \). We find that a non-trivial solution that vanishes at \( t_1 \) is \( u(t) = \sin \left( \frac{k}{L} (t - t_1) \right) \). By the periodicity of \( u \), the next zero will be \( t_1 + \left( \frac{l}{K} \right) \frac{R}{\pi} \) and by the Sturm Comparison Theorem the next zero \( t_2 \) occurs before this, which means \( t_2 < t_1 + \left( \frac{l}{K} \right) \frac{R}{\pi} \) or \( t_2 - t_1 < \left( \frac{l}{K} \right) \frac{R}{\pi} \). The other direction follows using a similar argument.

Now to prove \( \left( \frac{k}{L} \right) \frac{R}{\pi} < n < \left( \frac{K}{l} \right) \frac{R}{\pi} \), we realize by paritioning the domain we can obtain \( n \) subintervals between \( n + 1 \) zeroes. So by the inequality we just proved, we get \( R - 0 = R \), which is the sum of the lengths of the \( n \) subintervals, and less than \( n \left( \frac{l}{K} \right) \frac{R}{\pi} \). Therefore, by algebraically rearranging the terms we get \( \left( \frac{k}{L} \right) \frac{R}{\pi} < n \). The other direction of the inequality follows using a similar argument.

\[\blacksquare\]

**Theorem 2.1.3** Consider the following differential equation
\[
p(t)u'' + \lambda q(t)u = 0, \quad (2.1)
\]
where \( p(t), q(t) \) are continuous positive functions on a closed interval \([0, R]\). For each \( \lambda \), let \( u_\lambda \) be the unique solution of Eq. (2.1) that satisfy the initial conditions \( u_\lambda(0) = 0 \) and \( u'_\lambda(0) = 1 \). Then we can find an increasing sequence of positive values
\[
\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots
\]
that goes to infinity with the property that \( u_\lambda(R) = 0 \) if and only if \( \lambda \) equals to one of the \( \lambda_n \)'s. Moreover, \( u_{\lambda_n} \) has exactly \( n - 1 \) zeros in the open interval \((0, R)\). ([9])

**Proof.** Since \( p(t), q(t) \) are continuous on a compact interval \([0, R]\), there exist positive numbers \( k, K, l, L \) such that \( 0 < k^2 < q(t) < K^2 \) and \( 0 < l^2 < p(t) < L^2 \). By the Sturm comparison theorem \( u_\lambda \) oscillates more rapidly on \([0, R]\) than the solutions to \( Lu'' + \lambda k^2 u = 0 \) but less rapidly than the solutions to \( lu'' + \lambda K^2 u = 0 \). Then by the Lemma 2.1.2, when \( \lambda > 0 \) and small enough, we have \( \frac{\pi}{\sqrt{\lambda}} K \geq R \) so that the function \( u_\lambda \) will not have any zeroes in \([0, R]\) to the right of 0. Now, when \( \lambda \) increases to the point where \( \frac{\pi}{\sqrt{\lambda}} k \leq R \) then \( u_\lambda \) has at least one zero. Hence, as \( \lambda \) goes to infinity, the number of zeroes of \( u_\lambda \) on \([0, R]\) goes to infinity as well. It follows by Lemma 2.1.1 that the \( nth \) zero of \( u_\lambda \) to the right of 0 moves to the left continuously as \( \lambda \).
increases. As a result, as \( \lambda \) starts from 0 and goes to infinity there are infinitely many eigenvalues \( \lambda_1, \lambda_2, \lambda_3, ..., \lambda_n, ... \) for which a zero of \( u_\lambda \) reaches \( R \) and is in the interval so that \( u_{\lambda_n}(0) = u_{\lambda_n}(R) = 0 \). Moreover, it will have \( n - 1 \) zeroes in \( (0, R) \).

Applying Lemma 2.1.2 to the sequence \( \lambda_1, \lambda_2, \lambda_3, ..., \lambda_n, ... \) we obtain the following bounds

\[
\frac{\sqrt{\lambda_n}kR}{L\pi} < n < \frac{\sqrt{\lambda_n}KR}{l\pi}.
\]

Rearranging the terms yields

\[
\frac{l^2n^2\pi^2}{K^2R^2} < \lambda_n < \frac{L^2n^2\pi^2}{k^2R^2}.
\]

If \( p(0) = 0 \) the resulting Sturm Liouville equation is called singular. By applying first Theorem 1.2.3 and then the Sturm-Picone theorem after the first zero we also have the following, for the case that \( u'(0) = 0 \). Reid’s theorem may also be used to get a similar result in the singular case.

**Theorem 2.1.4** Let Eq. (2.1) be given where \( p(t), q(t) \) are continuous positive functions on a closed interval \([0, R]\). For each \( \lambda \), let \( u_\lambda \) be the unique solution of Eq. (2.1) that satisfy the initial conditions \( u_\lambda(0) = 1 \) and \( u'_\lambda(0) = 0 \). Then we can find an increasing sequence of positive values

\[
\lambda_1 < \lambda_2 < \lambda_3 < ... < \lambda_n < ...
\]

that goes to infinity with the property that \( u_\lambda(R) = 0 \) if and only if \( \lambda \) equals to one of the \( \lambda'_n \)'s. Moreover, \( u_{\lambda_n} \) has exactly \( n - 1 \) zeros in the open interval \( (0, R) \).

Finally, for the first eigenvalue \( \lambda_1 \) Reid’s Theorem works for singular Sturm Liouville problems to get the following theorem.

**Theorem 2.1.5** Consider the boundary conditions to the following eigenvalue problem.

\[
(py')' + \lambda qy = 0, \quad y'(0) = y(R) = 0 \tag{2.2}
\]

and

\[
(\bar{p}y')' + \lambda \bar{q}y = 0, \quad y'(0) = y(R) = 0 \tag{2.3}
\]

where \( p, \bar{p}, q, \bar{q} \) are nonnegative functions on \([0, R]\). Let \( \tilde{\lambda} \) and \( \bar{\lambda} \) be the first eigenvalues of Eq. (2.2) and Eq. (2.3) respectively. If \( \frac{q}{\bar{p}} \leq \frac{\bar{q}}{p} \) and \( \frac{q'}{\bar{p}'} \leq \frac{\bar{q}'}{p'} \) then \( \tilde{\lambda} \leq \bar{\lambda} \).

**Proof.** If \( \tilde{\lambda} \) is the eigenvalue for Eq. (2.3) along with if Eq. (2.2) has a zero before \( R \) then its corresponding eigenvalue \( \lambda \) for Eq. (2.2) is greater than \( \lambda \). In the figure below,
we see that the solution to Eq.(2.3) has a zero before Eq.(2.2) and shows that this is a contradiction to Reid’s Theorem because zeros before $R$ must have eigenvalues less than $\tilde{\lambda}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.3.png}
\caption{Proof of Theorem 2.1.5}
\end{figure}

We must decrease $\lambda$ for $y(R)$ to be 0.
Chapter 3

The Laplacian, Volume Comparison Properties and Inequalities

3.1 Finding the Laplacian and Sectional Curvature for a Sphere

Definition 3.1.1 A \textit{warped product} is a metric on \((0, R) \times S^{n-1}\) given by

\[ g = dr^2 + \phi(r) dS^{n-1}. \]

That is, the coordinate vector fields satisfy

\[ g \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) = 1 \]
\[ g \left( \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_i} \right) = 0 \]
\[ g \left( \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_i} \right) = \phi^2(r). \]

Figure 3.1: Warped Product

We derive the Laplacian on the warped product. Koszul’s Formula says the following for the connection:

\[ 2g(\nabla_x Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \]
\[ + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X). \]

We consider the orthonormal basis \(e_1 = \frac{\partial}{\partial r}, e_2 = \frac{\partial}{\partial \theta_2}, ..., e_n = \frac{\partial}{\partial \theta_n} \). Now, because \(e_1 = \frac{\partial}{\partial r}\) is a unit vector tangent to the geodesics, \(\nabla_{e_1} e_1 = 0\). Using properties of lie brackets, we easily get that \([e_1, \frac{\partial}{\partial r}] = \frac{\partial \phi}{\partial r} e_i\). By Koszul’s formula,

\[ g \left( \nabla \frac{\partial}{\partial r}, e_i \right) = 0. \]
\[ g(e_i, e_j) = \delta_{ij} \]
\[ [e_1, e_j] = \frac{\phi'}{\phi} \frac{\partial}{\partial \theta_i} \text{ for } j \geq 2 \]
\[ g \left( \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_i} \right) = \phi^2 \delta_{ij} \]

Table 3.1: The following results are placed into Koszul’s Formula.

This gives us the covariant derivatives recorded in the table below.

<table>
<thead>
<tr>
<th>(X)</th>
<th>(Y)</th>
<th>(\frac{\partial}{\partial r})</th>
<th>(\frac{\partial}{\partial \theta_i})</th>
<th>(e_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{\partial}{\partial r})</td>
<td>0</td>
<td>(\frac{\phi'}{\phi} \frac{\partial}{\partial \theta_i})</td>
<td>(\frac{\phi'}{\phi} e_i)</td>
<td></td>
</tr>
<tr>
<td>(\frac{\partial}{\partial \theta_i})</td>
<td>(\frac{\phi'}{\phi} \frac{\partial}{\partial \theta_i})</td>
<td>(\times)</td>
<td>(\times)</td>
<td></td>
</tr>
<tr>
<td>(e_i)</td>
<td>(\frac{\phi'}{\phi} e_i)</td>
<td>(\times)</td>
<td>(\times)</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.2: \(\nabla_X Y\) for basis vector fields

Hence, all the terms of Koszul’s formula become 0, which means that \(\nabla \frac{\partial}{\partial \theta_i} = 0\). We now derive the Laplacian on a function \(f\), using the definition

\[ \Delta f = \text{tr}(\text{Hess}_f) \]

where \(\text{Hess}_f(X, Y) = g(\nabla_X f, Y)\) is the Hessain Tensor.

We clearly see by the definition of the Hessian, \(\text{Hess}_f(X) = \nabla_X \nabla f\), that

\[ \text{Hess}_f(e_1, e_1) = g(\nabla_{e_1} \nabla f, e_1). \]

Continuing for \(i = 2, 3...n\) we get

\[ \text{Hess}_f(e_i, e_i) = \frac{1}{\phi^2} g \left( \nabla \frac{\phi}{\phi r} \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta_i} \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial \theta_i} \right). \]

Now we can get that,

\[ g \left( \nabla \frac{\phi}{\phi r} \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta_i} \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial \theta_i} \right) = 0 + \frac{1}{\phi^2} \frac{\partial f}{\partial r} g \left( \nabla \frac{\phi}{\phi r} \frac{\partial f}{\partial r} \frac{\partial}{\partial \theta_i} \right) \]

Hence,

\[ g \left( \nabla \frac{\phi}{\phi r} \frac{\partial f}{\partial r} \frac{\partial}{\partial r} \right) = \frac{\phi'}{\phi} \frac{\partial}{\partial r}. \]

For the \(\frac{\partial}{\partial \theta_i}\) parts, we get

\[ \sum_{i=2}^{n} \text{Hess}_f \left( \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_i} \right). \]
It follows from the definition of the gradient and the fact that since the sum starts at \( i = 2 \) up to \( n \), we are summing \( n - 1 \) components of the second order partials which gives us the Laplacian hence the \( \Delta_{S^{n-1}} \). Therefore, we get

\[
\Delta f = \frac{\partial^2 f}{\partial r^2} + (n - 1) \frac{\phi'}{\phi} \frac{\partial f}{\partial r} + \frac{1}{\phi^2} \Delta_{S^{n-1}} f.
\]

Finally, we compute the sectional curvature on \((0, R) \times S^{n-1}\). The sectional curvature \( \text{Sec}(u, v) \) is defined as \( \text{Sec}(u, v) = R(u, v, v, u) \) for an orthonormal pair of vectors \( u, v \). The sectional curvature \( \text{Sec}(e_i, \frac{\partial}{\partial r}) \) is the radial curvature and since we have an orthonormal pair \( e_1, e_i \) can compute the following

\[
R \left( e_i, \frac{\partial}{\partial r} \right) \frac{\partial}{\partial r} = \nabla_{e_i} \left( \frac{\phi'}{\phi} \right) - \nabla_{\frac{\phi'}{\phi}} \nabla_{e_i} \frac{\partial}{\partial r} - \nabla_{[e_i, \frac{\phi}{\phi}]} \frac{\partial}{\partial r}.
\]

Using the results from the table above, we get

\[
R \left( e_i, \frac{\partial}{\partial r} \right) \frac{\partial}{\partial r} = \nabla_{e_i} (0) - \nabla_{\frac{\phi'}{\phi}} \left( \frac{\phi'}{\phi} e_i \right) - \nabla_{\left( \frac{\phi'}{\phi} e_i \right)} \frac{\partial}{\partial r}.
\]

Applying the results from the table once more we obtain

\[
R \left( e_i, \frac{\partial}{\partial r} \right) \frac{\partial}{\partial r} = \left( \frac{\phi'}{\phi} \right)' e_i + \frac{\phi'}{\phi} (0) + \left( \frac{\phi'}{\phi} \right) e_i.
\]

Expanding the derivative using the Quotient Rule,

\[
\left( -\phi'' \phi + (\phi')^2 \phi^2 - \frac{\phi'}{\phi^2} \right) e_i.
\]

Hence,

\[
-\frac{\phi''}{\phi} e_i
\]

Therefore,

\[
\text{Sec} \left( e_i, \frac{\partial}{\partial r} \right) = R \left( e_i, \frac{\partial}{\partial r}, \frac{\partial}{\partial r}, e_i \right) = -\frac{\phi''}{\phi}.
\]

3.2 Logarithmic Volume Growth

**Definition 3.2.1** We define the **volume form** \( h = \frac{\omega'}{\omega} \) where \( \omega = \phi^{n-1} \). Thus,

\[
h = \frac{\omega'}{\omega} = \frac{d}{dt}(\log(\omega)) = \frac{(n - 1)\phi^{(n-2)}\phi'}{\phi^{(n-1)}} = (n - 1)\frac{\phi'}{\phi}
\]

where \( \frac{d}{dt}(\log(\omega)) \) is called the **Logarithmic Volume Growth**.

Now, we establish a relationship with the volume form and the sectional curvature. We are able to find the volume of the cap using the volume form \( \omega = \phi^{(n-1)} \). Hence,
we can the volume of the cap by
\[ \int_0^R \omega \, dt \]
then by substitution,
\[ \int_0^R \phi^{(n-1)}(t) \, dt \]
where \( \phi \) controls the volume.

**Theorem 3.2.2** Let \( h = \frac{\omega'}{\omega} \) be the volume form, where \( \omega = \phi^{n-1} \) and let \( k \) and \( K \) be given curvatures and let the sectional curvature be given. If \( K \leq -\frac{\phi''}{\phi} \leq k \). Then,
\[-(n-1)K \leq h' + \frac{h^2}{n-1} \leq -(n-1)k.\]

**Proof.** Take the Logarithmic Volume Form Growth \( h = (n-1)\frac{\phi'}{\phi} \). Then
\[ h' = (n-1) \left( \frac{\phi'}{\phi} \right)' = (n-1) \frac{\phi'' \phi - (\phi')^2}{\phi^2} = (n-1) \frac{\phi''}{\phi} - (n-1) \left( \frac{\phi'}{\phi} \right)^2 \]
Hence,
\[ h' + \frac{h^2}{(n-1)} = (n-1) \frac{\phi''}{\phi} = -(n-1) \left( -\frac{\phi''}{\phi} \right) \]
Note, that \(-(n-1)\frac{\phi''}{\phi}\) is called the Ricci curvature. Therfore, by assumption since, \( K \leq -\frac{\phi''}{\phi} \leq k \) then
\[-(n-1)K \leq h' + \frac{h^2}{n-1} \leq -(n-1)k.\]

If \( -\frac{\phi''}{\phi} \geq K \) then \( \phi'' \frac{\phi}{\phi} \leq -K \). So, we saw from Theorem 3.2.2
\[ h' + \left( \frac{1}{n-1} \right) h^2 \leq -(n-1)k \]
and as a result,
\[ h' + \left( \frac{1}{n-1} \right) h^2 \leq h'_k + \left( \frac{1}{n-1} \right) h_k^2. \]
From comparison theory of first order ODE’s \( h(t) \) and \( h_k(t) \) don’t have finite initial values at \( t = 0 \) if \( h_k(t) \) and \( h(t) \sim \frac{(n-1)}{t} \) as \( t \to 0 \). Thus,
\[ \lim_{t \to 0} th(t) = n - 1 \]
\[ \lim_{t \to 0} th_k(t) = n - 1 \]
and we have that
\[ h \leq h_k \]
([8].)
We will refer to the above as mean curvature comparison. Now we can use the definition of \( h \) and \( h \leq h_k \) we obtain
\[ \frac{\omega'}{\omega} \leq \frac{\omega_k'}{\omega_k}. \]
By integration,
\[ \log(\omega) \leq \log(\omega_k). \]
Therefore,
\[ \omega \leq \omega_k. \]
We will refer to the above as volume comparison.

3.3 The Laplacian and Separation of Variables

Definition 3.3.1 Let
\[ B_p(R) = \{ x \in M | d(p, x) \leq R \}. \]
In coordinates, \( B_p(R) \) is the set of points \((r, \theta)\) with \(0 \leq r \leq R\).

We now solve the eigenvalue problem on a disc in a warped product by separation of variables.

We use the following definition of the Laplacian from before:
\[ \Delta f = \frac{\partial^2 f}{\partial r^2} + (n-1)\frac{\phi'}{\phi} \frac{\partial f}{\partial r} + \frac{1}{\phi^2} \Delta_{S^{n-1}} f. \]

We start with the substitution of \( f(r, \theta) = y(r)\Theta(\theta) \). Taking the derivatives with respect to \( r \) we obtain \( \frac{\partial f}{\partial r} = y'(r)\Theta(\theta) \) and \( \frac{\partial^2 f}{\partial r^2} = y''(r)\Theta(\theta) \). Now suppose \( \Theta \) is an eigenfunction of \( \Delta_{S^{n-1}} \), so that
\[ \Delta \Theta = -l(n+l-2)\Theta. \]

Here we have used that \( \lambda_n(S^{n-1}) = l(n+l-2) \). Plugging all of this into the Laplacian gives
\[ y(r)''\Theta(\theta) + (n-1)\frac{\phi'}{\phi} (y(r)\Theta(\theta)) - \frac{1}{\phi^2} (l(n+l-2)y(r)\Theta(\theta)) = -\lambda y(r)\Theta(\theta). \]

We can factor out and cancel \( \Theta \) to get an ODE for \( y \).
\[ y(r)'' + (n-1)\frac{\phi'}{\phi} (y(r)') + \left( \frac{-1}{\phi^2} (l(n+l-2)) + \lambda \right) y(r) = 0 \]
Proposition 3.3.2 The first eigenvalue of the Dirichlet problem on the ball $B_p(R) \subseteq \mathbb{R}^n$ is

$$\lambda_1(B_p(R)) = \frac{j_{n-2}^2}{R^2}.$$ 

Proof. We know that the Laplacian in $\mathbb{R}^n$ is

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{n-1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}}.$$ 

The first eigenfunction $f$ is positive, and therefore $\Theta = 1$ and $\Delta_{S^{n-1}} f = 0$. So the function $y(r)$ gives the first eigenfunction if

$$y'' + \frac{n-1}{t} y' + \lambda y = 0 \quad (3.1)$$

with boundary values $y'(0) = 0$, and $y(R) = 0$. By the chain rule we see if $y_1$ solves $y_1'' + \frac{n-1}{t} y_1' + y_1 = 0$ then $y = y_1(kt)$ solves Eq.(3.1).

$$y_1'' + \frac{n-1}{t} y_1' + k^2 y = 0.$$ 

We want to compare $y_1$ to a Bessel function $J_\nu(x)$ which solves

$$z'' + \frac{1}{t} z' + \left(1 - \frac{\nu^2}{t^2}\right) z = 0.$$ 

Applying the Liouville transformation, we have $Y_1 = t^{n/2}y_1$ and $Z = t^{\frac{3}{2}}z$ solve

$$Y_1'' + \left(1 - \frac{n^2 - 4n + 3}{4t^2}\right) Y_1 = 0$$

$$Z'' + \left(1 - \frac{4\nu^2 - 1}{4t^2}\right) Z = 0.$$ 

Setting $n^2 - 4n + 3 = 4\nu^2 - 1$ gives $\nu = \frac{n-2}{2}$. Therefore,

$$t^{\frac{n-1}{2}} y_1 = t^{\frac{1}{2}} J_{\frac{n-2}{2}}(x).$$

Or,

$$y_1(x) = t^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(x) = \frac{J_\nu(x)}{t^\nu}$$

where $\nu = \frac{n-2}{2}$. 


Therefore, a solution to Eq. (3.1) with \( y(0) = 0 \), and \( y(R) = 0 \) is

\[
\frac{J_\nu(kt)}{(kt)^\nu}
\]

where \( kR = j_\nu \) and

\[
\lambda_1 = \lambda^2 = \left( \frac{j_\nu}{R} \right)^2.
\]

The following table lists the first eigenvalues for the Laplacian on a disc.
Table 3.3: Eigenvalues $\lambda_1 = j_\nu^2$ on the unit disc

<table>
<thead>
<tr>
<th>$n$</th>
<th>$j_\nu$</th>
<th>$j_\nu^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.404825558</td>
<td>5.783185964</td>
</tr>
<tr>
<td>3</td>
<td>3.141592654</td>
<td>9.869604404</td>
</tr>
<tr>
<td>4</td>
<td>3.831705970</td>
<td>14.681970641</td>
</tr>
<tr>
<td>5</td>
<td>4.493409458</td>
<td>20.190728557</td>
</tr>
<tr>
<td>6</td>
<td>5.139622302</td>
<td>26.374616429</td>
</tr>
<tr>
<td>7</td>
<td>5.763459197</td>
<td>33.217461915</td>
</tr>
<tr>
<td>8</td>
<td>6.380161896</td>
<td>40.706465819</td>
</tr>
<tr>
<td>9</td>
<td>6.987932001</td>
<td>48.831193651</td>
</tr>
<tr>
<td>10</td>
<td>7.588342435</td>
<td>57.582940911</td>
</tr>
</tbody>
</table>

3.4 The Rayleigh Quotient and Cheng’s Theorem

Remark 3.4.1 If

$$\Delta f = -\lambda_1 f$$

and $f = 0$ on $\partial B_p(r)$ then by Green’s Theorem

$$\frac{\int_{B_p(r)} \| \nabla f \|^2 dV}{\int_{B_p(r)} f^2 dV} = -\frac{\int_{B_p(r)} f \Delta f dV}{\int_{B_p(r)} f^2 dV}.$$ 

Therefore, if $\Delta f = \lambda_1 f$,

$$\frac{\int_{B_p(r)} \| \nabla f \|^2 dV}{\int_{B_p(r)} f^2 dV} = \frac{\int_{B_p(r)} f (\lambda_1 f dV}{\int_{B_p(r)} f^2 dV} = \lambda_1 \left( \frac{\int_{B_p(r)} f^2 dV}{\int_{B_p(r)} f^2 dV} \right) = \lambda_1.$$

Fact 3.4.1 (Rayleigh Quotient) $\lambda_1$ is characterized by

$$\lambda_1(B_p(R)) = \inf_{f \in C} \frac{\int_{B_p(R)} (\nabla f)^2 dV}{\int_{B_p(R)} f^2 dV},$$

where

$$C = \{ f \mid f \text{ satisfies } BC's \ f|_{\partial B_p(R)} = 0 \}.$$

Using the Rayleigh Quotient we now prove an important result concerning the first eigenvalue which is known as Cheng’s Theorem.
**Theorem 3.4.2 (Cheng)** Let $M$ be a Riemannian Manifold and Sec($M$) $\geq K$. Let $B_K(R)$ be the ball of radius $r$ in a manifold of constant sectional curvature $K$ and let $B_p(R)$ be given. Then,

$$\lambda_1 (B_p(R)) \leq \lambda_1^K (R).$$

**Proof.** Suppose $\phi$ is the first eigenfunction of $B_K(r)$. Then by definition of the Riemannian metric, we have $\phi'' + h_K \phi' + \lambda^K_1 (r) \phi = 0$ where $h_K$ is the volume form. By the Rayleigh quotient we have,

$$\frac{\int_{B_p(R)} |\nabla \phi|^2 \omega dt}{\int_{B_p(R)} \phi^2 \omega dt}.$$  

Moreover,

$$\frac{\int_{B_p(R)} |\nabla \phi|^2 \omega dt}{\int_{B_p(R)} \phi^2 \omega dt} \geq \lambda_1 (B_p(R))$$

because

$$\lambda_1 (B_p(R)) = \inf \frac{\int_{B_p(R)} |\nabla f|^2 \omega dt}{\int_{B_p(R)} f^2 \omega dt}.$$  

Now, examining the numerator to the Rayleigh quotient, $\int_{B_p(R)} |\nabla \phi|^2 \omega dt$, if we use integration by parts we get

$$- \int_{B_p(R)} \phi \Delta \phi \omega dt.$$  

By the definition of the Laplacian operator we see,

$$\int_{B_p(R)} \phi (\phi'' + h \phi') \omega dt.$$  

Using mean curvature comparison leads to

$$\int_{B_p(R)} \phi (\phi'' + h \phi') \omega dt \leq \int_{B_p(R)} \phi (\phi'' + h_K \phi') \omega dt.$$  

Using the Riemannian metric, we can rewrite the right side of the inequality as

$$\lambda^K_1 (r) \int_{B_p(R)} \phi^2 \omega dt.$$  

Therefore,

$$\frac{\int_{B_p(R)} |\nabla \phi|^2 \omega dt}{\int_{B_p(R)} \phi^2 \omega dt} \leq \lambda^K_1 (R).$$  

Consequently,

$$\lambda_1 (B_p(R)) \leq \frac{\int_{B_p(R)} |\nabla \phi|^2 \omega dt}{\int_{B_p(R)} \phi^2 \omega dt} \leq \lambda^K_1 (R).$$
Proof 2. Now, we will also prove Cheng’s Theorem using Reid’s Theorem, to demonstrate how the material from before can be used. If we let \( p = q = \omega \) and \( \bar{p} = \bar{q} = \omega_k \) then clearly,

\[
1 = \frac{p}{q} \leq \frac{\bar{p}}{\bar{q}} = 1.
\]

In addition, from Reid’s Theorem \( \frac{\dot{p}}{p} \leq \frac{\dot{q}}{q} \). So by definiton of the Laplacian we can get that \( \lambda_1(B_p(R)) \) satisfies

\[
(\omega u')' + \lambda \omega u = 0
\]

with \( u'(0) = u(R) = 0 \) and \( \lambda_1^{\text{k}} \) satisfies

\[
(\omega_k v')' + \lambda \omega_k v = 0
\]

with \( v'(0) = v(R) = 0 \). Now using mean curvature comparison we see that

\[
h \leq h_k
\]

and hence,

\[
\frac{\omega'}{\omega} \leq \frac{\omega'_k}{\omega_k}.
\]

Then by Proposition 2.1.5,

\[
\lambda_1(B_p(R)) \leq \lambda_1^{K}(R).
\]
3.5 Symmetrization of Functions

We now assume that $M$ is a warped product of curvature $\geq 1$. That is, $-\frac{\phi''}{\phi} \geq 1$.

Then by volume comparison, we see that

$$\frac{\text{vol}(B_p(r))}{\text{vol}(M)} \geq \frac{\text{vol}(B_k(r))}{\text{vol}(S^{n-1})}.$$ 

Moreover, by volume comparison we see that $\frac{\text{vol}(B_p(r))}{\text{vol}(B_k(r))}$ is a decreasing. Thus,

$$\frac{\text{vol}(B_p(r))}{\text{vol}(B_k(r))} \geq \frac{\text{vol}(M)}{\text{vol}(S^{n-1})}.$$ 

Since $\text{vol}(B_k(r))$ is increasing, we have that for each $r$ there is an $r^* \geq r$ such that

$$\frac{\text{vol}(B_p(r))}{\text{vol}(M)} = \frac{B_p(r^*)}{\text{vol}(S^{n-1})}.$$ 

We now discuss how to symmetrize a function $f$ on a Riemannian Manifold. By the
volume comparison we have above we have that
\[ \int_0^t \frac{\omega(s) ds}{\text{vol}(M)} \geq \int_0^t \frac{\omega(s) ds}{\text{vol}(S^{n-1})}. \]

Then there exists \( u(t) \geq t \) such that
\[ \int_0^t \frac{\omega(s) ds}{\text{vol}(M)} = \int_0^u \frac{\omega_k(s) ds}{\text{vol}(S^{n-1})}. \quad (3.2) \]

We want to find \( \frac{du}{dt} \), so we take the derivative of both sides to get
\[ \frac{1}{\text{vol}(M)} \frac{d}{dt} \int_0^t \omega(s) ds = \frac{1}{\text{vol}(S^{n-1})} \frac{d}{dt} \int_0^u \omega_k(s) ds. \]

Using the Fundamental Theorem of Calculus and the chain rule, we see that
\[ \frac{1}{\text{vol}(M)} \omega(t) = \frac{1}{\text{vol}(S^{n-1})} \frac{du}{dt} \int_0^u \omega_1(s) ds. \]

By the Fundamental Theorem of Calculus and Eq.(3.2) we see that the right hand side of the preceding equation can be rewritten as
\[ \frac{du}{dt} \omega_1(u) \left( \frac{1}{\text{vol}(S^{n-1})} \right). \]

Therefore,
\[ \frac{du}{dt} = \frac{\omega(t)}{\omega_1(u)} \frac{\text{vol}(S^{n-1})}{\text{vol}(M)}. \]

Now, given \( f : B_p(R) \to \mathbb{R}^+ \) we define \( f^* : B_{p^*}(R^*) \to \mathbb{R}^+ \) by \( f^*(u) = f(t) \). Then we have for any integrable \( f \),
\[ \frac{1}{\text{vol}(M)} \int_0^R f(t) \omega(t) dt \]
\[ = \int_0^{R^*} f(u) \frac{du}{dt} \omega(t) du \]
\[ = \frac{1}{\text{vol}(S^{n-1})} \int_0^{R^*} f(u) \omega_1(u) du. \]

Similarly,
\[ \frac{1}{\text{vol}(M)} \int_0^R f^2(t) \omega(t) dt = \frac{1}{\text{vol}(S^{n-1})} \int_0^{R^*} (f^*(u))^2 \omega_1(u) du \]

which will be important for later.
3.6  The Gromov-Levy Isoperimetric Inequality

Let $M$ be a Riemannian manifold where we divide $M$ into a disjoint union of connected sets $B(r)$ and $B^c(r)$. Let $B_1(\bar{r})$ and $B_1^c(\bar{r})$ be the connected open sets on $S^n$.

Using the mean curvature comparison for the sphere,

$$\text{vol}(B(r)) \leq \frac{\omega(r)}{\omega_1(\bar{r})} \int_0^\bar{r} \omega_1(t)dt,$$

$$\text{vol}(B^c(r)) \leq \frac{\omega(r)}{\omega_1(\bar{r})} \int_\bar{r}^\pi \omega_1(t)dt,$$

where $\bar{r}$ is such that $h_1(\bar{r}) = h(r)$. Hence,

$$\omega(r) \geq \frac{\text{vol}(B(r))}{\text{vol}(B_1(\bar{r}))},$$

$$\omega(r) \geq \frac{\text{vol}(B^c(r))}{\text{vol}(B_1^c(\bar{r}))}.$$

Since we can’t control $\bar{r}$, we need to take the worst possible case

$$\frac{\omega(r)}{\omega_1(\bar{r})} \geq \inf_{\bar{r}} \max \left\{ \frac{\text{vol}(B(r))}{\text{vol}(B_1(\bar{r}))}, \frac{\text{vol}(B^c(r))}{\text{vol}(B_1^c(\bar{r}))} \right\}.$$

This happens when

$$\frac{\text{vol}(B(r))}{\text{vol}(B_1(\bar{r}))} = \frac{\text{vol}(B^c(r))}{\text{vol}(B_1^c(\bar{r}))},$$

or if and only if

$$\frac{\text{vol}(B(r))}{\text{vol}(B_1(\bar{r}))} = \frac{\text{vol}(M) - \text{vol}(B(r))}{\text{vol}(S^n) - \text{vol}(B_1(\bar{r}))}.$$
Solving this gives

\[
\frac{\text{vol}(B(r))}{\text{vol}(M)} = \frac{\text{vol}(B_1(\bar{r}))}{\text{vol}(S^n)}.
\]

Therefore,

\[
\omega(r) \geq \frac{\text{vol}(M)}{\text{vol}(S^n)} \omega_1(\bar{r}).
\]

### 3.7 The Faber-Krahn and Talenti’s Inequalities

Although Cheng’s Theorem gives an upper bound for \(\lambda_1(B_p(R))\) by \(\lambda^K_1(R)\), the Faber-Krahn inequality establishes a lower bound for \(\lambda_1(B_p(R))\) by \(\lambda_1\) of a small ball \(B_1(R^*)\).

Before proving the Faber-Krahn Inequality, we need to establish a fundamental result that we will need to use to prove the Faber-Krahn Inequality. The inequality in question is known as Talenti’s Inequality. We first provide the background that is required to prove Talenti’s Inequality.

Let \(B(t)\) be a ball in \(M\) and \(B(\tau)\) be a ball in \(S^n\) with

\[
\frac{\text{vol}(B(t))}{\text{vol}(M)} = \frac{\text{vol}(B^*(\tau))}{\text{vol}(S^n)}.
\]

By the Isoperimetric Inequality,

\[
\omega(t) \geq \omega_1(\tau)
\]

where \(\omega_1(\tau)\) is the volume form on \(\sin^{n-1}(\tau)\). Furthermore, let

\[
f : B(t) \to \mathbb{R}
\]

and if \(f^*\) is defined on \(B^*(\tau)\) by \(f^*(\tau) = f(t)\) In addition, \(u\) solves \(\Delta u = f\) and \(u^*\) is defined by \(u^*(\rho) = u(r), \rho > 0\) as well as \(v\) on \(B^*(\tau)\) be defined by \(\Delta v = f^*\). We see that

\[
-\frac{1}{\omega}(\omega u')' = f \text{ on } B(t)
\]

and

\[
-\frac{1}{\omega_1}(\omega_1 u')' = f \text{ on } B^*(t).
\]

Therefore,

\[
u(r) = \int_r^R \left( \int_0^s f(t)\omega(t)dt \right) \frac{1}{\omega(s)}ds
\]

\[
v(\rho) = \int_\rho^{R^*} \left( \int_0^\sigma f(\tau)\omega_1(\tau)d\tau \right) \frac{1}{\omega(\sigma)}d\sigma.
\]
By the definition of \( \tau \) we have that \( \omega(t)dt = \omega_1(\tau)d\tau \), so

\[
    u(r) = \int_{\rho}^{R} \left( \int_{0}^{s} f^*(\tau)\omega_1(\tau)d\tau \right) \frac{1}{\omega(s)} ds
    = \int_{\rho}^{R^*} \left( \int_{0}^{\sigma} f^*(\tau)\omega(\tau)d\tau \right) \frac{1}{\omega(s)} \frac{\omega_1(\sigma)}{\omega(s)} d\sigma.
\]

**Theorem 3.7.1 (Talenti Inequality)** Let \( f \) be a continuous and positive function. Furthermore, let \( u(t) \) be the solution to \( \Delta u = f \) with \( u'(0) = u(R) = 0 \) and let \( u^* \) be defined. Then there exists a function \( v(\rho) \) such that

\[
    u^*(\rho) \leq v(\rho).
\]

**Proof.** Recall that the Isoperimetric Inequality says that

\[
    \omega(s) \geq \omega_1(\sigma).
\]

As a result,

\[
    \frac{\omega_1(\sigma)}{\omega(s)} \leq 1
\]

and

\[
    \frac{1}{\omega(s)} \leq \frac{1}{\omega_1(\sigma)}.
\]

We know that

\[
    u(r) = \int_{\rho}^{R^*} \left( \int_{0}^{\sigma} f^*(\tau)\omega_1(\tau)d\tau \right) \frac{1}{\omega(s)} \frac{\omega_1(\sigma)}{\omega(s)} ds.
\]

By the definition of \( u^* \), we get that

\[
    u^*(\rho) = \int_{\rho}^{R^*} \left( \int_{0}^{\sigma} f^*(\tau)\omega_1(\tau)d\tau \right) \frac{1}{\omega(s)} \frac{\omega_1(\sigma)}{\omega(s)} ds.
\]

Applying the Isoperimetric Inequality,

\[
    u^*(\rho) = \int_{\rho}^{R^*} \left( \int_{0}^{\sigma} f^*(\tau)\omega_1(\tau)d\tau \right) \frac{1}{\omega(s)} \frac{\omega_1(\sigma)}{\omega(s)} ds
\]

\[
    \leq \int_{\rho}^{R^*} \left( \int_{0}^{\sigma} f^*(\tau)\omega_1(\tau)d\tau \right).
\]

We let

\[
    v(\rho) = \int_{\rho}^{R^*} \left( \int_{0}^{\sigma} f^*(\tau)\omega_1(\tau)d\tau \right),
\]

which completes the proof.

Having proven Talenti’s Inequality, we now are able to state and prove the Faber-
Krahn Inequality.

**Theorem 3.7.2 (Faber-Krahn Inequality)** Let $B_p(R)$ be given. Define $\Omega^* \subseteq S^n$ by

\[
\frac{\text{vol}(B_p(r))}{\text{vol}(M)} = \frac{\text{vol}(\Omega^*)}{\text{vol}(S^n)}.
\]

Then

\[
\lambda_1(B_p(R)) \geq \lambda_1(\Omega^*).
\]

**Proof.** Consider the integral

\[
\int_0^R (f(t))^2 \omega(t) dt
\]

where we make the following substitution $z = z(t), z(0) = 0, z(R) = R^*$. Evidently,

\[
\int_0^R (f(z))^2 \omega(z) \frac{dt}{dz} dz.
\]

We define $z(t)$ in a way such that

\[
\frac{dz}{dt} = \frac{\omega(z) \text{vol}(S^n)}{\omega_1(z) \text{vol}(M)}.
\]

Then,

\[
\int_0^R (f(t))^2 \omega(t) dt = \xi \int_0^R (f(z))^2 \omega_1(z) dz
\]

where $\xi = \frac{\text{vol}(M)}{\text{vol}(S^n)}$. In a similar manner, we can find

\[
\int_0^R \left( \frac{df}{dt} \right)^2 \omega(t) dt.
\]

By using the chain rule to get $\frac{dt}{dz} = \frac{df}{dz} \frac{dz}{dt}$ and then using the volume comparison gives us that $\frac{dz}{dt} \geq 1$. As a result, by Talenti’s Inequality

\[
\int_0^R \left( \frac{df}{dt} \right)^2 \omega(t) dt \geq \xi \int_0^R (f(z))^2 \omega_1(z) dz.
\]

Now if we let $f$ be the eigenfunction on $B_p(R)$, then we see that

\[
\lambda_1(B_p(R)) = \frac{\int_0^R \left( \frac{df}{dt} \right)^2 \omega(t) dt}{\int_0^R (f(t))^2 \omega(t) dt}.
\]

Moreover,

\[
\lambda_1(\Omega^*) = \inf_{\phi} \frac{\int_0^{R^*} \left( \frac{d\phi}{dz} \right)^2 \omega_1(z) dz}{\int_0^{R^*} (\phi(z))^2 \omega_1(z) dz}.
\]
Therefore,

\[ \lambda_1(\Omega^*) \leq \lambda_1(B_p(R)). \]
Chapter 4

Fourth Order Eigenvalue Problems

4.1 Analyzing the Oscillatory Properties for a Fourth Order Eigenvalue Problem

As we did with the second order eigenvalue problem in Chapter 2, we begin this chapter by examining the simplest case of the type of eigenvalue problem we want to discuss.

Example 3 Consider

\[ u'''' - \lambda u = 0 \]

on \([0, R]\) with the given boundary conditions

\[ u(0) = 0, u'(0) = 0, u(R) = 0, u'(R) = 0. \]

We want to show that this ODE has non-trivial solutions.

Solving the auxiliary equation for the above ODE yields

\[ u = c_1 \cosh (kt) + c_2 \sinh (kt) + c_3 \cos (kt) + c_4 \sin (kt) \]

where \(k = \frac{\lambda}{4}\). Using the boundary condition \(u(0) = 0\), we get

\[ c_1 + c_3 = 0 \text{ thus, } c_3 = -c_1. \]

In addition, using the boundary condition \(u'(0) = 0\), we get

\[ c_2 + c_4 = 0 \text{ thus, } c_4 = -c_2. \]

Hence,

\[ u = c_1 (\cosh (kt) - \cos (kt)) + c_2 (\sinh (kt) - \sin (kt)). \]

Now using the other boundary conditions \(u(R) = 0, u'(R) = 0\),

\[ c_1 (\cosh (kR) - \cos (kR)) + c_2 (\sinh (kR) - \sin (kR)) = 0 \]

\[ c_1 (\sinh (kR) + \sin (kR)) + c_2 (\cosh (kR) - \cos (kR)) = 0. \]

We want to find non-trivial solutions for \(c_1, c_2\). We consider the following matrix

\[
\begin{bmatrix}
\cosh (kR) - \cos (kR) & \sinh (kR) - \sin (kR) \\
\sinh (kR) + \sin (kR) & \cosh (kR) - \cos (kR)
\end{bmatrix}
\cdot
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \end{bmatrix}.
\]
We can find the nontrivial solutions when the determinant of
\[
\begin{bmatrix}
\cosh (kR) - \cos (kR) & \sinh (kR) - \sin (kR) \\
\sinh (kR) + \sin (kR) & \cosh (kR) - \cos (kR)
\end{bmatrix}
\]
is zero. When we take the determinant, we obtain
\[
(cosh (kR) - cos (kR))^2 - (sinh (kR) - sin (kR))(sinh (kR) + sin (kR)) = 0.
\]
Expanding the expression, we obtain
\[
cosh^2 (kR) - 2cosh (kR) \cos (kR) + \cos^2 (kR)
\]
\[
- \sinh^2 (kR) - \sinh (kR) \sin (kR) + \sin (kR) \sinh (kR) + \sin^2 (kR) = 0.
\]
Using,
\[
sinh^2 (kR) - \cosh^2 (kR) = 1, \quad \sin^2 (kR) + \cos^2 (kR) = 1
\]
we obtain
\[-2cosh (kR) \cos (kR) = -2.
\]
Therefore,
\[
cosh (kR) \cos (kR) = 1.
\]
Moreover, when we solve for \(c_1, c_2\) we see that \(c_1 = 1\) and
\[
c_2 = \frac{- (cosh (kR) - cos (kR))}{(sinh (kR) - sin (kR))}.
\]
Solving for \(k\) by letting \(R = 1\), we obtain \(k = 4.73004074486270, k = 7.85320462409584, k = 10.9956078380017\) and \(k = 14.1371654912575\). The figures below represent the solutions with respect to each \(k\) value.

Figure 4.1: Solution \(y_1\) for \(k = 4.73004074486270\)
Remark 4.1.1 The following graphs show that the solution to the above example will not have zeros outside of [0, 1]. Moreover, we see that the curves approach infinity outside [0, 1]. We generalize and prove this behavior in the next chapter.
There is a larger variety of fourth order boundary value problems to consider, in comparison to the second order case. The previous example is roughly analogous to the Dirichlet problem. In the next example, we show that changing a boundary condition for this eigenvalue problem gives us only the trivial solution.

**Example 4** Consider

\[ u''' - \lambda u = 0 \]

on \([0, R]\) with the given boundary conditions

\[ u(0) = 0, u(R) = 0, u'(R) = 0, u''(R) = 0. \]

We want to show that this ODE has no nontrivial solutions.

Using the same procedure as in the previous example we see that using the fact that \(u(0) = 0\)

\[ u = c_1 \cosh(kt) - c_1 \cos(kt) + c_2 (\sinh(kt) + c_4 \sin(kt)). \]
Using \( u(R) = 0, u'(R) = 0, u''(R) = 0 \), we obtain the following system:

\[
\begin{align*}
c_1 (\cosh (kR) - \cos (kR)) + c_2 \sinh (kR) + c_4 \sin (kR) &= 0 \quad (4.1) \\
c_1 \sinh (kR) + c_1 \sin (kR) + c_2 \cosh (kR) + c_4 \cos (kR) &= 0 \quad (4.2) \\
c_1 \cosh (kR) + c_1 \cos (kR) + c_2 \sinh (kR) - c_4 \sin (kR) &= 0. \quad (4.3)
\end{align*}
\]

Subtracting Eq.(4.1) and Eq.(4.3) we get

\[ c_1 \cos (kR) = 0. \]

If \( c_1 = 0 \) then \( \tanh (kR) = -\tan (kR) \) where this is true if and only if \( c_2 = c_3 = c_4 = 0 \).

Now, if \( \cos (kR) = 0 \) then \( \sin (kR) \neq 0 \). Hence,

\[
\begin{align*}
c_1 \cosh (kR) + c_2 \sinh (kR) + c_4 \sin (kR) &= 0 \\
c_1 \cosh (kR) + c_2 \sinh (kR) - c_4 \sin (kR) &= 0.
\end{align*}
\]

Subtracting the above rows, we get that \( c_4 = 0 \) and consequently, \( c_1 = c_2 = c_3 = 0 \) which means that the trivial solution is the only solution.

### 4.2 Fourth Order Conjugate Points

We now examine the following fourth order ODE

\[
(r(t)y'')'' + (q(t)y')' + p(t)y = 0, \quad r(t), p(t) > 0. \quad (4.4)
\]

The definition of conjugate point must be modified from the second order case. From the above examples we see that there are a wider variety of boundary conditions to consider. We should note that \( q(t) \) in Eq.(4.4) can be eliminated as outlined in section 12 of ([6]). Hence, we will only work with the case of \( q(t) = 0 \).

**Definition 4.2.1** We say that \( \eta \) is a conjugate point for the fourth order ODE if

\[ \eta = \inf \{ t \mid \text{a solution to Eq.}(4.4) \text{ that has } \geq 4 \text{ zeros on } [0, t] \}. \]

**Definition 4.2.2** We say that \( z_{13} \) to be the smallest number such that there is a solution to Eq.(4.4) with \( u(0) = 0 \) and \( u(z_{13}) = u'(z_{13}) = u''(z_{13}) = 0 \).

**Definition 4.2.3** We say that \( z_{22} \) to be the smallest number such that there is a solution to (Eq.4.4) with \( u(0) = u'(0) = 0 \) and \( u(z_{22}) = u'(z_{22}) = 0 \).

Now we rewrite Eq.(4.4) into

\[
(ry(t)''')'' - p(t)y = 0, \quad r(t) > 0, p(t) > 0 \quad (4.5)
\]

[6]. Let us now consider the following lemma.
Lemma 4.2.4 Let $y(t)$ be a solution to Eq. (4.5) and let $y, y', y''$ and $(ry'')'$ be non-negative but not all zero for $t = t_0$. Then $y(t), y'(t), y''(t)$ and $(ry''(t))'$ are positive for $t > t_0$. ([6].)

Proof. We consider the case of which the initial values are non-zero because if the initial values are all zero, then by the Existence and Uniqueness Theorem for ODE’s, $y(t)$ reduces to the trivial solution. We see that $y(t_0) > 0$ or the first nonvanishing derivative at $t_0$ must be positive. As a result, for all cases $y(t)$ will be positive on some open interval where the left boundary point is $t_0$. If $y(t) < 0$ for all $t > t_0$ then by the the Intermediate Value Theorem there exists a point $R$ such that $y(R) = 0$ and $y(t) > 0$ for when $t \in (t_0, R)$. Applying the Fundamental Theorem of Calculus to Eq. (4.5) we get the following

\[ r(t)y''(t) = ry''|_{t=t_0} + (t-t_0) (ry'')'|_{t=t_0} + \int_{t_0}^{t} (t-s)p(s)y(s)ds. \]

By the hypothesis to the lemma, we clearly see that the right side of the above calculation must be positive for when $y(t) > 0$. Hence, $y''(t)$ and $(r(t)y''(t))'$ are positive for $t \in (t_0, R)$. Since $y'(t_0) \geq 0$ then $y'(t)$ will be positive in $(t_0, R)$. Thus, $0 \leq y(t_0) < y(R) = 0$ which is a contradiction. Therefore, we conclude that $y(t), y'(t), y''(t)$ and $(ry''(t))'$ must be positive for $t > t_0$.

This lemma shows us that Eq. (4.5) has nonoscillatory solutions and indicates the number of generated nonoscillatory solutions for Eq. (4.5). We can use the lemma that we just proved to look at the behavior if solutions of Eq. (4.5) for $t < t_0$. This leads to the next lemma.

Lemma 4.2.5 Let $y(t)$ be a nontrivial solution to Eq. (4.5) and let $t_0$ be a positive value. If $y(t_0) \geq 0, y''(t_0) \geq 0, y'(t_0) \leq 0$ and $(ry'')'|_{t=t_0} \leq 0$ then $y(t)$ and $y''(t)$ are positive and $y'(t)$ and $(r(t)y''(t))'$ are negative for $0 < t < t_0$. ([6])

Proof. Let $w$ be a positive number. Consider the following substitution for $t$. If we let $t = t_0 + w - s$, then plugging this into Eq. (4.5) gives us an ODE with the independent variable being $s$. Now by Lemma 4.2.4, if we let $s = w$ then the result follows.

Lemma 4.2.6 Let $y(t)$ be a solution that is not trivial for Eq. (4.5) and let $t_0, x_0, z_0$ be values such that $0 < t_0 < x_0 < z_0$. If $y(t_0) = y(x_0) = y(z_0) = 0$, then $y'(x_0) \neq 0$. ([6])

Proof. Let us assume otherwise. Suppose $y(x_0) = y'(x_0) = 0$. As a result, the second derivative cannot be zero so without a loss of generality we can assume that $y''(x_0) \geq 0$. Consider $(ry'')'|_{t=x_0}$. If $(ry'')'|_{t=x_0} \geq 0$ then by Lemma 4.2.4 $y(t) > 0$
when \( t > t_0 \). However, by assumption \( t_0 < z_0 \) and \( y(z_0) = 0 \), which is a contradiction. Using a similar argument, if \((ry'')'_{|t=t_0} \leq 0\), then this contradicts Lemma 4.2.5 for \( t < t_0 \) and the assumption that \( y(t_0) = 0 \).

Under the assumptions of Lemma 4.2.4 for \( t > t_0 \) we see that \( y, y', y'', (ry'')' \) are nonzero so we can extend Lemma 2.3 to the following result to establish a more general statement. This theorem tells us that the solution to Eq.(4.5) will not have zeros when \( t < t_0 \) and \( t > R \). The solution outside the interval \([0, R]\) will in fact go to infinity.

**Theorem 4.2.7** Let \( y(t) \) be a non-trivial solution to Eq.(4.5) and let \( t_0 < R < z_0 \). If \( y(R) = y'(R) = 0 \) then in at least one of the following two intervals \((0, R), (R, \infty)\) \( y, y', y'', (ry'')' \) are non zero. ([6])

**Proof.** Since \( y(R) = y'(R) = 0 \) we consider two cases about the behavior of \((ry'')'_{|t=R}, y''(R)\) The first case is if \( y''(R) \geq 0 \) and \((ry'')'_{|t=R} \geq 0\). We should note that both cannot be zero because by the Existence and Uniqueness Theorem for ODE’s, all the functions must be zero. So without loss of generality, suppose \( y''(R) > 0 \) and \((ry'')'_{|t=R} \geq 0\). Then by Lemma 4.2.4 \( y > 0 \) for all \( t > R \). If \((ry'')'_{|t=R} \leq 0\) then by Lemma 4.2.5 \( y' < 0 \) for \( t < R \). However, this is a contradiction to the fact that \( y \) has a minimum on \([0, R]\). The second case is if \( y''(R) \leq 0 \) and \((ry'')'_{|t=R} \leq 0\). By Lemma 4.2.4 \( y \) is negative for \( t > R \). If \((ry'')y' \geq 0\) then by Lemma 4.2.5 \( y' > 0 \) for \( t < R \) which contradicts \( y \) having a maximum value on \([0, R]\)

**Proposition 4.2.8** Let \( y(t) \) be a nontrivial solution to Eq.(4.5). Then on \((0, R)\) all zeros of \( y \) are simple, which means that \( y'(x_0) \neq 0 \) for some \( x_0 \in (0, R) \). ([6])

**Proof.** If we assume otherwise, then \( y(t) \neq 0 \) for all \( t > x_0 \). However, this is a contradiction of Theorem 4.2.7 if we replace \( R \) by \( x_0 \).

### 4.3 The Separation Theorem

We now develop the Separation Theorem for fourth order ODE’s but to do so requires the following lemmas.

**Lemma 4.3.1** Let \( g(t) \) and \( h(t) \) be \( C^1 \) functions on \((a,b)\) as well as \( h(t) \) having a constant sign in \((a,b)\). If \( g(t) \) has two distinct zeros \( z_1 \) and \( z_2 \) in \((a,b)\), then the function \( f(t) = h(t)g'(t) - g(t)h'(t) \) cannot have a constant sign in \((0,R)\) bounded by \( z_1, z_2 \). ([6])
**Proof.** Suppose on \((a,b)\) we have \(a < z_1 < z_2 < b\). If \(f(t) > 0\) on \((z_1, z_2)\), then
\[
0 = \frac{g(z_2)}{h(z_2)} = \int_{z_1}^{z_2} \frac{f(t)}{h^2(t)} dt > 0.
\]
This establishes a contradiction, which proves the lemma.

Restating the lemma that we just proved so that we may apply it can be formed in the following way.

**Lemma 4.3.2** Let \(g(t)\) and \(h(t)\) be \(C^1\) functions on \((a,b)\) as well as \(h(t)\) having a constant sign in \((a,b)\). If \(g(t)\) has two distinct zeros \(z_1\) and \(z_2\) in \((a,b)\), then there exists a constant \(\alpha\) such that the function \(g(t) - \alpha h(t)\) has a zero where both the function and its derivative are zero in \((z_1, z_2)\). \((\ref{6})\)

**Proof.** For some point \(t_0\) let \(f(t_0) = 0\). Furthermore, suppose we have two constants \(\alpha, \beta\) where \(\alpha^2 + \beta^2 > 0\) so that
\[
\alpha g(t_0) - \beta h(t_0) = \alpha g'(t_0) - \beta h'(t_0) = 0.
\]
Because \(h(t)\) is never zero on \([z_1, z_2]\), that means \(\alpha\) cannot be zero so we can choose it to be 1 and thus we prove the lemma.

The next issue now the relationship between two nontrivial solutions to Eq.\((4.5)\) and their zeros, which is addressed in the next lemma.

**Lemma 4.3.3** Let \(y(t)\) and \(\bar{y}(t)\) be two nontrivial solutions to Eq.\((4.5)\) and have three zeros in common. Then the zeros are constant multiples of each other. \((\ref{6})\)

**Proof.** Let \(z_1, z_2, z_3\) be three distinct zeros such that \(z_1 < z_2 < z_3\). By Lemma 4.2.6 \(y'(z_2) \neq 0\) and \(\bar{y}'(z_2) \neq 0\). By definition if we let \(f(t) = \bar{y}'(z_2)y(t) - y'(z_2)\bar{y}(t)\), then \(f(t)\) will be a solution that satisfies the following conditions: \(f(z_1) = f(z_2) = f'(z_2) = f(z_3) = 0\). However, by Lemma 4.2.6 this can’t happen unless \(f(t) = 0\), which means that \(\bar{y}(t)\) is a constant multiple of \(y(t)\). If two zeros are the same, say \(z_1 = z_2\), then we take \(f(t) = \bar{y}''(z_1)y(t) - y''(z_1)\bar{y}(t)\). Now, \(y''(z_1)\) and \(\bar{y}''(z_1)\) cannot be zero otherwise by Lemma 4.2.4 \(y(t)\) and \(\bar{y}(t)\) can’t vanish at \(t = z_3\). As a result, \(f(z_1) = f'(z_1) = f''(z_1) = 0\) and \(f(z_3)\) but this is a contradiction to Lemma 4.2.4 unless \(f(t)\) is the trivial solution. If \(y(z) = y'(z) = y''(z) = 0\) and \(\bar{y}(z) = \bar{y}'(z) = \bar{y}''(z) = 0\) for some zero \(z\) then by the Existence and Uniqueness Theorem for Eq.\((4.4)\) we have a unique solution if such conditions are met; hence, \(y(t) = \bar{y}(t)\).

Now, we are ready to state our first separation theorem for fourth order ODE’s.

**Theorem 4.3.4** Let \(y(t)\) and \(\bar{y}(t)\) be two different nontrivial solutions to Eq.\((4.5)\) on \((a,b)\) where \(y(a) = \bar{y}(a) = y(b) = \bar{y}(b) = 0\). Then we have that the zeros of \(y(t)\) and \(\bar{y}(t)\) in \((a,b)\) separate each other. \((\ref{6})\)

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Proof. If \( y(t) \) and \( \bar{y}(t) \) have zeros that coincide then we can take \( \bar{y}(t) \) to be the solution where \( \bar{y}(a) = \bar{y}(t_0) = \bar{y}(b) = 0 \) where \( t_0 \) is a point where \( a < t_0 < b \) and \( y(t_0) \neq 0 \). So using Lemma 4.3.3, we see that no zero of \( y(t) \) can coincide with a zero of \( \bar{y}(t) \) on \((a,b)\). Now suppose the theorem is not true. Then that means we can find two consecutive zeros \( z_1, z_2 \) where \( a<z_1<z_2<b \) of \( y(t) \) where \( \bar{y}(t) \) is nonzero on \([z_1,z_2]\). Then by Lemma 4.3.2 there exists an \( M \) such that \( f(t) = y(t) - Mh(t) \) has a double zero in \((z_1,z_2)\). Since \( f(t) \) is clearly a solution to Eq.(4.5) where \( f(a) = f(b) = 0 \) then we have a contradiction to Lemma 4.2.6.

An important theorem that we need to establish about the eigenvalues on an interval \((a,b)\) is that the eigenvalues to

\[
(ry'')'' - \lambda py = 0, \hspace{1cm} y(a) = y'(a) = y(b) = y'(b) = 0 \tag{4.6}
\]

are simple, which means that for every \( \lambda \) there is one eigenfunction \( y(t) \) or \( cy(t) \) for some constant \( c \) that corresponds to \( \lambda \).

**Theorem 4.3.5** Let Eq.(4.6) be given. Then its eigenvalues are simple. ([6])

**Proof.** Let’s assume otherwise. Suppose that there are two eigenfunctions \( y(t), \bar{y}(t) \) that correspond to some eigenvalue \( \lambda \). Then consider

\[
Y(t) = \alpha y(t) + \beta \bar{y}(t)
\]

for some \( \alpha \) and \( \beta \) such that \( \alpha^2 + \beta^2 > 0 \). Clearly, \( Y(t) \) is an eigenfunction that corresponds to \( \lambda \), where \( Y''(a) = 0 \). Moreover, we also have that \( Y(a) = Y'(a) = 0 \), which means by Lemma 4.2.4, \( Y(b) \neq 0 \). However, this is a contradiction to the boundary conditions of Eq.(4.6) at \( t = b \).

**Corollary 4.3.6** Let Eq.(4.6) be given. Then the \( n \)th eigenfunction \( y_n(t) \) of Eq.(4.6) has \( n - 1 \) simple zeros in \((a,b)\). ([6])

**Proof.** The result follows from the previous result.

We know that we can find a countable set of \( \lambda \)'s that are positive and got to infinity as well as solve Eq.(4.6) with its boundary conditions. We also see that the first eigenvalue \( \lambda_1 \) is actually the minimum of the Rayleigh quotient. We will return to the Rayleigh quotient when we consider the Biplaplacian.
4.4 Comparison Theorems for Fourth Order Eigenvalue Problem

We wish to consider a fourth order analog to the Sturm comparison theorem which leads us to the following system.

\[
(r(t)y'')'' - p(t)y = 0, \quad r(t) > 0, p(t) > 0 \tag{4.7}
\]

and

\[
(\bar{r}(t)y'')'' - \bar{p}(t)y = 0, \quad \bar{r}(t) > 0, \bar{p}(t) > 0 \tag{4.8}
\]

where \(r(t) \geq \bar{r}(t)\) and \(p(t) \leq \bar{p}(t)\). Now, we establish a relationship between the first conjugate points for Eq.(4.7) and Eq.(4.8) in the next theorem.

**Theorem 4.4.1** Let \(\eta_1(a)\) and \(\bar{\eta}_1(a)\) be the first conjugate points of \(t=a\) for Eq.(4.7) and Eq.(4.8) respectively. Then

\[
\bar{\eta}_1(a) \leq \eta_1(a).
\]

(\cite{6})

**Proof.** Using the properties of the eigenvalues and Corollary 4.3.6, we see that the first eigenvalues of

\[
(r(t)y'')'' - \lambda p(t)y = 0, \quad y(a) = y'(a) = y(\eta_1) = y'(\eta_1) = 0 \tag{4.9}
\]

\[
(\bar{r}(t)y'')'' - \lambda \bar{p}(t)y = 0, \quad y(a) = y'(a) = y(\bar{\eta}_1) = y'(\bar{\eta}_1) = 0 \tag{4.10}
\]

are both equal to 1. Now substituting \(\eta_1\) for \(\bar{\eta}_1\) into Eq.(4.10) we obtain

\[
(\bar{r}(t)y'')'' - \lambda^* \bar{p}(t)y = 0, \quad y(a) = y'(a) = y(\eta_1) = y'(\eta_1) = 0. \tag{4.11}
\]

So for a given function \(y(t)\), the Rayleigh quotient with \(R = \eta_1\) doesn’t increase if \(r\) and \(p\) are replaced by \(\bar{r}\) and \(\bar{p}\) respectively. Then by properties of eigenvalues and using the Rayleigh quotient we see that \(\lambda_1^* \leq \lambda_1 = 1\). If we consider \((a, \ell]\), then by the Existence and Uniqueness of eigenvalues we can determine a unique point such that the corresponding first eigenvalue of Eq.(4.11) will be 1 by replacing the point with \(\eta_1\), which gives the interval \((a, \eta_1]\). However, the first eigenvalue to Eq.(4.10) is also 1; hence, the determined point must coincide with \(\bar{\eta}_1\). Evidently, this means that \(\bar{\eta}_1\) must be in \((a, \eta_1]\). Therefore, \(\bar{\eta}_1 \leq \eta_1\).

\[\blacksquare\]

**Remark 4.4.1** We can extend this argument for the \(n\)th conjugate point. The argument is done in [6].
Chapter 5

The Bilaplacian Operator on Warped Products

5.1 Bilaplacian

**Definition 5.1.1** An operator $L$ is called self-adjoint with respect to a measure $\omega$ if

$$\int_a^b (Ly)v \omega dx = \int_a^b y(Lv) \omega dx$$

for all $y, v$ satisfying the given boundary conditions.

**Example 5** Boundary conditions which make the operator

$$L = (Ry)'' + (Qy)' + Py$$

self adjoint with respect to $\omega = 1$ are

$y(a) = y'(a) = 0$ and $y(b) = y'(b) = 0$.

**Proof.**

$$\int_a^b ((Ry)'' + (Qy)' + Py) v dx$$

$$= \int_a^b (Ry)''v dx + \int_a^b (Qy)'v dx + \int_a^b Pyv dx$$

$$= -\int_a^b (Ry)'v' dx + (Ry)''v|_a^b$$

$$- \int_a^b Qy'v' dx + Qy'|_a^b + \int_a^b Pyv dx$$

$$= \int_a^b Ry''v'' dx + Ry''v'|_a^b - \int_a^b Qy'v'dx + \int_a^b Pyv dx$$

$$= \int_a^b Ry''v'' - Qy'v' + Pyv dx$$

Similarly integrating by parts twice shows

$$\int_a^b ((Rv)'' + (Qv)' + Pv) y dx$$

is also equal to the above.

We now try to examine the volume form $\frac{\omega'}{\omega}$ and try to establish a connection with the Bilaplacian operator $\Delta \Delta f$. 
Definition 5.1.2 The Bilaplacian of a function \( y(t) \) is defined as,
\[
\Delta\Delta y = (y'' + hy'')'' + h(y'' + hy')'
\]
where \( h = \frac{\omega'}{\omega} \).

Proposition 5.1.3 Let the Bilaplacian be defined as above. Then
\[
\omega(\Delta\Delta y) = (\omega y'')'' + (\omega h'y')'.
\]

Proof. Expanding \( \Delta\Delta y \) using the Product Rule, we see that
\[
\Delta\Delta y = y'''' + 2hy''' + (2h' + h^2)y'' + (h'' + hh')y'
\]
Multiplying the Bilaplacian by \( \omega \),
\[
\omega[(y'' + hy')'' + h(y'' + hy')'].
\]
By the Product Rule,
\[
\omega y''' + \omega h''y' + 2\omega'y''' + (2h' + h^2)\omega y'' + \omega h'y'.
\]
Substituting, \( h = \frac{\omega'}{\omega} \), we get
\[
\omega y''' + 2\omega'y''' + (\omega'' + \omega h')y'' + \omega h'y' + \omega h''y'.
\]
Recollecting terms using both the Product Rule and Quotient Rule, we see that
\[
\omega(\Delta\Delta y) = (\omega y'')'' + (\omega h'y')'.
\]}

Now we consider the clamped plate problem on the disc. Since \( y(r) \) must extend to an even function on \((-R, R)\), all odd derivatives of \( y \) at 0 are 0. Hence, we can take the boundary conditions at the left endpoint which are \( y'(0) = 0, y''(0) = 0 \). In general, such boundary conditions are not self adjoint, but we will show that they are self adjoint with respect to \( \omega \).

Proposition 5.1.4 Suppose we have the following boundary conditions \( y'(0) = y''(0) = 0, y(R) = y'(R) = 0 \). Moreover, let \( v' \) and \( \omega \) have the same boundary conditions as above on an interval \([0, R]\). Then,
\[
\int_0^R \omega(\Delta\Delta y)vdt = \int_0^R \omega y(\Delta\Delta v)dt.
\]
Proof.

\[
\int_0^R (\Delta \Delta y)v \omega dx = \int_0^R ((\omega y'')'' + (h' \omega y')') v dx
\]

\[
= - \int_0^R (\omega y')'v' dx + (\omega y'')'v|_0^R - \int_0^R h' \omega y' v' dx + \omega h' y' v'|_0^R
\]

\[
= \int_0^R y''v'' \omega dx - \int_0^R h' y' v' \omega dx
\]

\[
+ (\omega y'')'v|_0^R + \omega h' y' v|_0^R + \omega y'' v'|_0^R
\]

Since \(\omega(0) = 0\), \(v(R) = v'(R) = 0\), the only term that is not obviously 0 is \((\omega y'')'(0)\).

But

\[
(\omega y'')' = \omega y'' + \omega y'''
\]

\[
= h \omega y'' + \omega y'''
\]

and we can use \(\omega(0) = 0\) to show that this term is also zero.

\[
\square
\]

5.2 Eliminating the Middle Term

If we recall

\[
(r(t)y'')'' + (q(t)y')' + p(t)y = 0
\]

we wish to eliminate the middle term of the Bilaplace operator so that it has a form similar to Eq.(4.5). In section 12 of ([6]) if \(\sigma\) solves

\[
(\omega \sigma')' + h' \omega \sigma = 0,
\]

then the change of variables \(\tau = \int_t^R \sigma(s) ds\) reduces Eq.(4.4) to

\[
(\omega \sigma^3 y') + \frac{1}{\sigma} \omega y = 0
\]

where \(ij\) denotes differentiation with respect to \(\tau\). By the Product Rule,

\[
\dot{\omega} \dot{\sigma} + \omega \ddot{\sigma} + h' \omega \sigma = 0.
\]

Dividing through by \(\omega\) and recollecting terms, we get

\[
(h \sigma) + \ddot{\sigma} = 0.
\]
Integrating through,
\[ h\sigma + \dot{\sigma} = c. \]
Now, we have a first order linear ODE with the following integrating factor
\[ e^{\int h dt} = e^{\int \frac{\omega'}{\omega} dt} = e^{\ln \omega} = \omega. \]
Hence,
\[ (\omega \dot{\sigma}) = c\omega. \]
Integrating once more and solving for \( \sigma \)
\[ \omega \sigma = \int_0^t c \omega ds \]
and as a result,
\[ \sigma = \frac{1}{\omega} \int_0^t \omega ds. \]
Now we plug our solution of \( \sigma \) back into the fourth order normal form
\[ (\omega \sigma^2 \ddot{y}) + \frac{1}{\sigma} \lambda \omega y = 0. \]
Thus,
\[ \left( \frac{\left( \int_0^t \omega ds \right)^3}{\omega^2} \ddot{y} \right) + \frac{\lambda \omega^2}{\int_0^t \omega ds} y = 0. \]
For \( \Delta \Delta \) on \( \mathbb{R}^n \), going back to Eq.(4.4) taking \( r = \omega \) and \( p = -\lambda \omega \) we do the following computation. Let \( \omega = t^{n-1} \) and \( h = \frac{n-1}{t} \). Taking the derivative of \( h \), we see that \( h' = -\frac{(n-1)}{t^2} \). Using the result that we found for \( \sigma \) we see that
\[ \sigma = \int_0^t s^{n-1} ds = \frac{t^n}{n} = \frac{t}{n}. \]
Substituting this into the normal form for \( \sigma \) we have
\[ \left( t^{n-1} \left( \frac{t}{n} \right)^3 y'' \right)'' - \frac{n}{t} \lambda t^{n-1} y = 0. \]
We can rewrite this as
\[ \frac{1}{n^3} \left( t^{n+2} y'' \right)'' - n\lambda t^{n-2} y = 0. \]
We guess that the following is a solution for some \( \alpha \)
\[ y = t^\alpha. \]
By substitution
\[ \frac{1}{n^3} \left( t^{n+2} \alpha (\alpha - 1) t^{\alpha - 2} \right)'' - n \lambda t^{\alpha + n - 2} = 0. \]

As a result,
\[ \frac{\alpha (\alpha - 1)}{n^3} (t^{\alpha + n})'' - n \lambda t^{\alpha + n - 2} = 0. \]

By the Product Rule,
\[ \frac{\alpha (\alpha - 1)(\alpha + n)(\alpha + n - 1)}{n^3} t^{\alpha + n - 2} - n \lambda t^{\alpha + n - 2} = 0. \]

We see that the \( t^{\alpha + n - 2} \) cancels and hence,
\[ \frac{\alpha (\alpha - 1)(\alpha + n)(\alpha + n - 1)}{n^3} = n \lambda. \]

Finally,
\[ \alpha (\alpha - 1)(\alpha + n)(\alpha + n - 1) = n^4 \lambda \quad (5.1) \]

We use a graphical solution to show that \( \alpha \) must be greater than 1.

Figure 5.1: Solution for Eq.(5.1)

5.3 Solution of Clamped Plate Equation by Factoring

For fixed \( k \),
\[ \Delta \Delta f = k^4 f \]

is a fourth order ODE with four linearly independent solutions. We can factor \( \Delta \Delta f = k^4 f \) to obtain the following equations.
\[ \Delta f_1 + k^2 f_1 = 0 \quad (5.2) \]

and
\[ \Delta f_2 - k^2 f_2 = 0 \quad (5.3) \]
To find the solutions, we can solve separately the equations Eq.(5.2) and Eq.(5.3), which give two solutions each. In the next proposition, we show that indeed the solutions to Eq.(5.2) and Eq.(5.3) are linearly independent.

**Proposition 5.3.1** Let $y_1, y_2$ be solutions to Eq.(5.2) and $y_3, y_4$ be solutions to Eq.(5.3). Then $y_1, y_2, y_3, y_4$ is a linearly independent set.

**Proof.** Consider the following linear combination,

$$c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4 = 0 \quad (5.4)$$

for some constants $c_1, c_2, c_3, c_4$. Now apply the Laplace operator, $\Delta$, to both sides of the equation to get

$$-k^2c_1y_1 - k^2c_2y_2 + k^2c_3y_3 + k^2c_4y_4 = 0 \quad (5.5)$$

for $k \neq 0$. Adding Eq.(5.4) and Eq.(5.5) we see that, $c_1y_1 + c_2y_2 = 0$ and $c_3y_3 + c_4y_4 = 0$. Hence, $c_1, c_2 = 0$ and $c_3, c_4 = 0$.

---

**Example 6 (Bessel Functions, $k = 0$)** Consider Eq.(5.2) and Eq.(5.3) on $0$ to $R$ with boundary conditions $y(R) = 0, y'(R) = 0$. For $k = 0$ we see that Eq.(5.2) becomes

$$y'' + \frac{1}{t}y' + k^2y = 0. \quad (5.6)$$

and Eq.(5.3) becomes

$$y'' + \frac{1}{t}y' - k^2y = 0. \quad (5.7)$$

In fact, Eq.(5.6) is a Bessel function with solution $J_0(t)$, and Eq.(5.7) is a modified Bessel function with solution with $I_0(t)$.

In general, Eq.(5.2) has the solution

$$y_1(t) = \frac{J_\nu(kt)}{(kt)^\nu} \quad (5.8)$$

and Eq.(5.3) has the solution

$$y_2(t) = \frac{I_\nu(kt)}{(kt)^\nu}. \quad (5.9)$$

We want to solve the following system

$$c_1y_1(R) + c_2y_2(R) = 0$$

$$c_1y_1'(R) + c_2y_2'(R) = 0.$$
This has nontrivial solutions when
\[
\begin{bmatrix}
y_1(R) & y_2(R) \\
y'_1(R) & y'_2(R)
\end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
and taking the determinant yields
\[y_1(R)y'_2(R) - y'_1(R)y_2(R) = 0.\]

Using Eq.(5.8) and Eq.(5.9) as well as the chain rule, we obtain
\[
\left( \frac{J_\nu(t)}{t^\nu} \right) \left( \frac{I_\nu(t)}{t^\nu} \right)' - \left( \frac{J_\nu(t)}{t^\nu} \right)' \left( \frac{J_\nu(t)}{t^\nu} \right) = 0
\]
which we want to find the zeros for. In addition, we have the following Bessel Function identities
\[
\frac{1}{t} \frac{d}{dt} \left( \frac{J_\nu(t)}{t^\nu} \right) = -\frac{J_{\nu+1}}{t^{\nu+1}}
\]
and
\[
\frac{1}{t} \frac{d}{dt} \left( \frac{I_\nu(t)}{t^\nu} \right) = \frac{I_{\nu+1}}{t^{\nu+1}}.
\]
By the Bessel Function identities we get that
\[J_\nu(kR)I_{\nu+1}(kR) + J_{\nu+1}(kR)I_\nu(kR) = 0.\]
The above equation can be rewritten as
\[J_\nu I_{\nu+1} + J_{\nu+1} I_\nu = k_\nu. \tag{5.10}\]
The zeroes of Eq.(5.10) are recorded below.
We now prove an important theorem that deals with the eigenvalue of $\Delta \Delta f = k^4 f$ for general warped products.

**Theorem 5.3.2** Let $y_k$ solve $\Delta y = -k^2 y$ and $z_k$ solve $\Delta z = k^2 z$. Then $k^4$ is an eigenvalue of
\[
\Delta \Delta f = k^4 f, \quad f(R) = f'(R) = 0
\]
if and only if
\[
\int_0^R y_k(t)z_k(t)\omega(t)dt = 0
\]

**Proof.** Consider the following system
\[
y'' + hy' + k^2 y = 0 \tag{5.11}
\]
and
\[
z'' + hz' - k^2 z = 0 \tag{5.12}
\]
with $y_k$ as a solution to Eq.(5.11) and $z_k$ as a solution to Eq.(5.12). Both $y_k$ and $z_k$ extend to even functions on $(−R, R)$, so any linear combination of $y_k$ and $z_k$ satisfies the boundary conditions at 0. (That is, all odd derivatives are zero at $t = 0$.)

$\Delta \Delta f = k^4 f, \quad f(R) = f'(R) = 0$

has $k^4$ as an eigenvalue if and only if $k$ is such that

$y_k(R)z_k'(R) - y_k'(R)z_k(R) = 0$.

If
\[
F(x) = y_k(x)z_k(x) - y_k'(x)z_k(x)
\]
then taking its derivative gets us
\[
F'(x) = y_k(x)z_k''(x) - y_k''(x)z_k(x).
\]

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<tr>
<td>10</td>
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<td>4853.327967253</td>
</tr>
</tbody>
</table>

Table 5.1: The zeroes of $J_\nu I_{\nu+1} + J_{\nu+1} I_\nu = k_\nu$
Using Eq.(5.11) and Eq.(5.12) we see that

\[ F'(x) = y_k(-hz_k' + k^2z_k) - (-h'y_k' - k^2y_k)z_k. \]

Hence,

\[ F' + hF = 2k^2y_kz_k. \]

By the Product Rule and the fact that \( \omega' = h\omega \),

\[ (\omega F)' = 2k^2y_kz_k\omega. \]

By the Fundamental Theorem of Calculus,

\[ \omega F(R) = 2k^2 \int_0^R y_k(t)z_k(t)\omega(t)dt. \]

Since \( F(R) = 0 \), we conclude that

\[ \int_0^R y_k(t)z_k(t)\omega(t)dt = 0. \]

The following graphs below help to analyze the eigenvalues for the clamped plate.

**Figure 5.2: Eigenvalues for Curvatures, \( K = -1, 0, 1 \) (\( n = 4 \))**
Figure 5.3: Eigenvalues for Curvatures, $K = 0, -1, 1$ ($n = 10$)

In addition, the next figures shows the eigenfunctions for the clamped plate of curvatures $K = -1, 1$.

Figure 5.4: Eigenfunctions for Curvature $K = -1$

Indeed, we see that the curves for Figure 6.3 approach $\pi$. This is due to the fact that the maximum radius on the sphere is $\pi$. Finally, we have the eigenfunctions for the clamped plate of curvature $K = 1$. 

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Remark 5.3.1 We proved Cheng’s Theorem in Chapter 4. Now we will attempt to prove Cheng’s Theorem for the Clamped Plate.

Let \( y \) solve

\[
(\omega_k y '')'' + (h_k y' \omega_k)' + \lambda_k(R) y \omega_k = 0.
\]

We know that \( \frac{-\phi''}{\phi} \geq k \) and hence,

\[
\lambda_1(B_p(R)) \leq \frac{\int_0^R ((\omega_k y '')'' + (h_k y' \omega_k)') y dt}{\int_0^R y^2 \omega dt}.
\]

Using integration by parts,

\[
\frac{\int_0^R ((\omega_k y '')'' + (h_k y' \omega_k)') y dt}{\int_0^R y^2 \omega dt} = \frac{\int_0^R ((y'')^2 \omega_k - (y')^2 h_k' \omega_k) dt}{\int_0^R y^2 \omega dt}.
\]

We know that

\[
\omega \leq \omega_k
\]

but \( h', h'_k \) are typically negative and they can be much larger than \( \omega \). So we can have

\[
|h'| \geq |h'_k|.
\]
Thus, we can’t get the inequality we need on the second term of the numerator to get Cheng’s Theorem.
Cheng’s Theorem would imply for $K = -1, 0$

$$\lambda_0^0(R) \leq \lambda_1^{-1}(R).$$

However, we see in Figure 6.5, that this is not true due to the fact that the graphs cross each other.

### 5.4 Szego’s Inequality

We now are ready to prove the main result of this thesis. Before we prove Szego’s Inequality, let us recall Talenti’s Inequality.

If $f : B_p(r) \to \mathbb{R}^+$ is a positive function, then the symmetric rearrangement $f^* : \Omega^* \to \mathbb{R}^+$ which can be thought of as the function which minimizes energy

$$\int_{\Omega^*} \|\nabla f^*\|^2 dV$$

over all functions satisfying Dirichlet boundary conditions and

$$\int_{\Omega^*} (f^*)^2 dV = \int_{B_p(r)} f^2 dV.$$
Theorem 5.4.1 (Talenti’s Inequality) Let $u : B_p(r) \rightarrow \mathbb{R}^+$ solve
$$\Delta u = f$$
and $u^* : \Omega^* \rightarrow \mathbb{R}^+$ be the symmetric rearrangement of $u$. If $f^*$ is the symmetric rearrangement of $f$ and $v : \Omega^* \rightarrow \mathbb{R}^+$ solve
$$\Delta v = f^*,$$
then
$$u^* \leq v$$
on $\Omega^*$.

Szegő’s Inequality is a lower bound for the first eigenvalue of the clamped plate corresponding to the Faber-Krahn Inequality

Theorem 5.4.2 (Szegő’s Inequality) Let $M$ be the warped product of curvature $\geq 1$. Consider
$$\Delta \Delta u = \lambda u$$
with $u$ as its solution. Define $\Omega^* \subseteq S^n$ by
$$\frac{\text{vol}(B_p(r))}{\text{vol}(M)} = \frac{\text{vol}(\Omega^*)}{\text{vol}(S^n)}.$$
If $v$ is the eigenfunction on $\Omega^*$ where $\Delta u = \Delta v$, then
$$\lambda_1(B_p(r)) \geq \lambda_1(\Omega^*).$$

Proof. On $B_p(r)$, we know that the first eigenvalue is
$$\lambda_1(B_p(r)) = \inf_u \frac{\int_{B_p(r)} (\Delta u)^2}{\int_{B_p(r)} u^2}.$$  

By Talenti’s Inequality,
$$\frac{\int_{B_p(r)} (\Delta u)^2}{\int_{B_p(r)} u^2} \geq \frac{\int_{B_p(r)} (\Delta u)^2}{\int_{\Omega^*} v^2} = \frac{\int_{\Omega^*} (\Delta v)^2}{\int_{\Omega^*} v^2} = \lambda_1(\Omega^*).$$
Therefore, on $\Omega^*$ with
$$\frac{\text{vol}(B_p(r))}{\text{vol}(M)} = \frac{\text{vol}(\Omega^*)}{\text{vol}(S^n)}$$
$$\lambda_1(B_p(r)) \geq \lambda_1(\Omega^*).$$
Bibliography


