SAN FERNANDO VALLEY STATE COLLEGE

METHODS FOR A TRANSIENT SOLUTION TO A QUEUEING PROBLEM

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by

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# CONTENTS

1. INTRODUCTION. ............................................. 1

2. DISCUSSION OF THE PROBLEM ............................... 4
   2.1 Development of the Problem ......................... 4
   2.2 Formulation of the Mathematical Model. .......... 5
   2.3 The Moments of the Number of Parked Cars as a Function of Time .......... 8

3. METHODS OF SOLUTION ....................................... 12
   3.1 General Remarks. .................................. 12
   3.2 Solution by the Laplace Transform. ............... 12
   3.3 Solution by the Eigenvalue-Eigenvector Method. ... 16
   3.4 Solution by the Method of a Probability Generating Function ............. 22
   3.5 Solution by the Method of a Weighted Probability Generating Function ... 27
   3.6 Solution for the Case of N = 1 ..................... 32

4. APPENDIX .................................................. 38
   4.1 The Stay-Time and Arrival Distribution Functions .. 38
   4.2 The Probability Mass Functions of the Number of Car Arrivals .......... 40
   4.3 Properties of the Transformed Transition Probability Matrix .................. 41
   4.4 The Partial Derivatives of the Weighted Probability Generating Function ... 46

5. BIBLIOGRAPHY ................................................ 49
ABSTRACT

METHODS FOR A TRANSIENT SOLUTION

TO A QUEUEING PROBLEM

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The transient solution of a parking lot problem with a finite number of parking spaces is considered when, in a given time interval, the number of cars arriving are Poisson distributed with identical parameters. If the parking lot is full, all arriving cars depart immediately; thus, allowing no queue to be formed.

The system is expressed in terms of its transition probabilities which form a finite system of linear differential equations. New methods involving the application of generating functions are developed and described along with some standard methods for solving the transient queueing problem.
1. INTRODUCTION

In recent years, much interest has been devoted to queueing problems. As a consequence, a considerable amount of theoretical literature has been written on the subject. This thesis will study, in detail, several methods of solution to a particular queueing situation; namely, a parking lot problem with no queue allowed. It is hoped that many pertinent concepts of theory will be brought forth and clarified.

When one refers to a queue, the implication is that there exists a line-up or file of individuals, automobiles, animals, etc. Thus, a queue is often called a waiting line, whose behavior is dynamically that of a larger system. This larger system may be broken down into two basic stages:

- An input stage, into which items are fed into the system
- A servicing stage wherein the items in the system are serviced in some manner and then discharged from the system

The waiting line mentioned previously refers to the input supply which arrived in the system but has not yet been serviced. Queueing problems, then, are concerned with the study of the waiting line and the number of units in the system as it fluctuates and changes with time. To study this dynamic system, we need to specify three phases of the operation.

a) **Input Process**

This process refers to the individuals arriving in the system and is specified by the probability mass function, \( Q_k(t) \), where \( k \) is the number of arrivals occurring within the time interval \( t \). In this
problem, it is assumed that the interval of time between successive arrivals at the parking lot is a random variable, whose distribution is a negative exponential. Thus, during the period $t$, the number of arrivals is given by the Poisson probability mass function

$$Q_k(t) = (\lambda t)^k e^{-\lambda t} / k!; \quad k = 0, 1, 2...$$

where $\lambda$ is the average arrival rate. The various properties of the Exponential and Poisson distributions will be discussed in the Appendix Sections 4.1 and 4.2.

b) **Queue Discipline**

This refers to the rule by which units are placed to form a queue, and the manner in which they are moved while waiting. In this study, if the parking lot is full, an arriving car will depart immediately, therefore, a queue will not be formed.

c) **Service Mechanism**

This not only refers to the manner in which a unit is serviced, but also to the time required to service an individual. That is, many problems reflect the case of multiple channel service situations wherein there are several servicing agents in parallel, series, or both. The parking lot problem is considered to be a multiple parallel channel service situation, each channel being a parking lot space with a corresponding stay-time distribution function. The stay time, $t$, of a car is the negative-exponential distribution

$$F(t) = e^{-\mu t}; \quad t \geq 0$$
where $1/\mu$ is the average stay-time of a car parked on the lot. Sometimes, one wishes to know the probability that $\lambda$ parked cars depart within an interval of time of duration $t$. In this problem, the number of departures, $\lambda$, may also be given by the Poisson probability mass function

$$D_\lambda(t) = (\mu t)^\lambda e^{-\mu t}/\lambda! ; \lambda = 0, 1, 2...$$

The system will be assumed to be a collection of discrete objects; namely, cars. The number of cars in the system is denoted by $n$, a discrete set of numbers ($n = 0, 1, 2, ..., N$), where $N$ is the total number of parking spaces.

The characteristic of the system as it fluctuates with time is characterized by $P_n(t) = \text{Prob}(n \text{ cars parked at time } t)$. Some attributes, defining system behavior, are the number of empty parking spaces, the number of cars parked, and the probability that the parking lot is full. The two most frequently studied characteristics are the first two moments of the parked cars, which serve as measures of system fluctuation.

This thesis will be concerned with the number of parked cars under transient conditions and will develop several methods of solution that describe the process for any time. Still, some attention will be devoted to the steady-state solution which is the solution for the number of cars parked when $t$ approaches infinity. Thus, the random variable, number of cars parked, which is described by the probability mass function, $P_n(t)$ (where $n = 0, 1, 2, ...,$ and $P_n(0) = 1$), denoting the probability of $n$ cars parked at time $t$, will be studied together with its moments when investigating these different methods of solution.
2. DISCUSSION OF THE PROBLEM

2.1 DEVELOPMENT OF THE PROBLEM

A parking lot consists of \( N \) parking spaces with one or more entrances or exits. It is a dynamic system described by the characteristic, \( P_n(t) \), which represents a probability mass function of time. The number of parked cars form a stochastic process \( \{ X(t), t \in T \} \) indexed by the time parameter, \( t \), in an index set \( T \). A typical sample function, \( X(t) \), is graphically shown below.

![Graph showing the number of cars parked over time](image)

For a given point in time, the random variable, \( X(t) \), is a discrete-valued function with the number of cars parked as its values. The probability mass function of \( X(t) \) is \( P_n(t) \), which, represents the probability that \( n \) cars are in the parking lot at time \( t \). This relationship may be expressed, in standard notation, as

\[
P_n(X(t) = n) = P_n(t)
\]
The objective of this thesis then is to develop or demonstrate techniques by which the probability mass function, \( P_n(t) \), or the moments may be obtained.

2.2 FORMULATION OF THE MATHEMATICAL MODEL

Cars that arrive at the parking lot follow a Poisson distribution function, \( Q_\lambda(t) \), in time with some known average arrival rate \( \lambda \). Arriving vehicles park as long as space is available; however, if the parking lot is full, arriving cars will immediately depart. The stay-time of the parked cars form a negative exponential distribution function, \( F(t) \), with some known average departure rate, \( \mu \). A more precise formulation is as follows:

Let \( N \) stand for the total number of parking spaces. The parking lot can be described by noting the number of cars parked, and hence, the number of spaces still empty. Let \( E_n \) (where \( n = 0, 1, 2, \ldots, N \)) denote the state of having \( n \) cars parked. It is assumed that the system changes only through transitions to adjacent states (from \( E_n \) to \( E_{n+1} \) if \( 1 \leq n \leq N-1 \), from \( E_0 \) to \( E_1 \), and from \( E_N \) to \( E_{N-1} \)). To derive an expression for \( P_n(t + \Delta t) \), the probability of being in state \( E_n \) at time \( t + \Delta t \), it is necessary to consider only simple transitions of a single car.

Remember that the parking lot consists of \( N \) parking spaces with equal exponentially distributed stay-times each with a mean departure \( \mu \). The probability that a particular parked car departs within a time interval of length \( \Delta t \) is \( D_\lambda(\Delta t) = \mu e^{-\mu \Delta t} \), when neglecting higher powers of \( \Delta t \). However, if \( n \leq N \) cars are parked, then the probability that any one of the cars will depart in time \( \Delta t \) is \( n\mu \Delta t \), since any one of the \( n \) parking spaces may become empty during this time.
interval $\Delta t$. The probability, therefore, that none of the $n$ parked cars depart in the time interval $\Delta t$ is $1 - n\mu\Delta t$ to first order in $\Delta t$.

The probability that a car arrives in the interval $\Delta t$ is $Q_1(\Delta t) = \lambda\Delta t e^{-\lambda\Delta t}$ and the probability that no car arrives during the same time interval is $Q_0(\Delta t) = e^{-\lambda\Delta t} - 1 - \lambda\Delta t$, when neglecting higher powers of $\Delta t$.

The various possibilities and their probabilities (to terms of order $\Delta t$), for $1 \leq n \leq N - 1$ are:

a) At time $t$, the system is in state $E_n$, with probability $P_n(t)$, and during $(t, t + \Delta t)$ no changes occur with probability $(1 - \lambda\Delta t)(1 - n\lambda\Delta t) = 1 - \lambda\Delta t - n\mu\Delta t + O(\Delta t^2)$.

b) At time $t$, the system is in state $E_{n-1}$, with probability $P_{n-1}(t)$, and a transition to $E_n$ occurs with probability $\lambda\Delta t$.

c) At time $t$, the system is in state $E_{n+1}$, with probability $P_{n+1}(t)$, and a transition to $E_n$ occurs with probability $(n + 1)\mu\Delta t$.

Since the above contingencies $E_{n-1}$, $E_n$, and $E_{n+1}$ are mutually exclusive events,

\[ P_n(t + \Delta t) = [1 - \lambda\Delta t - n\mu\Delta t] P_n(t) + \lambda\Delta t P_{n-1}(t) + P_{n+1}(t) (n+1)\mu\Delta t. \] \hspace{1cm} (1)

Proceeding similarly for the special cases $n = 0, N$ results in

\[ P_0(t + \Delta t) = [1 - \lambda\Delta t] P_0(t) + \mu\Delta t P_1(t) \] \hspace{1cm} (2)

and

\[ P_N(t + \Delta t) = \lambda\Delta t P_{N-1}(t) + (1 - N\mu\Delta t) P_N(t). \] \hspace{1cm} (3)
Expressions (1), (2), and (3) can be used to determine an associated system of linear differential equations in the usual manner:

\[
P_0'(t) = -P_0(t) + P_1(t),
\]

\[
P_n'(t) = -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n + 1)\mu P_{n+1}(t), \quad 0 < n < N
\]

\[
P_N'(t) = \lambda P_{N-1}(t) - N\mu P_N(t),
\]

It is the purpose of this thesis to explore and develop techniques to solve this system of differential equations and other associated problems. It should be noted here that the differential equations (4), (5), and (6) are also known as the "forward" equations because the equations were derived by prolonging the time interval \([0, t]\) to \([0, t + \Delta t]\) and considering the possible changes during the short time interval \([t, t + \Delta t]\). One could as well have prolonged the interval \([0, t]\) in the direction of the past and considered the changes during \([-\Delta t, 0]\). In this way a new system of differential equations, known as the "backward" equations, is obtained. Standard methods are applied for solving the system of linear differential equations (4), (5), and (6). In Section 3.2, the Laplace transform procedure of the differential equations is described in detail. The transformation of the differential equations results in a system of rational functions. To invert the system of rational functions, each function is expanded in terms of partial fractions, thereby reducing each term to a known form. The Eigenvalue - Eigenvector method is used in Section 3.3. In the Appendix, Section 4.3, it is shown that the roots of the characteristic equation of the matrix of the transformed transition probabilities are all distinct. This simplifies the canonical decomposition of the
matrix of the transition probabilities. Then, given the roots of the characteristic equation, the probability mass function of the number of parked cars is obtained. Finally, methods involving the use of probability-generating functions (which is a special case of the method of characteristic functions) are developed in Sections 3.4 and 3.5 for the transient queueing problem. These methods are applicable to discrete random variables such as the number of cars parked which assumes only integral values.

2.3 THE MOMENTS OF THE NUMBER OF PARKED CARS AS A FUNCTION OF TIME

The moments of the probability mass function \( P_n(t) \) are the expected values of the powers of the random variable \( X(t) \) representing the number of parked cars at time \( t \). The \( k \)-th moment of \( X(t) \) is denoted by

\[
M_k(t) = \sum_{n=0}^{N} n^k P_n(t), \quad k = 0, 1, 2, \ldots
\]  

(7)

In particular \( M_0(t) = 1 \). For a given \( t \), the moments are a set of descriptive constants of \( P_n(t) \) which are useful in measuring the properties of \( P_n(t) \) or possibly for specifying it. For this purpose, therefore, a knowledge of \( P_n(t) \); equivalent, that is, in the sense that it should be possible theoretically to exhibit all properties of \( P_n(t) \) in terms of its moments. As is shown in Section 3.5, equally important is the fact that, given the set of moments of \( P_n(t) \), one can uniquely determine the probability mass function \( P_n(t) \) in terms of its moments \( M_k(t), \ k = 0, 1, 2, \ldots \).
Differentiating (7) with respect to time results in

\[ M'_k(t) = \sum_{n=0}^{N} n^k p'_n(t). \] (8)

Multiplying (5) by \( n \), and summing for \( n = 1, 2, \ldots, N-1 \) and adding (6), after multiplying it by \( N \), one obtains:

\[ M'_1(t) = -\mu M_1(t) + \lambda - \lambda p'_N(t) \]

\[ = \sum_{j=1}^{N} (-1)^j C^k_j [\mu M_{2-j}(t) - \lambda M_{1-j}(t) + \lambda p_N(t) N^{1-j}] \]

where \( C^k_j \) is the binomial coefficient: \( C^k_j = \frac{k!}{(k-j)!j!} \). For \( k=2 \), (5) is multiplied by \( n^2 \) and summed over \( n = 1, 2, \ldots, N-1 \). Adding (6), after multiplying it by \( N^2 \), results in:

\[ M'_2(t) = -2 \mu M_2(t) + \mu M_1(t) + 2 \lambda M_1(t) - \lambda + \lambda p_N(t) - 2 \lambda N p_N(t) \]

\[ = \sum_{j=1}^{N} (-1)^j C^2_j [\mu M_{3-j}(t) - \lambda M_{2-j}(t) + \lambda p_N(t) N^{2-j}] . \]

In general, it can be shown easily by induction that the derivative of the \( k \)-th moment is:

\[ M'_k(t) = \sum_{j=1}^{N} (-1)^j C^k_j [\mu M_{k+1-j}(t) - \lambda M_{k-j}(t) + \lambda p_N(t) N^{k-j}] . \] (9)

For \( k=1 \), \( M_1(t) \) satisfies a first-order linear differential equation whose solution is:

\[ M_1(t) = e^{-\mu t} \int_0^t \lambda(1 - p_N(x)) e^{\mu x} dx + C e^{-\mu t} . \]
To be determined in Section 3.1.1, $P_N(t)$, the probability that the parking lot is full, is given by:

$$P_N(t) = \sum_{i=0}^{N} B_{N,i} e^{r_{N,i} t}$$

where the $B_{N,i}$'s are constants and the $r_{N,i}$'s are real negative numbers; therefore, $M_1(t)$ reduces to

$$M_1(t) = \lambda (1 - e^{-\mu t})/\mu - \sum_{i=0}^{N} B_{N,i} (e^{r_{N,i} t} - e^{-\mu t})/(r_{N,i} + \mu) + Ce^{-\mu t}.$$ 

But

$$M_1(0) = \sum_{n=0}^{N} nP_n(0) = 0$$ since at $t=0$, $\{P_n(0)\} = \{1, 0, \ldots, 0\}$; therefore, $C=0$.

and

$$M_1(t) = (1 - e^{-\mu t})/\mu - \sum_{i=0}^{N} B_{N,i} (e^{r_{N,i} t} - e^{-\mu t})/(r_{N,i} + \mu).$$

Similarly, for $k=2$, $M_2'(t)$ is a first-order linear differential equation whose solution is

$$M_2(t) = \mu e^{-2\mu t} \int_0^t M_1(x) e^{2\mu x} \, dx$$

$$+ e^{-2\mu t} \sum_{j=1}^{2} (-1)^j C_j^2 \int_0^t \left[ -\lambda M_{2-j}(x) + \lambda P_N(x) N^{2-j} \right] e^{2\mu x} \, dx$$

$$+ Ce^{-2\mu t}.$$ 

But $M_2(0) = 0$ which implies $C=0$, thus
\[ M_2(t) = \mu e^{-2\mu t} \int_0^t M_1(x)e^{2\mu x} \, dx \]
\[ + e^{-2\mu t} \sum_{j=1}^{2} (-1)^j \frac{C_j^2}{j} \int_0^t [-M_{2-j}(x) + P_N(x)N^{2-j}]e^{2\mu x} \, dx. \]

In general, \( M_k(t) \) is a first-order linear differential equation with the initial condition that \( M_k(0) = 0 \); therefore, the particular solution is:

\[ M_k(t) = \mu e^{-k\mu t} \sum_{j=2}^{k} (-1)^j \frac{C_j^k}{j} \int_0^t M_{k+1-j}(x)e^{k\mu x} \, dx \]
\[ + \lambda e^{-k\mu t} \sum_{j=1}^{k} (-1)^j \frac{C_j^k}{j} \int_0^t [-M_{k-j}(x) + P_N(x)N^{k-j}]e^{k\mu x} \, dx, \] (10)

for \( k = 1, 2, 3, \ldots \).
3. METHODS OF PROBLEM SOLUTION

3.1 GENERAL REMARKS

The methods for solving the system of differential equations or its transform are several. No particular attention is given to the order in which these techniques are investigated. In all cases, some numerical analysis is required to obtain numerical results.

3.2 SOLUTION BY LAPLACE TRANSFORM

Letting $P_n^*(s)$ denote the Laplace transform of $P_n(t)$ where $P_n^*(s) = \int_0^\infty e^{st} P_n(t) \, dt$, then by definition, the Laplace transform of $P_n'(t)$ is

$$\mathcal{L} P_n'(t) = sP_n^*(s) - P_n(0).$$

Applying this transform to equations (4), (5), and (6) results in a system of algebraic equations:

$$P_0^*(s) (s + \lambda) - P_1^*(s) \mu = 1 \quad (11)$$

$$P_n^*(s) (s + \lambda + n\mu) - P_{n-1}^*(s) \lambda - P_{n+1}^*(s) (n + 1)\mu = 0; \ 1 \leq n \leq N \quad (12)$$

$$P_N^*(s) (s + N\mu) - P_N^*(s) \lambda = 0 \quad (13)$$

where $P_0(0) = 1, P_n(0) = 0; n = 1, 2, \ldots, N$ are the initial conditions.

In matrix notation, the equations become

$$S_N \mathbf{P}^*(s) = \mathbf{P}(0) \quad (14)$$

where $\mathbf{P}^*(s)$ and $\mathbf{P}(0)$ are two column vectors $[P_0^*(s), \ldots, P_N^*(s)]$ and $[1, 0, \ldots, 0]$ respectively, with the latter giving the initial conditions. The transformed transition probability matrix:
\[
S_N = \begin{bmatrix}
    s + \lambda & -\mu & 0 & 0 & \cdots & 0 \\
    -\lambda & (s + \lambda + \mu) & -2\mu & 0 & \cdots & 0 \\
    0 & -\lambda & (s + \lambda + 2\mu) & -3\mu & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & s + N\mu
\end{bmatrix}
\]

is a tridiagonal matrix.

Supposing that the inverse \( S_N^{-1} \) exists, then from (14) one obtains

\[
P^*(s) = S_N^{-1} P(0)
\]

Since the initial condition vector \( P(0) \) contains all zeros except for \( P_0(0) = 1 \), the first column vector of \( S_N^{-1} \) is of interest only. The elements of this column vector are the cofactors \( C_{0n}(s) \); \( n = 0, 1, 2, \ldots, N \), of the first row vector of \( S_N \) divided by the determinant \( |S_N| \). By replacing each element of the first row by the sum of its corresponding column, one sees that \( |S_N| \) has \( s = 0 \) as a root, thus \( |S_N| \) can be written in the form \( sT_N(s) \), where \( T_N(s) \) is an \( N \)-th degree polynomial in \( s \), and

\[
P^*(s) = C(s)
\]

where \( C(s) \) is a column vector

\[\begin{bmatrix} C_{00}(s)/sT_N(s), C_{01}(s)/sT_N(s), \ldots, C_{0N}(s)/sT_N(s) \end{bmatrix}.\]

Both \( C_{0n}(s) \) and \( sT_N(s) \) are polynomials, the order of \( sT_N(s) \) being greater than the order of \( C_{0n}(s) \), \( n = 0, 1, 2, \ldots, N \). In Section 4.3, it will be shown that \( sT_N(s) \) has \((N+1)\) distinct real roots of which
N are negative and one is zero. Thus, \( sT_N(s) \) is completely factorable into the unrepeat linear factors:

\[
(s - r_{N,0}), (s - r_{N,1}), \ldots, (s - r_{N,N}), (r_{N,0} = 0)
\]

which implies that the inverse Laplace transform \( \mathcal{L}^{-1}p_n(s) \) is

\[
p_n(t) = \sum_{i=0}^{N} c_{0,n}(r_{N,i}) e^{r_{N,i}t} \quad (r_{N,i} T_N(r_{N,i})); \quad n = 0, 1, \ldots, N
\]

\[
= \sum_{i=0}^{N} B_{N,i} e^{r_{N,i}t}
\]

where

\[
B_{N,i} = c_{0,n}(r_{N,i})/(r_{N,i} T_N(r_{N,i})).
\]

By investigating \( sT_N(s) \), note that for \( N = 1 \)

\[
s_{T_1}(s) = s^2 + (\lambda + \mu) s,
\]

and for \( N = 2 \)

\[
s_{T_2}(s) = s^3 + (2\lambda + 3\mu) s^2
\]

\[
+ (\lambda^2 + 2\lambda\mu + 2\mu^2) s.
\]

In general, it will be shown in Section 4.2 that for any \( N \geq 2 \)

\[
s_{T_N}(s) = a_{N0}s^{N+1} + a_{N1}s^N + \ldots + a_{NN}s
\]

\[
= s[T_{N-1}(s) (s + \lambda + N\mu) - (N-1)\lambda\mu T_{N-2}(s)]
\]

where

\[
a_{NK} = \sum_{j=0}^{k} \alpha(N)_{kj} \lambda^{k-j} \mu^j
\]
and

\[ \alpha(N)_{kj} = \begin{cases} 
C_k^N N+i-k \sum_{m=j}^{N} \alpha(m)_{j, j-1}; & j = 1, 2, 3, \ldots \\
C_k^N \alpha(N)_{0, 0}; & j = 0 
\end{cases} \]

where \( \alpha(N)_{0, 0} = 1 \), and \( \alpha(N)_{N, N} = N! \).

Thus, for any \( N \), \( sT_N(s) \) can be constructed and its roots determined numerically with a digital computer.

Similarly, to obtain values for \( C_{0n}(s) \) at \( s = r_{N, i} \), \( i = 0, \ldots, N \) first one notes that \( C_{0N}(s) = \lambda^N \). That is, the cofactor \( C_{0N}(s) \) reduces to a triangular matrix whose determinant is \( \lambda^N \). By triangularizing each cofactor, one easily obtains, by iteration, the following result:

\[ C_{0n}(s) = \lambda^n \prod_{j=n+1}^{N} b_{j, j}(s); \quad n = 0, 1, 2, \ldots, N-1 \]  \( (18) \)

where

\[ b_{n+1,n+1}(s) = s + \lambda + (n+1)\mu, \quad b_{N,N}(s) = s + N\mu \]

and

\[ b_{n+k,n+k}(s) = s + \lambda + (n+k)\mu - (n+k)\lambda\mu / b_{n+k-1,n+k-1}(s). \]

Finally,

\[ C_{0n}(r_{N,i}) = \lambda^n \prod_{j=n+1}^{N} b_{j, j}(r_{N,i}); \quad i = 0, 1, \ldots, N. \]  \( (19) \)

To obtain the steady state solution for \( P_n(t) \), MORSE\(^1\)), for fixed \( N \), has shown that
\[ \lim_{t \to \infty} P_n(t) = P_n = \frac{(\lambda/\mu)^n}{n!} \sum_{i=0}^{N} \frac{(\lambda/\mu)^i}{i!}; \]

but

\[ \lim_{s \to 0} sp^*(s) = \lim_{t \to \infty} P_n(t) \]

therefore,

\[ \lim_{s \to 0} sP^*_n(s) = \lim_{s \to 0} sC_{0,n}(s)/sT_N(s). \]

For example,

\[ \lim_{s \to 0} sP^*_N(s) = \frac{(\lambda/\mu)^N}{N!} \sum_{i=0}^{N} \frac{(\lambda/\mu)^i}{i!}. \]

In general, the steady state solution for \( P_n(t) \) is

\[ P_n = \frac{(\lambda/\mu)^n}{n!} \sum_{i=0}^{N} \frac{(\lambda/\mu)^i}{i!}. \]

3.3 SOLUTION BY THE EIGENVALUE-EIGENVECTOR METHOD

To obtain a solution to the system of linear-differential equation directly, it is advantageous to consider the system in matrix notation. That is, if

\[ \vec{P}'(t) = [P'_0(t), P'_1(t), ..., P'_N(t)] \]

and

\[ \vec{P}(t) = [P_0(t), P_1(t), ..., P_N(t)] \]
are two column vectors, then the equations can be written as:

\[
P'(t) = AP(t)
\]

(20)

where

\[
A = \begin{bmatrix}
-\lambda & \mu & 0 & \ldots & 0 \\
\lambda & -(\lambda+\mu) & 2\mu & 0 & \ldots & 0 \\
0 & \lambda & -(\lambda+2\mu) & 3\mu & 0 \\
0 & 0 & \ldots & \vdots \\
\vdots & \vdots & \ddots & N\mu \\
0 & 0 & \ldots & \lambda & -(\lambda+N\mu)
\end{bmatrix}
\]

If one can show that \(A\) is a matrix with distinct eigenvalues, then \(A\) is similar to a diagonal matrix \(\Omega\) of these eigenvalues and a new system of linear differential equations with separated variables is obtained, that is, the \(n\)-th equation of the new system contains only one variable.

To show that the eigenvalues of \(A\) are distinct, it must be shown that the roots \(r\) of the equation

\[
|A-\lambda I| = 0
\]

are distinct. In Section 4.3, it is shown that the polynomial \(sT_N(s)\) has \(N\) distinct negative real roots and one root with a value of zero. But

\[
|A-\lambda I| = -rT_N(r);
\]

therefore, the eigenvalues \(r_{N,0}, r_{N,1}, \ldots, r_{N,N}\) of \(A\) are distinct as shown in Section 4.3. Thus \(A\) is similar to the diagonal matrix \(\Omega\) of the eigenvalues; that is
\[ \Omega = M_R^{-1} A M_R \]  

(21)

where \( M_R = [\alpha_0, \alpha_1, \ldots, \alpha_N] \) is an eigenvector matrix (i.e., the columns \( \alpha_j \) of \( M_R \) are the column eigenvectors of \( A \)).

To separate the variables \( P_n(t) \), one notes first that

\[ AM_R = M_R \Omega \]  

(22)

or

\[ M_R^{-1} A = \Omega M_R^{-1} \]  

(23)

The from (20),

\[ (M_R^{-1} P(t))^' = M_R^{-1} A P(t) \]

\[ = \Omega M_R^{-1} P(t) = \Omega(M_R^{-1} P(t)) \]  

(24)

Let

\[ U(t) = M_R P(t) \]

then

\[ U'(t) = (M_R^{-1} P(t)) = \Omega U(t) \]  

(25)

Thus, the new variables are separated, where \( U_n(t) = r_n U(t) \), and the solution can be written directly as

\[ M_R^{-1} P(t) = U(t) = e^{\Omega t} U(0) \]

\[ = e^{\Omega t} M_R^{-1} P(0) \]  

(26)

where
Hence

\[ \Omega = \begin{bmatrix} e^{r_{N,0}t} \\ e^{r_{N,1}t} \\ \vdots \\ e^{r_{N,N}t} \end{bmatrix} \]

\[ P(t) = M_R e^{\Omega t} M_R^{-1} P(0) \]  

(27)

where

\[ P(0) = [1, 0, 0, \ldots, 0] \] is a column vector.

In order to obtain the solution (27), it is necessary to obtain the eigenvalues \( r_{N,j} \) and the corresponding eigenvectors \( \alpha_j \) (the columns of \( M_R \)). The procedure is straightforward and is given below. To obtain \( M_R^{-1} \) for use in (27), rather than inverting \( M_R \), it is more efficient to obtain the left eigenvector matrix \( M_L \) of \( A \) (\( M_R \) being the right eigenvector matrix of \( A \)) which, except for an appropriate scalar, can be shown to be equal to \( M_R^{-1} \).

It should be noted that the rows \( \beta_j \) of \( M_L = [\beta_0, \beta_1, \ldots, \beta_N] \) are the left-eigenvectors of \( A \) in contrast to the columns \( \alpha_j \) of \( M_R \).

In writing equation (22) with the \( M_R \) matrix replaced with the left-eigenvector matrix \( M_L \), one obtains

\[ M_L A = \sigma M_L \]

But from (23), one sees that \( M_R^{-1} = M_L \), up to an appropriate constant scalar multiplier of \( M_L \).

Thus,
To obtain $M_R$ and $M_L$, let $[\alpha_{0,j}, \ldots, \alpha_{N,j}] = \alpha_j$ be the jth column vector of $M_R$ and $[\beta_{j,0}, \beta_{j,1}, \ldots, \beta_{j,N}] = \beta_j$ be the jth row vector of $M_L$. The matrix equation

$$A\alpha = r\alpha$$

and

$$\beta A = r\beta$$

have solutions $\alpha$ and $\beta$ other than the zero vector since $r$ is an eigenvalue of the matrix $A$. Consider $A\alpha_j = r_{N,j}\alpha_j$ where $\alpha_j = [\alpha_{0,j}, \alpha_{1,j}, \ldots, \alpha_{N,j}]$. This matrix equation is equivalent to the following system of linear equations:

$$-\lambda \alpha_{0,j} + \mu \alpha_{1,j} = r_{N,j} \alpha_{0,j}$$  \hspace{1cm} (30)

$$\lambda \alpha_{n-1,j} - (\lambda + \mu) \alpha_{n,j} + (n+1) \mu \alpha_{n+1,j} = r_{N,j} \alpha_{n,j}$$ \hspace{1cm} (31)

$$\lambda \alpha_{N-1,j} - N\mu \alpha_{N,j} = r_{N,j} \alpha_{N,j}$$ \hspace{1cm} (32)

To solve (30), (31), and (32), let $\alpha_{0,j} = 1$, then

$$\alpha_{1,j} = (r_{N,j} + \lambda)/\mu,$$

$$\alpha_{n,j} = \{(r_{N,j} + \lambda + (n-1)\mu) \alpha_{n-1,j} - \lambda \alpha_{n-2,j}\}/n\mu, \hspace{1cm} 1<n<N.$$  

Finally, for $n = N$

$$\alpha_{N,j} = \lambda \alpha_{N-1,j}/(r_{N,j} + N\mu).$$
Similarly, for \( \beta_j A = r_{N,j} \beta_j \) where \( \beta_j = [\beta_{j,0}, \beta_{j,1}, \ldots, \beta_{j,N}] \). First let \( \tilde{\beta}_j = [1, \tilde{\beta}_{j,1}, \tilde{\beta}_{j,2}, \ldots, \tilde{\beta}_{j,N}] \)

where

\[
\tilde{\beta}_{j,0} = 1
\]

\[
\tilde{\beta}_{j,1} = \frac{r_{N,1} + \lambda}{\lambda}
\]

\[
\tilde{\beta}_{j,n} = \frac{(r_{N,j} + \lambda + (n-1)\mu) \tilde{\beta}_{j,n-1} - (n-1)\mu \beta_{j,n-2}}{\lambda}; \quad 1 \leq n < N
\]

and

\[
\tilde{\beta}_{j,N} = \frac{N\mu}{r_{N,j} + N\mu}
\]

is the solution of the system of equations:

\[
-n\lambda \tilde{\beta}_{j,0} + \lambda \tilde{\beta}_{j,1} = r_{N,j} \tilde{\beta}_{j,0},
\]

\[
n\mu \tilde{\beta}_{j,n-1} - (\lambda + n\mu) \tilde{\beta}_{j,n} + n\lambda \tilde{\beta}_{j,n+1} = r_{N,j} \tilde{\beta}_{j,n}; \quad 1 \leq n < N
\]

\[
N\mu \tilde{\beta}_{j,N-1} - N\mu \tilde{\beta}_{j,N} = r_{N,j} \tilde{\beta}_{j,N}.
\]

To obtain the row vector \( \beta_j \), note that

\[
\tilde{\beta}_j \cdot \alpha_j = \sum_{i=0}^{N} \beta_{j,i} \cdot \alpha_{i,j} = K_j,
\]

therefore,

\[
\frac{1}{K_j} \tilde{\beta}_j \cdot \alpha_j = 1,
\]

and hence,

\[
\beta_j = \frac{1}{K_j} \tilde{\beta}_j: \quad \text{that is, } \beta_{j,k} = \frac{1}{K_j} \tilde{\beta}_{j,k} \quad k = 0, 1, 2, \ldots, N
\]

or
\[ \beta_j = \left[ \frac{1}{k_j}, \frac{\beta_j^1}{k_j^1}, \frac{\beta_j^2}{k_j^2}, \ldots, \frac{\beta_j^N}{k_j^N} \right]. \]

Since \( P(0) = [1, 0, \ldots, 0] \), from Equation (29)

\[ P_n(t) = \sum_{j=0}^{N} a_{n,j} \beta_j^j 0^e r_{N,j}^t \quad n = 0, 1, \ldots, N. \quad (33) \]

3.4 SOLUTION BY THE METHOD OF THE PROBABILITY GENERATING FUNCTION

The probability generating function of \( P_n(t) \) is defined by:

\[ P(z, t) = \sum_{n=0}^{N} P_n(t) z^n. \quad (34) \]

Note that \( P(1, t) = 1 \) and \( P(0, t) = P_0(t) \). In addition \( P(z, 0) = 1 \) satisfies the initial condition at \( t = 0 \); for, \( P_n(0) = 0, n = 1, \ldots, N \) and \( P_0(0) = 1 \).

Differentiating Equation (34) with respect to \( z \) results in

\[ \frac{\partial P(z, t)}{\partial z} = \sum_{n=0}^{N} P_n(t) z^{n-1} \]

\[ = \sum_{n=1}^{N} n P_n(t) z^{n-1} \quad (35) \]

\( P_1(t) \) can now be obtained by evaluating the partial at \( z = 0 \); that is,

\[ \frac{\partial P(0, t)}{\partial z} = P_1(t). \quad (36) \]

In general,

\[ \frac{\partial^n P(0, t)}{\partial z^n} = n! P_n(t) \quad (37) \]
or
\[
\frac{1}{n!} \frac{\partial^n P(O,t)}{\partial z^n} = P_n(t): n = 1, 2, \ldots, N
\]
which is the desired probability mass function.

By multiplying (4), (5), and (6) by appropriate powers of \( z \), one obtains for \( n = 0, 1, 2, \ldots, N \):

\[
P'_0(t) = -\lambda P_0(t) + \mu P_1(t)
\]
\[
z P'_1(t) = z \lambda P_0(t) - z(\lambda + \mu) P_1(t) + z^2 \mu P_2(t)
\]
\[
\vdots
\]
\[
z^{N-1} P'_{N-1}(t) = z^{N-1} \lambda P_{N-2}(t) - z^{N-1} (\lambda + (N-1)\mu) P_{N-1}(t) + z^{N-1} N\mu P_N(t)
\]
\[
z^N P'_N(t) = z^N \lambda P_{N-1}(t) - z^N N\mu P_N(t).
\]

Summing the above equations over \( n \) results in:

\[
\sum_{n=0}^{N} P'_n(t)z^n = \lambda \sum_{n=1}^{N} P_n(t)z^n - \lambda \sum_{n=0}^{N-1} P_n(t)z^n - \mu \sum_{n=1}^{N} nP_n(t)z^n
\]
\[
+ \mu \sum_{n=1}^{N} nP_n(t)z^{n-1} = \lambda(z-1) \sum_{n=0}^{N} P_n(t)z^n
\]
\[
+ \mu(1-z) \sum_{n=1}^{N} nP_n(t)z^{n-1} = \lambda(z - z^{N-1})P_N(t).
\]

But

\[
\sum_{n=0}^{N} P'_n(t)z^n = \frac{\partial P(z,t)}{\partial t}, \sum_{n=1}^{N} nP_n(t)z^{n-1} = \frac{\partial P(z,t)}{\partial z} \text{ and,}
\]
remembering that \( \sum_{n=0}^{N} P_n(t)z^n = P(z,t) \), then (38) reduces to the partial differential equation;

\[
\frac{\partial P(z,t)}{\partial t} = \lambda(z-1) P(z,t) + \mu(1-z) \frac{\partial P(z,t)}{\partial z} + \lambda(z^N - z^{N+1})P_N(t) .
\]  

(39)

Since the Laplace transform of \( P(z,t) \) is:

\[
\mathcal{L}\{P(z,t)\} = P^*(z,s) = \sum_{n=0}^{N} P_n^*(s)z^n
\]

(40)

and

\[
\mathcal{L}\left[ \frac{\partial P(z,t)}{\partial t} \right] = sP^*(z,s) - P(z,0)
\]

\[
= sP^*(z,s) - 1 ,
\]

(41)

the partial differential equation reduces to a first-order linear differential equation

\[
sP^*(z,s) - 1 = \lambda(z-1) P^*(z,s) + \mu(1-z) \frac{\partial P^*(z,s)}{\partial z} + \lambda(z^N - z^{N+1})P_N^*(s)
\]

or

\[
\frac{\partial P^*(z,s)}{\partial z} = P^*(z,s) f(z,s) - g(z) - h(z) P_N^*(s) ,
\]

(42)

where

\[
f(z,s) = \frac{s}{\mu(1-z)} + \frac{\lambda}{\mu}
\]

\[
g(z) = 1/\mu(1-z)
\]

\[
h(z) = \frac{\lambda}{\mu} z^N
\]
Rather than solve Equation (42) and then obtain the inverse Laplace transform \( \mathcal{L}^{-1}(P(z,s)) = P(z,t) \), one observe that from Equation (42) at \( z = 0 \):

\[
\frac{\partial^n P^*(0,s)}{\partial z^n} = \sum_{i=0}^{n-1} c_1^{(n-1)} \frac{\partial^i f(0,s)}{\partial z^i} \frac{\partial^{n-1-i} P(0,s)}{\partial z^{n-1-i}} - \frac{(n-1)!}{\mu}
\]

\[
= \sum_{i=1}^{n-1} \frac{(n-1)!}{(n-1-i)! i!} \frac{i! s}{\mu} (n-1-i)! P_{n-1-i}^*(s) - \frac{(n-1)!}{\mu}
\]

\[
+ \frac{(s+\lambda)}{\mu} (n-1)! P_{n-1}^*(s) = (n-1)! \left\{ \frac{s}{\mu} \sum_{i=1}^{n-1} P_{n-1-i}^*(s) - \frac{1}{\mu} \right\}
\]

But

\[
\sum_{i=1}^{n-1} P_{n-1-i}^*(s) = \sum_{j=0}^{n-2} P_j^*(s)
\]

and from Equation (37)

\[
\mathcal{L} \left\{ \frac{\partial^n p(0,s)}{\partial z^n} \right\} = n! P_n^*(s) \quad \text{(44)}
\]

to therefore,

\[
n! P_n^*(s) = (n-1)! \left[ \frac{s}{\mu} \sum_{j=0}^{n-2} P_j^*(s) + \left( \frac{s+\lambda}{\mu} \right) P_{n-1}^*(s) - \frac{1}{\mu} \right]
\]

or

\[
n! P_n^*(s) = \left( \frac{s+\lambda}{\mu} \right) P_{n-1}^*(s) + \frac{s}{\mu} \sum_{j=0}^{n-2} P_j^*(s) - \frac{1}{\mu} \quad \text{(45)}
\]
The Laplace-Transform of \[ \sum_{n=0}^{N} P_n(t) = 1; \text{ thus } \sum_{j=0}^{N} P_j^*(s) = \frac{1}{s} \] and (45) becomes

\[ nP^*(s) = \left( \frac{s+\lambda}{\mu} \right) P_{n-1}^*(s) + \frac{s}{\mu} \left[ \frac{1}{s} - \sum_{j=n-1}^{N} P_j^*(s) \right] - \frac{1}{\mu} \] (46)

\[ = \frac{\lambda}{\mu} P_{n-1}^*(s) - \frac{s}{\mu} \sum_{j=n}^{N} P_j^*(s) \]

which upon solving for \( P_{n-1}^*(s) \) is the desired recursive formula:

\[ P_{n-1}^*(s) = \frac{1}{\lambda} \left\{ (\mu n + s) P_n^*(s) + s \sum_{j=n+1}^{N} P_j^*(s) \right\} . \] (47)

Let \( n=N \), then

\[ P_{N-1}^*(s) = \left( \frac{\mu N + s}{\lambda} \right) P_N^*(s) ; \] (48)

but \( P_N^*(s) \) was determined previously (see Equation (15)); that is, it was found that

\[ P_N^*(s) = \frac{\lambda^N}{sT_N(s)} , \]

therefore;

\[ P_{N-1}^*(s) = \left( \frac{\mu N + s}{\lambda} \right) \frac{\lambda^N}{sT_N(s)} \]

\[ = \lambda^{N-1} \frac{\mu N + s}{sT_N(s)} . \] (49)

For \( n=N-1 \),
Continuing this process one obtains next

\[ p^{*}_{N-2}(s) = \frac{1}{\lambda} \left\{ (\mu(N-1) + s) p^{*}_{N-1}(s) + s p^{*}_{N} \right\} \]

\[ = \lambda^{N-2} \left( \frac{1}{s^{N-1}} \right) \frac{p^{*}_{N-2}(s)(\mu(N-1) + s) + s^{N-1} p^{*}_{N}}{s T_{N}(s)}. \]  

(50)

Continuing this process one obtains next

\[ p^{*}_{N-3}(s) = \frac{r_{N-2}(s)}{s T_{N}(s)}, \]

(51)

and finally

\[ p^{*}_{0}(s) = \frac{r_{1}(s)}{s T_{N}(s)}, \]

(52)

where \( r_{1}, r_{2}, \ldots, r_{N} \) are polynomials in \( s \). For example, \( r_{N}(s) = \lambda^{N-1}(\mu N + s) \).

Applying the inverse Laplace-Transform to \( p^{*}_{n}(s), n = 0, 1, 2, \ldots, N \), yields the desired probability mass function \( p_{n}(t) \).

3.5 SOLUTION BY THE METHOD OF THE WEIGHTED PROBABILITY GENERATING FUNCTION

To obtain the probability mass function \( p_{n}(t) \); the solution to the system of first order linear differential equations, known techniques were applied as described in the previous three sections. In all cases, Equations (4), (5) and (6) were used directly. The method to be developed here leads to an expression of \( p_{n}(t) \)'s in terms of the moments \( M_{k}(t) \) of \( X(t) \), the random variable for the number of parked cars.

Let \( \overline{M}(t) = [M_{1}(t), M_{2}(t), \ldots, M_{N}(t)] \) be a column vector, each element being a moment \( M_{k}(t) \) of \( X(t) \). Then

\[ \overline{M}(t) = E \overline{P}_{n}(t), \]

(53)
where

\[
E = \begin{bmatrix}
1 & 2 & 3 & \ldots & N \\
1 & 2^2 & 3^2 & \ldots & N^2 \\
& \cdots & \cdots & \cdots & \cdots \\
1 & 2^N & 3^N & \ldots & N^N
\end{bmatrix}
\]

is a Vandermonde\(^{(3)}\) type matrix, with distinct matrix elements, except for the first column, \(E\) is non-singular, and \(\overline{P(t)} = [P_1(t), P_1(t), \ldots, P_N(t)]\) is the probability mass function of the number of parked cars. The solution \(\overline{P(t)}\) can now easily be written down, namely,

\[
\overline{P(t)} = E^{-1} \overline{M(t)}
\]

(54)

The moments are already determined in Section 2.3; therefore, it only remains to invert the matrix \(E\). Rather than obtain the inverse of the matrix \(E\), a simpler method of solving the same matrix Equation (53) is derived next.

Since \(X(t)\) assumes only integral values \(n = 0, 1, \ldots, N\) a weighted probability generating function is used in developing the method. Let

\[
M(z,t) = \sum_{n=0}^{N} nP_n(t)z^n ;
\]

(55)

be the generating function of the first moment \(M_1(t) = M(1,t)\) of the random variable \(X(t)\). Note that for \(z = 0\), \(M(0,t) = 0\). Since \(M(z,t)\) can also be expanded in a Taylor series about \(z = 1\), it will be shown that by equating the \(n\)-th partial derivative \(\frac{\partial^n M(z,t)}{\partial z^n}\) of both expressions evaluated at \(z = 0\), one obtains the probability function \(P_n(t)\).
Differentiating Equation (55) with respect to $z$ results in:

$$\frac{\partial M(z,t)}{\partial z} = p_1(t) + 2^2 p_2(t)z + \ldots + N^2 p_N(t)z^{N-1},$$  \hspace{1cm} (56)

which at $z = 0$ reduce to

$$\frac{\partial M(0,t)}{\partial z} = 1(1!) p_1(t).$$  \hspace{1cm} (57)

Differentiating Equation (56) again with respect to $z$ results in:

$$\frac{\partial^2 M(z,t)}{\partial z^2} = 2^2(1) p_2(t) + 3^2(2) p_3(t)z + \ldots + N^2(N-1) p_N(t)z^{N-2},$$  \hspace{1cm} (58)

which at $z = 0$ gives

$$\frac{\partial^2 M(0,t)}{\partial z^2} = 2(2!) p_2(t).$$  \hspace{1cm} (59)

In general, one obtains:

$$\frac{\partial^n M(z,t)}{\partial z^n} = n^2(n-1)(n-2) \ldots (2)(1) p_n(t) (n+1)^2(n)(n-1) \ldots (2) p_{n+1}(t)z^+$$

$$\ldots + N^2(N-1)(N-2) \ldots N-n + 1) p_N(t)z^{N-n}$$

$$= \sum_{k=0}^{N-n} \frac{(n+k)!}{k!} p_{n+k}(t)z^k,$$  \hspace{1cm} (60)

and for $z = 0$

$$\frac{\partial^n M(0,t)}{\partial z^n} = n(n!) p_n(t).$$  \hspace{1cm} (61)

Similarly, for $n = N$

$$\frac{\partial^N M(z,t)}{\partial z^N} = N(N!) p_N(t) = \frac{\partial^N M(0,t)}{\partial z^N}.$$  \hspace{1cm} (62)
Not let \( M(z,t) \) be expressed as a Taylor series about \( z = 1 \); that is:

\[
M(z,t) = \sum_{j=0}^{N} \frac{(z-1)^j}{j!} \frac{\partial^j M(1,t)}{\partial z^j},
\]

(63)

where \( \frac{\partial^j M(1,t)}{\partial z^j} \) is the \( j \)-th order partial of the weighted probability generating function \( M(z,t) \) with respect to \( z \) evaluated as \( z = 1 \).

For \( z = 0 \),

\[
M(0,t) = \sum_{j=0}^{N} \frac{(-1)^j}{j!} \frac{\partial^j M(1,t)}{\partial z^j}.
\]

(64)

Differentiating the Taylor expression (63) with respect to \( z \) results in:

\[
\frac{\partial M(z,t)}{\partial z} = \frac{\partial M(1,t)}{\partial z} + (z-1) \frac{\partial^2 M(1,t)}{\partial z^2} + \ldots + \frac{(z-1)^{N-1}}{(N-1)!} \frac{\partial^N M(1,t)}{\partial z^N},
\]

(65)

which upon substitution of \( z = 0 \) yields:

\[
\frac{\partial M(0,t)}{\partial z} = \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} \frac{\partial^{j+1} M(1,t)}{\partial z^{j+1}}.
\]

(66)

Differentiating Equation (65) again results in:

\[
\frac{\partial^2 M(z,t)}{\partial z^2} = \frac{\partial^2 M(1,t)}{\partial z^2} + (z-1) \frac{\partial^3 M(1,t)}{\partial z^3} + \ldots + \frac{(z-1)^{N-2}}{(N-2)!} \frac{\partial^N M(1,t)}{\partial z^N}
\]

which for \( z = 0 \) gives:

\[
\frac{\partial^2 M(0,t)}{\partial z^2} = \sum_{j=0}^{N-2} \frac{(-1)^j}{j!} \frac{\partial^{j+2} M(1,t)}{\partial z^{j+2}}.
\]

(67)

Continuing in the same manner one obtains:
\[
\frac{\partial^n M(z,t)}{\partial z^n} = \frac{\partial^n M(1,t)}{\partial z^n} + (z-1) \frac{\partial^{n+1} M(1,t)}{\partial z^{n+1}} + \ldots + (z-1)^{N-n} \frac{\partial^N M(1,t)}{(N-n)!} \frac{\partial^n M(1,t)}{\partial z^n} \\
= \sum_{j=0}^{N-n} \frac{(z-1)^j}{j!} \frac{\partial^{n+j} M(1,t)}{\partial z^{n+j}}.
\] (68)

Substituting \( z = 0 \) in Equation (69) results in

\[
\frac{\partial^n M(0,t)}{\partial z^n} = \sum_{j=0}^{N-n} \frac{(-1)^j}{j!} \frac{\partial^{n+j} M(1,t)}{\partial z^{n+j}}.
\] (69)

For \( n = N \) results in:

\[
\frac{\partial^N M(z,t)}{\partial z^N} = \frac{\partial^N M(1,t)}{\partial z^N} = \frac{\partial^N M(0,t)}{\partial z^N}.
\] (70)

From Equations (57) and (66):

\[
P_1(t) = \frac{1}{n(1!)} \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} \frac{\partial^{1+j} M(1,t)}{\partial z^{1+j}}.
\] (71)

Similarly, combining Equations (59) and (67) results in:

\[
P_2(t) = \frac{1}{2(2!)} \sum_{j=0}^{N-2} \frac{(-1)^j}{j!} \frac{\partial^{2+j} M(1,t)}{\partial z^{2+j}}.
\] (72)

In general,

\[
P_n(t) = \frac{1}{n(n!)} \sum_{j=0}^{N-n} \frac{(-1)^j}{j!} \frac{\partial^{n+j} M(1,t)}{\partial z^{n+j}}; \quad n = 1, 2, \ldots, N
\] (73)

In the Appendix, Section 4.4, it will be shown that

\[
\frac{\partial^{n+j} M(1,t)}{\partial z^{n+j}} = \sum_{i=1}^{n+j} \frac{(-1)^{i+1}}{i!} a_{n+j+2-1} m_{n+j+2-i}(t)
\] (74)
where the $a_{n+j+2-i}$ are constants determined in the same Appendix.

Hence:

$$P_n(t) = \frac{1}{n(n!)} \sum_{j=0}^{N-n} \frac{(-1)^j}{j!} \sum_{i=1}^{n+j} (-1)^{i+1} a_{n+j+2-i} M_{n+j+2-i}(t), \quad (75)$$

$n = 1, 2, \ldots, N$.

To determine $P_0(t)$, note that $\sum_{n=0}^{N} P_n(t) = 1$; therefore,

$$P_0(t) = 1 - \sum_{n=1}^{N} P_n(t)$$

$$= 1 - \sum_{n=1}^{N} \frac{1}{n(n!)} \sum_{j=0}^{N-n} \frac{(-1)^{i+1}}{j!} a_{n+j+2-i} M_{n+j+2-i}(t)(76)$$

3.6 SOLUTION FOR THE CASE OF $N = 1$

In Section 3.2, the solution by Laplace-Transform was described.

Utilizing the matrix Equation (14), one obtains for $N = 1$:

$$\begin{pmatrix} s+\lambda & -\mu \\ -\lambda & s+\mu \end{pmatrix} \begin{pmatrix} P_0^*(s) \\ P_1^*(s) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ 

To get expressions for $P_0^*(s)$ and $P_1^*(s)$, one must evaluate Equation (18). For $n = 0$, this results in;

$$C_{00}(s) = s+\mu$$

and for $n = 1$

$$C_{01}(s) = \lambda ;$$

therefore,
\[ P_0^*(s) = \frac{s+\mu}{sT_1(s)} \]

and

\[ P_1^*(s) = \frac{\lambda}{sT_1(s)} \]

where \( sT_1(s) = s^2 + s(\lambda+\mu) \) is the determinant of the above matrix and defined by Equation (17).

The determinant \( sT_1(s) \) has as its roots \( r_{1,0} = 0 \) and \( r_{1,1} = -(\lambda+\mu) \); therefore, from Equation (16) one obtains:

\[ P_0(t) = \sum_{i=0}^{1} C_{0,i}(r_{1,i}) e^{r_{1,i}t} \]

and

\[ P_1(t) = \sum_{i=0}^{1} C_{0,i}(r_{1,i}) e^{r_{1,i}t} \]

Substituting the roots \( r_{1,0} = 0 \) and \( r_{1,1} = -(\lambda+\mu) \) in the above expression results in:

\[ P_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \]

and

\[ P_1(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \]

which is the desired probability mass function.

Taking the limit of \( P_0(t) \) and \( P_1(t) \) as \( t \) approaches infinity, the steady state solution \( P_0 \) and \( P_1 \) is obtained; that is,

\[ P_0 = \frac{\mu}{\lambda+\mu} \]

and

\[ P_1 = \frac{\lambda}{\lambda+\mu} \]
which agrees with the results of MORSE.

The solution by the Eigenvalue-Eigenvector method was considered in Section 3.3. For \( N = 1 \), the matrix Equation (20) becomes

\[
\begin{pmatrix}
P_0'(t) \\
P_1'(t)
\end{pmatrix}
= 
\begin{pmatrix}
-\lambda & \mu \\
\lambda & -\mu
\end{pmatrix}
\begin{pmatrix}
P_0(t) \\
P_1(t)
\end{pmatrix}.
\]

Applying Equations (30), (31), and (32) results in a matrix \( \alpha \):

\[
\alpha = \begin{pmatrix}
1 & 1 \\
\lambda/\mu & -1
\end{pmatrix}.
\]

Similarly, the matrix \( \beta \) contains the elements:

\[
\beta = \begin{pmatrix}
1 & 1 \\
1 & -\mu/\lambda
\end{pmatrix}.
\]

To obtain the matrix \( \beta \), one notes that:

\[
\beta_0 \cdot \alpha_0 = (1,1) \cdot (1,\frac{\lambda}{\mu}) = \frac{\lambda+\mu}{\mu} = K_0
\]

and

\[
\beta_1 \cdot \alpha_1 = (1,-\frac{\mu}{\lambda}) \cdot (1,-1) = \frac{\lambda+\mu}{\mu} = K_1;
\]

therefore,

\[
\beta = \begin{pmatrix}
\frac{1}{K_0} & \frac{1}{K_0} \\
\frac{1}{K_1} & -\mu/\lambda K_1
\end{pmatrix} = \begin{pmatrix}
\frac{\mu}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\
\frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu}
\end{pmatrix}.
The probability mass function $P_n(t)$ can now easily be obtained from Equation (33); that is:

$$P_n(t) = \sum_{n=0}^{1} \alpha_n \beta_{n,0} e^{r_n t} ; n = 0, 1.$$  

For $n = 0$

$$P_0(t) = \alpha_0 \beta_{0,0} e^{r_0 t} + \alpha_0 \beta_{0,1} e^{r_1 t} = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

and for $n = 1$

$$P_1(t) = \alpha_1 \beta_{1,0} e^{r_0 t} + \alpha_1 \beta_{1,1} e^{r_1 t} = \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t}$$

which is the desired result.

In Section 3.4, the method of the probability generating function was discussed and described. It was found that for $N = 1$ and Equation (48)

$$P_0^*(s) = \left(\frac{\mu + s}{\lambda}\right) P_1^*(s) ;$$

but

$$P_1^*(s) = \frac{\lambda}{s^2 + s(\lambda + \mu)} ;$$

therefore,

$$P_0^*(s) = \left(\frac{\mu + s}{\lambda}\right) \frac{\lambda}{s^2 + s(\lambda + \mu)} = \frac{s + \mu}{s^2 + s(\lambda + \mu)}$$

The solution to these Laplace-Transforms has already been discussed previously.
Finally, in Section 3.5, the solution by the method of a weighted probability generating function was described. For \( N = 1 \), Equation (75) becomes

\[
p_n(t) = \frac{1}{n(n!)} \sum_{j=0}^{1-n} \frac{(-1)^j}{j!} \sum_{i=1}^{n+j} (-1)^{i+1} a_{n+j+2-i} \ M_{n+j+2-i}(t),
\]

Letting \( n = 1 \), \( P_1(t) \) becomes

\[
P_1(t) = a_2 \ M_2(t).
\]

Since \( n+j = 1 \), therefore \( k = 1 \) and from Equation (97) one concludes that \( a_2 = 1 \); thus \( P_1(t) = M_2(t) \). But for \( N = 1 \), \( M_2(t) = M_1(t) \); therefore

\[
P_1(t) = M_1(t)
\]

\[
= \lambda(1 - e^{-\mu t})/\mu - \lambda \sum_{i=1}^{1} \ B_{1,i} \left( e^{r_{1,i}t} - e^{-\mu t} \right)/(r_{1,i} + \mu),
\]

where \( M_1(t) \) was determined in Section 2.3. At the beginning of this section, it was found that for the roots \( r_{1,0} = 0 \), and \( r_{1,1} = -(\lambda + \mu) \) the coefficients:

\[
B_{1,0} = \frac{\lambda}{\lambda + \mu}
\]

and

\[
B_{1,1} = -\frac{\lambda}{\lambda + \mu}
\]

respectively; therefore,
\[
P_1(t) = \left(\frac{\lambda}{\mu}\right) (1 - e^{-\mu t}) - \lambda \left[ \frac{\lambda}{\mu(\lambda+\mu)} \left(1 - e^{-\mu t}\right) + \frac{\lambda}{\lambda+\mu} \left(e^{-(\lambda+\mu)t} - e^{-\mu t}\right) \right]
\]

\[
= \left(\frac{\lambda}{\mu}\right) (1 - e^{-\mu t}) \left(1 - \frac{\lambda}{\lambda+\mu}\right) - \left(\frac{\lambda}{\lambda+\mu}\right) \left(e^{-(\lambda+\mu)t} - e^{-\mu t}\right)
\]

\[
= \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t},
\]

but from Equation (76),

\[
P_0(t) = 1 - P_1(t)
\]

\[
= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}
\]

which is the desired result for the probability mass function \(P_n(t)\).
4. APPENDIX

4.1 THE STAY-TIME AND ARRIVAL DISTRIBUTION FUNCTIONS

The stay-time distribution of a car plays a central role in the formulation of the parking lot model. To determine the statistical nature of the stay-time distribution, a large random sample of recorded stay-time histories is required. Using this data, a sequence of recorded times in order of increasing length can be constructed. By plotting the number of stay-times that are shorter than a given time, divided by the total number of cases in the sample, a curve for the probability distribution function $F_0(t)$ that the stay-time for this class will be less than a certain time is obtained. If the curve follows the complement of a negative exponential function, then

$$P_r \{ \xi \leq t \} = F_0(t) = 1 - e^{-\mu t}$$  \hspace{1cm} (77)

where $1/\mu$ is the mean stay-time and $\xi$ denotes the stay-time random variable. The probability density function of $\xi$ is obtained by differentiating $F_0(t)$, thus:

$$\frac{dF_0(t)}{dt} = \mu e^{-\mu t}$$  \hspace{1cm} (78)

and the probability distribution function $F_0(t)$ is given by:

$$P_r \{ \xi > t \} = F_0(t) = e^{-\mu t}$$  \hspace{1cm} (79)

which gives the probability that the stay-time of a car will be greater than $t$ units of time after its arrival. From (78) the mean stay-time,
$T_\mu$ is obtained; integrating by parts gives:

$$T_\mu = \int_0^\infty t dF_0(t) = \int_0^\infty F_0(t) \, dt = \frac{1}{\mu}. \quad (80)$$

By a similar argument, one obtains the distribution function $Q_0(t)$, where

$$P_r \{ \xi \leq t \} = Q_0(t) = 1 - e^{-\lambda t}, \quad (81)$$

or

$$P_r \{ \xi > t \} = Q_0(t) = e^{-\lambda t} \quad (82)$$

and $\xi$ is the random variable denoting the time between successive car arrivals. The probability density function of $\xi$ is given by:

$$\frac{dQ_0(t)}{dt} = \lambda e^{-\lambda t}$$

$$= q(t), \quad (83)$$

where $\lambda$ is the mean arrival rate. The mean time between successive arrivals is; after integrating by parts:

$$T_\lambda = \int_0^\infty t dQ_0(t) = \int_0^\infty Q_0(t) \, dt = \frac{1}{\lambda}. \quad (84)$$
4.2 THE PROBABILITY MASS FUNCTION OF THE NUMBER OF CAR ARRIVALS

Now that the arrival of a single car has been characterized, it is possible to determine the probability that any number of cars arrive within a period of time of length \( t \). Let \( t \) be an arbitrary chosen time and \( x \) any time less than \( t \). Then the chance that, starting at time 0, the first car to appear arrives in the interval \([x, x + dx]\) is \( q(x) \, dx = \lambda e^{-\lambda x} \, dx \). The chance that no car arrives in \([t-x]\) is \( Q_0(t-x) \). Hence, the probability \( Q_1(t) \), that exactly one car arrived within an interval of length \( t \) is the product of the two probability integrated over all possible \( x \) satisfying \( x \leq t \). Therefore,

\[
Q_1(t) = \int_0^t q(x) \, Q_0(t-x) \, dx = \lambda t e^{-\lambda t}
\]  

(85)

Since the chance that one car arrived in \([t-x]\) is \( Q_1(t-x) \), the probability that exactly two cars arrived in this interval \([0, t]\) is given, in a similar manner, by the convolution:

\[
Q_2(t) = \int_0^t q(x) \, Q_1(t-x) \, dx = (\lambda t)^2 e^{-\lambda t/2!}
\]  

(86)

Continuing in the same manner, by recursion, one obtains

\[
Q_k(t) = \int_0^t q(x) \, Q_{k-1}(t-x) \, dx = (\lambda t)^k e^{-\lambda t/k!}
\]  

(87)

which is the Poisson distribution function with parameter time.
4.3 PROPERTIES OF THE TRANSFORMED TRANSITION PROBABILITY MATRIX

In Section 4.3, it is assumed that the N-th degree polynomial
\[ sT_n(s) = |S_N| \] had (N+1) real and distinct roots; permitting, therefore, to write down the solution in the form of Equation (16). To prove that the determinant \( |S_N| \) has (N+1) distinct real roots, it will be shown first that the recursive relationship \( T_N(s) = T_{N-1}(s) (s+A+N\mu) - (N-1)\lambda\mu T_{N-2}(s) \) holds.

**THEOREM 1:** Let \( S_k \) be a \((k+1)\times(k+1)\) tridiagonal matrix of the form
\[
\begin{bmatrix}
  s+\lambda & -\mu & 0 & 0 & \cdots & 0 \\
  -\lambda & (s+\lambda+\mu) & -2\mu & 0 & \cdots & 0 \\
  0 & -\lambda & (s+\lambda+2\mu) & -3\mu & \cdots & 0 \\
  & & \ddots & \ddots & \ddots & \ddots \\
  & & & 0 & & s+\lambda+(k-1)\mu & -k\mu \\
  0 & \cdots & 0 & -\lambda & s+k\mu
\end{bmatrix}
\]
then \( |S_k| \) is \( sT_k(s) = s \{ T_{k-1}(s) (s+\lambda+k\mu) - (k-1)\lambda\mu T_{k-2}(s) \} \), for \( k = 1, 2, 3, \ldots \) with \( T_0(s) = 1 \) and \( T_1(s) = s+\lambda+\mu \).

Proof: First transform the matrix \( S_k \) by performing the following sequence of operations: (i) Replace the last column of \( S_k \) by the difference of the last two columns. (ii) Replace the newly obtained second to last row by the sum of the last two rows of the transformed matrix. (iii) Replace the third to last row by the sum of itself and the newly obtained second to last row. By continuing this process of replacing the \( i \)-th to last row by the sum of itself and the newly obtained \((i+1)\)-st to last row, a matrix \( D_k \) is obtained which is of the form...
Since the row and column operations do not change the value of its determinant, on expanding \( \det(D_k) \) according to elements of its last row:

\[
\det(S_k) = \det(D_k) = (s+\lambda+k\mu) \det(D_{k-1}) - (k-1)\lambda\mu \det(D_{k-2}); \quad (90)
\]

but

\[
\det(D_{k-1}) = sT_{k-1}(s),
\]

and

\[
\det(D_{k-2}) = sT_{k-2}(s).
\]

Hence for \( k = N \)

\[
sT_N(s) = s \left\{ T_{N-1}(s) (s+\lambda+N\mu) - (N-1)\lambda\mu T_{N-2}(s) \right\}; \quad (91)
\]

which is the desired result.

To solve the system of linear-differential equations by the Laplace-transform, it was assumed that the polynomial \( sT_N(s) \) has \((N+1)\) real distinct roots \( \{ r_{N,N}, r_{N,N-1}, \ldots, r_{N,j}, \ldots, r_{N,0} \} \), the first subscript referring to the degree of the polynomial and the second subscript \( j \) being the \( j \)-th root that satisfies \( sT_N(s) = 0 \). To show that
the polynomial \( sT_N(s) \) has real distinct roots, one needs the following theorem.

**THEOREM 2:** Let \( S_k \) be an \((k+1) \times (k+1)\) tridiagonal transition matrix of the form (88); then the polynomial \( sT_k(s) = 0 \) has exactly \((k+1)\) distinct real roots \( \{ r_{k,k}, r_{k,k-1}, \ldots, r_{k,j}, \ldots, r_{k,0} \} \) such that for each fixed \( k \), \( r_{k,j-1} > r_{k,j}; \quad j = 1, 2, 3, \ldots, k \).

Proof: The root 0 will obviously be assigned to \( r_{k,0} \). Consider the equation \( T_k(s) = 0 \) only. The proof of Theorem 2 will be by induction on \( k \).

Suppose \( k=1 \); then by Theorem 1,

\[
T_1(s) = (s+\lambda+\mu).
\]

Thus, if \( r_{1,1} = -(\lambda+\mu) \), then

\[
T_1(r_{1,1}) = 0, \quad r_{1,1} \text{ is obviously real, and } r_{1,1} < r_{1,0} = 0. \quad \text{Let } k = 2, \text{ then}
\]

\[
T_2(s) = T_1(s)(s+\lambda+2\mu) - \lambda \mu T_0(s).
\]

For \( s = -\mu \), (remembering that \( T_0(s) = 1 \)) one obtains

\[
T_2(-\mu) = T_1(-\mu)(\lambda+\mu) - \lambda \mu = \lambda^2.
\]

Therefore, since \( T_2(r_{1,1}) = -\lambda \mu \), then there exists a real root \( r_{2,1} \) satisfying \( T_2(r_{2,1}) = 0, \) and \( r_{1,1} < r_{2,1} < -\mu \). Since the second degree polynomial \( T_2(s) \) has a leading coefficient of one, it follows that \( \lim_{s \to \infty} T_2(s) > 0. \) But \( T_2(r_{1,1}) < 0 \) and thus there exists a root \( r_{2,2} \) of \( T_2(s) \) such that \(-\infty < r_{2,2} < r_{1,1}. \) Thus, the roots \( \{r_{2,2}, r_{2,1} \} \) fall into the interior of the intervals
Suppose that for \( p = 1, 2, \ldots, k \)
\[
T_p(s) = T_{p-1}(s) \left(s + \lambda + p\mu\right) - (p-1)\lambda\mu T_{p-2}(s)
\]
has \( p \) distinct real roots satisfying the following inequalities:
\[
-\infty < r_{p,2} < r_{1,1} < r_{p,1} < -\mu.
\]
\[\cdots \]
\[
-\infty < r_{p,p} < r_{p-1,p-1} < r_{p,p-1} < -\mu\) (92)
\]
from which it follows that \( r_{k,k} < r_{k,k-1} < \cdots < r_{k,1} \).

Consider then
\[
T_{k+1}(s) = T_k(s) \left(s + \lambda + (k+1)\mu\right) - k\lambda\mu T_{k-1}(s);
\]
for any root \( r_{k,j} \) of \( T_k(s) \), it follows that
\[
T_{k+1}(r_{k,j}) = -k\lambda\mu T_{k-1}(r_{k,j}).\) (93)

By (92), \( T_{k-1}(s) \) has exactly one root between \( r_{k,j} \) and \( r_{k,j+1} \) and thus \( T_{k-1}(r_{k,j}) \) and \( T_{k-1}(r_{k,j+1}) \) have opposite signs and from (93) \( T_{k+1}(r_{k,j}) \) must have opposite signs. Therefore, \( T_{k+1}(s) \) and \( T_{k+1}(r_{k,j+1}) \) must have opposite signs. Therefore, \( T_{k+1}(s) \) has at least one real root \( r_{k+1,j} \) such that \( r_{k,j+1} < r_{k+1,j} < r_{k,j} \). Thus, at least \( k-1 \) distinct real roots exist. For \( s = -\mu \), \( T_{k+1}(-\mu) = \lambda^{k+1} \). By (92) and (93), \( T_{k+1}(r_{k,1}) < 0 \). Hence there exists a real root \( r_{k+1,1} \) for which \( T_{k+1}(r_{k+1,1}) = 0 \) and \( r_{k,1} < r_{k+1,1} < -\mu \). For \( k \) odd, \( T_{k+1}(r_{k,k}) < 0 \) and \( \lim_{s \to -\infty} T_{k+1}(s) > 0 \). Hence there exists a real root \( r_{k+1,k+1} \) for which
\[ T_{k+1}(r_{k+1,k+1}) = 0 \text{ and } -\infty < r_{k+1,k+1} < r_{k,k}. \] For \( k \) even, a similar arguments leads to the same conclusion.

Thus \((k+1)\) distinct real roots exist. Since the polynomial is of \((k+1)\)-st degree there exists exactly \((k+1)\) roots. The proof is thus complete.
4.4 THE PARTIAL DERIVATIVE OF THE WEIGHTED PROBABILITY GENERATING FUNCTION

In Section 3.5, a solution by the method of a weighted probability generating function was derived. The probability mass function \( P_n(t) \) of the number of cars parked at time \( t \) was expressed in terms of its moments \( M_k(t) \). It was shown that:

\[
\frac{\partial^n M(0,t)}{\partial z^n} = n(n!) \, P_n(t) = \sum_{j=0}^{N-n} \frac{(-1)^j}{j!} \frac{\partial^{n+j} M(1,t)}{\partial z^{n+j}}.
\]

We now show that

\[
\frac{\partial^{n+j} M(1,t)}{\partial z^{n+j}} = \sum_{i=0}^{n+j} (-1)^{i+1} a_{n+j+2-i} M_{n+j+2-i}(t).
\]

To show that the above relationship holds, one must show first that:

\[
\frac{\partial^k M(1,t)}{\partial z^k} = \sum_{n=1}^{N} n^2 \,(n-1) \,(n-2) \, \ldots \,(n-k+1) \, P_n(t).
\]

From Equation (56) one obtains for \( z = 1 \)

\[
\frac{\partial M(1,t)}{\partial z} = \sum_{n=1}^{N} n^2 \, P_n(t) = M_2(t),
\]

and from Equation (58) one obtains for \( z = 1 \)

\[
\frac{\partial^2 M(1,t)}{\partial z^2} = 2^2 \,(1) \, P_2(t) + 3^2 \,(2) \, P_3(t) + \ldots + N^2 \,(N-1) \, P_N(t)
\]

\[
= 1^2 \,(1-1) \, P_1(t) + 2^2 \,(2-1) \, P_2(t) + 3^2 \,(3-1) \, P_3(t)
\]

\[+ \ldots + N^2 \,(N-1) \, P_N(t) = \sum_{n=1}^{N} n^2 \,(n-1) \, P_n(t) = M_3(t) - M_2(t). \]  

(94)
In general, one can see that from (60) that:

\[
\frac{a^k M(1,t)}{a z^k} = 1^2 (1-1)(1-2) \cdots (1-k+1) P_1(t) + 2^2 (2-1)(2-2)(2-3) \cdots (2-k+1) P_2(t) + \cdots k^2 (k-1)(k-2) \cdots (1) P_k(t)
\]

\[+ \cdots + N^2 (N-1)(N-2) \cdots (N-k+1) P_N(t)\]

\[= \sum_{n=1}^{N} n^2 (n-1)(n-2) \cdots (n-k+1) P_n(t) . \quad (95)\]

Let

\[p(n) = (n-1)(n-2) \cdots (n-k+1) \quad (96)\]

then \(p(n) = u_{k-1} n^{k-1} - u_{k-2} n^{k-2} + u_{k-3} n^{k-3} \cdots (-1)^{k-1} u_0 , \]

where

\[
u_{k-1} = 1
\]

\[
u_{k-2} = 1 + 2 + 3 + \cdots + (k-1)
\]

\[
u_{k-3} = 1 \cdot 2 + 1 \cdot 3 + \cdots 1 \cdot (k-1) + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot (k-1)
\]

\[
+ \cdots (k-2) \cdot (k-3) + \cdots + (k-2) (k-1)
\]

\[
\]

\[
u_0 = 1 \cdot 2 \cdot 3 \cdots (k-1) .
\]

Thus, when multiplying \(p(n)\) by \(n^2\) one obtains

\[n^2 p(n) = u_{k-1} n^{k+1} - u_{k-2} n^{k} + u_{k-3} n^{k-1} - \cdots (-1)^{k-1} u_0 . \quad (97)\]
The subscript of each coefficient is adjusted to reflect the power of \( n \), so that

\[
n^2 p(n) = a_{k+1} n^{k+1} - a_k n^k + a_{k-1} n^{k-1} - \ldots \cdot (-1)^{k-1} a_2
\]

\[
= \sum_{i=1}^{k} (-1)^{i+1} a_{k+2-i} n^{k+2-i}.
\]

(98)

Substituting Equation (98) in Equation (95), results in:

\[
\frac{\partial^k M(1,t)}{\partial z^k} = \sum_{n=1}^{N} n^2 p(n) p_n(t)
\]

\[
= \sum_{n=1}^{N} \sum_{i=1}^{k} (-1)^{i+1} a_{k+2-i} n^{k+2-i} p_n(t)
\]

\[
= \sum_{i=1}^{k} (-1)^{i+1} a_{k+2-i} \sum_{n=1}^{N} n^{k+2-i} p_n(t)
\]

\[
= \sum_{i=1}^{k} (-1)^{i+1} a_{k+2-i} M_{k+2-i}(t).
\]

Hence, if \( k = n+j \)

\[
\frac{\partial^{n+j} M(1,t)}{\partial z^{n+j}} = \sum_{i=1}^{n+j} (-1)^{i+1} a_{n+j+2-i} M_{n+j+2-i}(t)
\]

(100)

which concludes the derivation.
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