MODULATIONAL INSTABILITY FOR A CLASS OF
NONLINEAR SCHRÖDINGER EQUATIONS AND FOR
A SYSTEM OF NEWELL-SEGEL-WHITEHEAD EQUATIONS

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DEDICATIONS

To my family and friends,

Thank you for all your support
I would like to thank my thesis advisor Dr. Yomba and co-chair Dr. Djellouli for their support and encouragement throughout my thesis preparation. Thanks also to my committee members Dr. Zakeri and Dr. Panferov.
# Table of Contents

Signature page ................................................................. ii  
Dedication ................................................................. iii  
Acknowledgement ............................................................... iv  
Abstract ...................................................................... vii  

Chapter 1  
Introduction ............................................................... 1  

Chapter 2  
Higher order nonlinear Schrödinger equation ......................... 3  
  2.1 Introduction ............................................................... 3  
  2.2 Analysis of the generalized nonlinear Schrödinger equation with cubic-quintic  
      nonlinearity ................................................................. 4  
    2.2.1 Problem Statement ....................................................... 4  
    2.2.2 Dispersion Relations ..................................................... 4  
    2.2.3 Numerical study ......................................................... 8  
    2.2.4 Summary and Conclusion ............................................. 12  
  2.3 Analysis for the generalized nonlinear Schrödinger equation with higher order  
      nonlinear and dispersion terms ....................................... 13  
    2.3.1 Problem Statement ....................................................... 13  
    2.3.2 Dispersion Relations ..................................................... 13  
    2.3.3 Numerical study ......................................................... 16  
    2.3.4 Summary and Conclusion ............................................. 20  

Chapter 3  
System of coupled nonlinear Schrödinger equations .................... 21  
  3.1 Introduction ............................................................... 21  
  3.2 Analysis for the coupled Schrödinger equations ..................... 21  
    3.2.1 Problem Statement ....................................................... 21  
    3.2.2 Dispersion Relations ..................................................... 22  
    3.2.3 Numerical study ......................................................... 26  
  3.3 Summary and Conclusion ............................................... 28
Chapter 4
System of coupled complex Newell-Segel-Whitehead equations 29
4.1 Introduction 29
4.2 Analysis for a system of complex coupled Newell-Segel-Whitehead Equa-
tions 29
   4.2.1 Problem Statement 29
   4.2.2 Dispersion Relations 30
   4.2.3 Numerical study 35
4.3 Summary and Conclusion 50

General Conclusion 51

Bibliography 51

Appendix 55
ABSTRACT

MODULATIONAL INSTABILITY FOR A CLASS OF NONLINEAR SCHRÖDINGER EQUATIONS AND FOR A SYSTEM OF NEWELL-SEGEL-WHITEHEAD EQUATIONS

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Master of Science in Mathematics

Partial differential equations (PDEs) describe many phenomena in the sciences and engineering. However most real world physical systems are modeled by nonlinear PDEs, that can rarely be solved analytically. Solving them numerically has been facilitated by the availability of fast computing power. However, numerical techniques often require addressing challenges including algorithmic and computational considerations chief among them the efficient solution of large-scale systems. We propose a procedure called Modulational Instability (MI). MI is a process in which the amplitude and phase modulations of a wave grow as a result of interplay between nonlinearity and dispersion. Hence, it describes the behavior of the solution without determining the close form of the solution or its numerical approximation. In this work, we employ MI to analyze the generalized nonlinear Schrödinger equation with cubic-quintic terms, as well as with higher order nonlinear and dispersion terms. Then, we extend MI to investigate a system of nonlinear Schrödinger equations and a system of Newell-Segel-Whitehead equations. Our investigations reveal that the MI gain exhibits specific patterns that could be determined by an appropriate choice of parameter values. It is expected that this study will provide engineers and applied mathematicians with the prerequisite knowledge and practical guidelines for selecting the discretization parameters of the numerical techniques to be employed for solving this class of problems efficiently.
Chapter 1

Introduction

Partial differential equations (PDEs) play a major role in modeling real world applications such as weather prediction, sonar and radar applications, digital image processing, ultrafast signal routing, data processing, etc. Due to their ubiquitous presence, PDEs have received a great deal of attention from mathematicians, scientists, and engineers. The quest for better understanding PDEs is ongoing. Scientists have developed an array of techniques for solving PDEs. These techniques can be roughly categorized into three groups: analytical methods, numerical methods, and qualitative methods.

Analytical methods provide an explicit expression of the solution to a given PDE. This category of techniques includes separation of variables [1], Fourier series expansion [2], Fourier transform [2], Laplace transform [3]. These techniques are exceptionally powerful analytical tools for solving PDEs. In spite of their nice features, analytical methods are limited to a very small range of PDEs (linear, constant coefficients, simple shaped computational domain).

Numerical methods are currently the primary candidates for solving PDEs. This category of methods provides numerical approximations of the solutions to the PDEs. Some of the most commonly used numerical algorithms include finite difference methods (FDM) [2], finite volume methods (FVM) [4], finite elements methods (FEM) [5], and spectral methods [6]. The efficiency of these techniques depends upon several considerations chief among them algorithmic issues, implementation complexity, computational cost, and computational capabilities.

The last category of methods for studying PDEs are qualitative methods. These techniques are conceptually simple to understand and easy to apply to any kind of PDE. These methods shed light on the local behavior of the considered equation without knowing its solution.

The goal of this study is to employ a qualitative method called Modulational Instability (MI) to investigate the behavior of the solutions of several types of PDEs. MI is a process in which the amplitude and phase modulations of a wave grow as a result of interplay between nonlinearity and dispersion [7], [8], [9]. It is considered to be a basic process that classifies the qualitative behavior of modulated waves (envelope waves) [10]. In this study, we first apply MI to the generalized nonlinear Schrödinger equation with cubic-quintic terms [11] as well as with higher order nonlinear and dispersion terms [9]. The Schrödinger equation is prominently used in nonlinear optics to model optical pulses [12]. Adding more nonlinear and dispersive terms to it enables a more accurate description of shorter optical pulses, which are used in telecommunications and ultrafast signal routing [13]. Then, we also analyze a system of coupled nonlinear Schrödinger equations [14]. This system models multimode fibers, which are commonly used in network applications [15]. Furthermore, we employ MI to investigate a system of complex coupled nonlinear Newell-Segel-Whitehead equations [16]. The Newell-Segel-Whitehead equation is a mathematical model that has been used for many years to describe the evolution of convection rolls [17]. It has been applied to a number of problems such as description of the disorder
of singularities of the amplitude and phase envelope fields, etc. [18].

The remainder of this manuscript is organized as follows. In chapter 2, we apply MI to the generalized nonlinear Schrödinger equation with cubic-quintic terms as well as with higher order nonlinear and dispersion terms. In chapter 3, we employ MI to analyze the system of nonlinear Schrödinger equations coherently coupled with cross phase modulation and four wave mixing. Chapter 4 is devoted to the analysis of a complex coupled system of nonlinear Newell-Segel-Whitehead equations. And finally, a summary and concluding remarks are reported in the last chapter of this manuscript.
Chapter 2

Higher order nonlinear Schrödinger equation

2.1 Introduction

The Schrödinger equation explains high water waves [19] as well as various phenomena in quantum mechanics [20]. It is one of the most prominently used mathematical models in optics [12]. Indeed, propagation of optical pulses through fibers is described by solving Maxwell’s equations. In the slowly varying envelope approximation, these equations lead to the following cubic Schrödinger equation [12].

\[ A_t = i(\beta_2 A_{xx} + \gamma |A|^2 A) - \delta A. \]  

(2.1)

where, \( A(x,t) \) represents a slowly varying envelope of a wave. Slowly varying envelope approximation is the assumption that the envelope of a forward-traveling wave pulse varies slowly in time and space compared to a period or wavelength. \( x \) and \( t \) are the normalized distance and time. The subscripts \( x \) and \( t \) denote the spatial and the temporal partial derivatives respectively. \( \beta_2 \) is a real number parameter representing the group velocity dispersion. The group velocity dispersion introduces frequency dependent delay of the different spectral components of a pulse.

The Schrödinger equation given by Eq. (2.1) is of great value in several applications. The slowly varying wave modulation approximation leading to Eq. (2.1) can explain the mathematics behind the applications such as specifying wave groups in the hydrodynamic forces on marine structures and three-dimensional waves where accurate solutions are not computable for more than a few wave periods [21]. Eq. (2.1) with a lattice potential models a periodic dilute-gas Bose-Einstein Condensates [22]. Some general properties of a superconductor, such as the Josephson effects, the Magnus force, and the Bogoliubov-Anderson mode can be obtained readily from this equation [23].

The Schrödinger equation is a versatile integrable nonlinear PDE. However, its applicability is restricted in terms of nonlinearity, magnitude, and spectral width. These disadvantages can be eliminated by adding higher order nonlinear and dispersive terms, as done in reference [19] leading to a higher order equation (see Eq. (2.2) in section (2.2)). The terms added in this equation are mainly dispersion terms that are crucial in optics. Indeed, the dispersion is a phenomenon in which the phase velocity of a wave depends on its frequency [24]. Furthermore, adding the higher-order dispersive terms to Eq. (2.2) provides a more accurate description of shorter pulses (see Eq. (2.26) in section (2.2)). The additional terms in this equation are mainly the dispersion terms that are crucial in optics. The resulting equation (see Eq. (2.26) in section (2.2)) is applicable to telecommunications and ultrafast signal-routing systems extensively in the weakly dispersive and nonlinear dielectrics with distributed parameters [25].
2.2 Analysis of the generalized nonlinear Schrödinger equation with cubic-quintic nonlinearity

2.2.1 Problem Statement

In this section, we consider the following Schrödinger equation:

\[ A_t = i(\beta_2 A_{xx} + g_1 |A|^2 A) + \beta_3 A_{xxx} + i \beta_4 A_{xxxx} + \alpha_1 (|A|^2 A)_x + i \gamma_2 |A|^4 A, \quad x \in \mathbb{R}, \ t > 0, \]  

(2.2)

where,

- \( A(x,t) \) is a function representing a slowly varying envelope.
- The subscript \( x \) (resp. \( t \)) denotes the spatial (resp. temporal) partial derivative.
- \( \beta_j; j = 2, 3, 4 \) are real numbers representing the dispersion coefficients.
- \( \gamma_1 \) is a real number representing the self phase modulation parameter.
- \( \gamma_2 \) is a real number representing the quintic nonlinearity parameter.
- \( \alpha_1 \) is a real number representing the self steepening parameter.

The dispersion parameter \( \beta_2 \) is obtained by taking one derivative of the group velocity, whereas \( \beta_3 \) and \( \beta_4 \) are the third and fourth order derivatives, which are called the third and fourth order dispersion [26]. The self phase modulation parameter \( \gamma_1 \) incurs a nonlinear phase modulation of the wave caused by its own intensity [12], [27]. It is often used for fast optical switching. Lastly, the self steepening parameter \( \alpha_1 \) gives rise to one of the important higher order effects. It causes a sharp drop at the trailing edge of the pulse resulting in a shock formation in analogy to an acoustic wave [28].

**Remark 1.** Note that the Schrödinger equation given by Eq. (2.2) is strongly nonlinear. It involves a nonlinearity of the fifth order. It also incurs the fourth order spatial derivative. This equation cannot be solved analytically, and its numerical approximation is not a trivial task. Hence, MI procedure appears to be an interesting alternative to shed some light on the behavior of the solutions of the Schrödinger equation given by Eq. (2.2). We must point out that this equation has been studied in [11]. We pursue this investigation by extending the range of the parameter values.

2.2.2 Dispersion Relations

In the following, we investigate the stability of the Schrödinger equation given by Eq. (2.2). To this end, we first consider the following plane wave:

\[ \phi(x,t) = Me^{j(kx-\omega t)}, \ x \in \mathbb{R}, \ t > 0, \]  

(2.3)

where,

- \( M \) is a positive real number representing the amplitude of the plane wave \( \phi \).
- \( k \) is a real number representing the wave number.
- \( \omega \) is a real number representing the angular frequency.

Wave numbers (resp. angular frequencies) with opposite signs indicate the opposite directions of the waves.
The next result, that is easy to verify, states the condition on the parameters of the wave $\phi$ given by Eq. (2.3), so that $\phi$ satisfies the Schrödinger equation given by Eq. (2.2).

**Lemma 1.** The plane wave $f$ given by Eq. (2.3) is a solution of the Schrödinger equation given by Eq. (2.2), if the following dispersion relation is satisfied:

$$\omega = -kM^2 + k^2\beta_2 + k^3\beta_3 - k^4\beta_4 - M^2\gamma_1 - M^3\gamma_2.$$  \hfill (2.4)

Next, we examine the stability behavior of the plane wave $f$ given by Eq. (2.3). Therefore, we perturb the wave $f$ as follows:

$$y(x,t) = (M + \varepsilon U(x,t))e^{i(kx - \omega t)}, \quad x \in \mathbb{R}, \quad t > 0,$$  \hfill (2.5)

where $\varepsilon$ is a very small positive number. $U$ is a superposition of ingoing and outgoing waves, that is:

$$U(x,t) = A_1e^{i(Kx - \Omega t)} + \bar{A}_2e^{-i(Kx - \Omega t)},$$  \hfill (2.6)

where,

- $K$ is a real number representing the wave number of $U$.
- $\Omega$ is a complex number representing the angular frequency of $U$.
- $A_1, A_2$ are complex numbers representing the amplitude of $U$.
- $\bar{A}_j; j = 1, 2$ denote the conjugate of $A_j$.

**Definition 1.** A plane wave $\Phi$ is said to be stable if there exists a positive constant $C$ such that

$$|\Phi(x,t)| \leq C, \quad \forall x \in \mathbb{R}, \forall t > 0,$$  \hfill (2.7)

Otherwise, the plane wave $\Phi$ is said to be unstable.

**Lemma 2.** The wave $U$ given by Eq. (2.6) is stable iff $\text{Im}(\Omega) \leq 0$.

**Proof.** It follows from Eq. (2.6) that

$$U(x,t) = [A_1e^{i(Kx - \text{Re}(\Omega)t)} + \bar{A}_2e^{-i(Kx - \text{Re}(\Omega)t)}]e^{\text{Im}(\Omega)t}.$$  \hfill (2.8)

If $\text{Im}(\Omega) = 0$, then

$$U(x,t) = A_1e^{i(Kx - \text{Re}(\Omega)t)} + \bar{A}_2e^{-i(Kx - \text{Re}(\Omega)t)}.$$  \hfill (2.9)

Therefore,

$$|U(x,t)| \leq |A_1| + |A_2|, \quad \forall x \in \mathbb{R}, \forall t > 0.$$  \hfill (2.10)

Moreover, if $\text{Im}(\Omega) < 0$, then

$$0 < e^{\text{Im}(\Omega)t} < 1, \quad \forall t > 0.$$  \hfill (2.11)

Hence,
\[ |U(x,t)| \leq |A_1| + |A_2|, \quad \forall x \in \mathbb{R}, \forall t > 0. \quad (2.12) \]

Consequently, if \( \text{Im}(\Omega) \leq 0 \), then the wave \( U \) given by Eq. (2.6) is stable.

On the other hand, if \( \text{Im}(\Omega) > 0 \), then

\[
\lim_{t \to \infty} e^{\text{Im}(\Omega)t} = \infty.
\]

Therefore,

\[
|U(x,t)| \to \infty, \quad \forall t > 0. \quad (2.13)
\]

Hence, if \( \text{Im}(\Omega) > 0 \), then the wave \( U \) given by Eq. (2.6) is unstable.

\[\square\]

**Corollary 1.** The perturbed wave \( \psi \) given by Eq. (2.5) is stable iff

\[
\text{Im}(\Omega) \leq 0.
\]

Next, we derive the dispersion relation so that the perturbed wave \( \psi \) given by Eq. (2.5) is a solution of the Schrödinger equation given by Eq. (2.2). Note that the obtained dispersion relation is a first order approximation with respect to \( \varepsilon \).

**Theorem 1.** The perturbed wave \( \psi \) given by Eq. (2.5) is a solution of the Schrödinger equation given by Eq. (2.2) if, in the first order approximation with respect to \( \varepsilon \), the following dispersion relation is satisfied:

\[
\Omega^2 + a\Omega + b = 0, \quad (2.14)
\]

where \( a \) and \( b \) are two complex numbers given by:

\[
a = v_1 - \bar{u}_2, \\
b = \bar{a}_1 v_2 - \bar{a}_2 v_1, \quad (2.15)
\]

and

\[
u_1 = M^2(k - K\alpha_1 + \gamma_1 + 2M^2\gamma_2) \\
u_2 = kM^2 - KM^2 - KM^2\alpha_1 + (2k - K)K\beta_2 + 3k^2K\beta_3 - 3kK^2\beta_3 + K^3\beta_3 - 4k^3K\beta_4 + 6k^2K^2\beta_4 - 4kK^3\beta_4 + K^4\beta_4 + M^2\gamma_1 + 2M^4\gamma_2 \\
\]

\[
u_1 = kM^2 + KM^2 + KM^2\alpha_1 - K(2k + K)\beta_2 - 3k^2K\beta_3 - 3kK^2\beta_3 - K^3\beta_3 + 4k^3K\beta_4 + 6k^2K^2\beta_4 + 4kK^3\beta_4 + K^4\beta_4 + M^2\gamma_1 + 2M^4\gamma_2 \\
\]

\[
u_2 = M^2(k + K\alpha_1 + \gamma_1 + 2M^2\gamma_2). \quad (2.16)
\]
Proof. We substitute the perturbed wave $\psi$ given by Eq. (2.5) into the Schrödinger equation given by Eq. (2.2), and solve it step by step to obtain the dispersion relation. First we obtain an expansion with respect to $\varepsilon$ of the form:

$$\psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \varepsilon^3 \psi_3 + \varepsilon^4 \psi_4 + \varepsilon^5 \psi_5 = 0.$$  \hspace{1cm} (2.17)

Note that, $\psi_0 = 0$ by Lemma 1. Furthermore, we neglect the higher orders of $\varepsilon$, and obtain that:

$$\psi_1 = 0,$$  \hspace{1cm} (2.18)

where,

$$\psi_1 = iM^2(k + \gamma_1 + 2M^2\gamma_2)U + iM^2(k + \gamma_1 + 2M^2\gamma_2)\bar{U} - U_t$$

$$+ (M^2 + M^2\alpha_1 - 2k\beta_2 - 3k^2\beta_3 + 4k^3\beta_4)U_x + M^2\alpha_1 \bar{U}_x$$

$$+ i(\beta_2 + 3k(\beta_3 - 2k\beta_4))U_{xx} + (\beta_3 - 4k\beta_4)U_{xxx} + i\beta_4 U_{xxxx}. \hspace{1cm} (2.19)$$

Note that, we have not included the expressions for $\psi_2, \psi_3, \psi_4,$ and $\psi_5$ because of their size, and more importantly, the corresponding terms are neglected. However, the interested reader can find these expressions in Appendix A.

Next, we substitute the expression of the perturbation $U$ given by Eq. (2.6) into Eq. (2.19). Hence we obtain that:

$$e^{-i(Kx - \bar{\Omega}t)}u + e^{i(Kx - \Omega t)}v = 0,$$  \hspace{1cm} (2.20)

where,

$$u = u_1\tilde{A}_1 + (u_2 - \bar{\Omega})\tilde{A}_2,$$

$$v = (v_1 + \Omega)A_1 + v_2A_2,$$  \hspace{1cm} (2.21)

and $u_1, u_2, v_1, v_2$ are given by Eq. (2.16).

By the properties of complex numbers:

$$u = 0 \Rightarrow \bar{u} = 0, \quad \text{and} \quad v = 0 \Rightarrow \bar{v} = 0.$$

Hence, we consider the system of equations:

$$\begin{cases} \bar{u} = 0, \\ v = 0. \end{cases} \hspace{1cm} (2.22)$$

That is:

$$\begin{cases} \bar{u}_1A_1 + (\bar{u}_2 - \Omega)A_2 = 0, \\ (v_1 + \Omega)A_1 + v_2A_2 = 0. \end{cases} \hspace{1cm} (2.23)$$
The system given by Eq. (2.23) can be expressed in matrix form as follows:

\[
\begin{pmatrix}
\bar{u}_1 & \bar{u}_2 - \Omega \\
v_1 + \Omega & v_2
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

(2.24)

Nontrivial solutions to system Eq. (2.24) exist iff the dispersion relation given by Eq. (2.14) holds.

**Remark 2.** Observe that the MI procedure transformed the study of the Schrödinger equation given by Eq. (2.2) to the study of the roots of a polynomial of the degree 2, given by Eq. (2.14).

**Lemma 3.** Let \( a \) and \( b \) be the complex numbers given by Eq. (2.15). Then, the roots \( \Omega_+ \) and \( \Omega_- \) of the polynomial given by Eq. (2.14) satisfy:

- if \( a^2 - 4b \geq 0 \), then
  \[
  \Omega_\pm = -a \pm \sqrt{a^2 - 4b}.
  \]
- if \( a^2 - 4b < 0 \), then
  \[
  \Omega_\pm = -a \pm i|a^2 - 4b|^{1/2}.
  \]

(2.25)

**Remark 3.** The roots \( \Omega_+ \) and \( \Omega_- \) depend upon the values of the parameters in the coefficients of the dispersion relation. By Corollary 1, the signs of \( \text{Im}(\Omega_+) \) and \( \text{Im}(\Omega_-) \) determine the stability of the solution of the Schrödinger equation given by Eq. (2.2).

### 2.2.3 Numerical study

In this section, we investigate numerically the sensitivity of the stability of the perturbed wave \( \psi \) given by Eq. (2.5) to the values of the parameters in the Schrödinger equation given by Eq. (2.2). To this end, we study numerically the dependence of the imaginary parts of roots of the dispersion relation given by Eq. (2.14). More specifically, we analyze numerically the dependence of \( \text{Im}(\Omega_+) \) and \( \text{Im}(\Omega_-) \) given by Eq. (2.25) with respect to the parameter values \( a \) and \( b \). Note that the regions of the instability are the regions where \( \text{Im}(\Omega_+) > 0 \) and \( \text{Im}(\Omega_-) > 0 \). These regions are commonly called MI gain or MI gain spectrum.

We have performed numerous experiments to analyze the effects of the parameters \( \beta_j, \gamma_j, \alpha_j \) on the stability of the perturbed wave \( \psi \) given by Eq. (2.5). For illustration purposes, we report here the obtained results of three experiments.

**Experiment 1.** In this experiment, we analyze the dependence of \( \text{Im}(\Omega_+) \) given by Eq. (2.25) to the the wave number \( K \) and the dispersion parameters \( \beta_j \). The results are reported
in Figure 1. These results indicate the following:

- The normal dispersion (positive group velocity dispersion coefficient) as well as anomalous dispersion (negative group velocity dispersion coefficient) has the same effect on the MI gain spectra.

- Figures 1(a), 1(b), and 1(c) indicate the sensitivity with respect to $\beta_4$. The MI gain can be observed in the region where $\beta_4 < 0$ in Figures 1(a)-1(b). It takes the form of two symmetric side lobes that are connected to each other for $k = -10$ in Figure 1(a). The reduction in the magnitude of $k$ brings about a gradual change in the shape of the lobes. The two lobes are separated for $k = -1$ in Figure 1(b).

- In Figure 1(c), the gain is visible for $\beta_4 > 0$. The variation in the values of $\gamma_1$ and $\gamma_2$ has caused this variation in the pattern of the gain.

- Figure 1(d) depicts the dependence with respect to $\beta_3$, the third order dispersion. The gain is observed for all values of $\beta_3$. It is in the form of two symmetric parallel bands.

- Figure 1(e) shows MI as a function of $K$ and $\beta_2$, the group velocity dispersion. The gain can be observed in the form of two symmetric side lobes for $\beta_2 > -20$. 
Figure 1: Sensitivity of the solution to the wave number $K$ and the dispersion coefficients $\beta_j$. 

(a) $M = \sqrt{500}, \beta_2 = 25, \beta_3 = 1/60, \gamma_1 = 1, \gamma_2 = 0.48, \alpha_1 = 0.02, k = -10.$

(b) $M = \sqrt{500}, \beta_2 = 25, \beta_3 = 1/60, \gamma_1 = 1, \gamma_2 = 0.48, \alpha_1 = 0.02, k = -1.$

(c) $M = \sqrt{15}, \beta_2 = 25, \beta_3 = 1/60, \gamma_1 = -1, \gamma_2 = -1, \alpha_1 = 0.02, k = -10.$

(d) $M = \sqrt{5}, \beta_2 = -25, \beta_4 = -1/2, \gamma_1 = 1, \gamma_2 = 1, \alpha_1 = 0.02, k = 10.$

(e) $M = \sqrt{5}, \beta_3 = 0.9, \beta_4 = 0.02, \gamma_1 = 1, \gamma_2 = -0.2, \alpha_1 = 0.02, \alpha_3 = 0.02, k = 10$. 


Figure 2: Sensitivity of the solution to the wave number $K$, the self modulation, and the quintic nonlinearity coefficients $g_j$. Case where: $M = \sqrt{5}, \beta_2 = 25, \beta_3 = 1/60, \beta_4 = 1/2, \alpha_1 = 0.02, k = -10$.

**Experiment 2.** In this experiment, we analyze the dependence of $\text{Im}(\Omega_+)$ given by Eq. (2.25) to the wave number $K$, the self modulation, and the quintic nonlinearity parameters $g_j$. The results are reported in Figure 2. These results indicate the following:

- The normal as well as anomalous dispersion regimes have the same effect on the MI gain.

- Figure 2(a) demonstrates the sensitivity with respect to $g_1$, the self phase modulation coefficient. The gain takes the form of two symmetric side bands for positive as well as negative values of $g_1$.

- Figure 2(b) indicates the sensitivity with respect to $g_2$, the quintic nonlinearity parameter. The gain is visible in the shape of two symmetric side bands for mostly negative values of $g_2$. 

(a) $g_2 = -1$ 

(b) $g_1 = 1$
Experiment 3. In this experiment, we analyze the dependence of $\text{Im}(\Omega_+)$ given by Eq. (2.25) to the the wave number $K$ and the self steepening parameter $\alpha_1$. The results are reported in Figure 3. These results indicate the following:

- The normal as well as anomalous dispersion regimes have the same effect on the MI gain spectra.
- The gain can be viewed in the form of two side bands for positive and negative values of $\alpha_1$.

2.2.4 Summary and Conclusion

We have derived the dispersion relation of the Schrödinger equation given by Eq. (2.2), and numerically analyzed its stability. The numerical study reveals that:

- The MI gain spectra exhibits a limited variety of patterns of lobes, bumps, and bands.
- The gain patterns are symmetric with respect to the sign of the wave number $K$. This indicates that the waves traveling in the opposite directions showed the same MI gain.
- The self phase modulation parameter $\gamma_1$ and the quintic nonlinearity parameter $\gamma_2$ together affect the magnitude of the MI gain.
2.3 Analysis for the generalized nonlinear Schrödinger equation with higher order nonlinear and dispersion terms

2.3.1 Problem Statement

This section is devoted to the stability analysis of the following Schrödinger equation:

\[
A_t = i(\beta_2 A_{xx} + \gamma_1 |A|^2 A) + \beta_3 A_{xxx} + i\beta_4 A_{xxxx} + \alpha_1 (|A|^2 A)_x \\
+ \alpha_2 (|A|^2)_x + i\gamma_2 |A|^4 A + \alpha_3 (|A|^4 A)_x + \alpha_4 A(|A|^4)_x, \\
x \in \mathbb{R}, \quad t > 0,
\]  

(2.26)

where,

- \(A(x,t)\) is a function representing a slowly varying envelope.
- The subscript \(x\) (resp. \(t\)) denotes the spatial (resp. temporal) partial derivative.
- \(\beta_j\); \(j = 2, 3, 4\) are real numbers representing the dispersion coefficients.
- \(\gamma_1\) is a real number representing the self phase modulation parameter.
- \(\gamma_2\) is a real number representing the quintic nonlinearity parameter.
- \(\alpha_1\) is a real number representing the self steepening parameter.
- \(\alpha_2\) is a real number representing the Raman scattering parameter.
- \(\alpha_3, \alpha_4\) are real numbers representing the dispersive nonlinearity parameters.

The extra terms in the Schrödinger equation given by Eq. (2.26) as compared to the Schrödinger equation given by Eq. (2.2) are \(\alpha_2, \alpha_3, \) and \(\alpha_4\). \(\alpha_2\) represents the self-frequency shift due to the stimulated Raman scattering. Stimulated Raman scattering greatly affects ultrashort pulses. It plays a vital role in communications [29], [30]. \(\alpha_3\) and \(\alpha_4\) are dispersive nonlinearity coefficients.

Remark 1. Note that the Schrödinger equation given by Eq. (2.26) is strongly nonlinear. It involves a nonlinearity of the fifth order. It also incurs the fourth order spatial derivative. This equation cannot be solved analytically, and its numerical approximation is not an easy task. Thus, MI procedure appears to be a more feasible alternative to demonstrate the behavior of the solutions of the Schrödinger equation given by Eq. (2.26). We should mention, that the Schrödinger equation given by Eq. (2.26) has been studied in [9]. We continue this investigation by extending the range of the parameter values.

2.3.2 Dispersion Relations

In the following section, we investigate the stability of the Schrödinger equation given by Eq. (2.26). To this end, we apply the approach used in section (2.2). Hence, we first consider the following plane wave:

\[
\phi(x,t) = Me^{i(kx-\omega t)}, \quad x \in \mathbb{R}, \quad t > 0,
\]  

(2.27)

where,

- \(M\) is a positive real number representing the amplitude of the plane wave \(\phi\).
- \(k\) is a real number representing the wave number.
- \(\omega\) is a real number representing the angular frequency.
The sign of the wave number (resp. the angular frequency) determines the direction of the wave.

The next lemma states the condition on the parameters of the plane wave $\phi$ given by Eq. (2.27), such that $\phi$ satisfies the Schrödinger equation given by Eq. (2.26). The result is easy to verify.

**Lemma 4.** The plane wave $\phi$ given by Eq. (2.27) is a solution of the Schrödinger equation given by Eq. (2.26) if the following dispersion relation is satisfied:

$$\omega = -kM^2 - kM^4 \alpha_3 + k^2 \beta_2 + k^3 \beta_3 - k^4 \beta_4 - M^2 \gamma_1 - M^4 \gamma_2. \quad (2.28)$$

Next, we examine the stability behavior of the plane wave $\phi$ given by Eq. (2.3). Therefore, we perturb $\phi$ as follows:

$$\psi(x,t) = (M + \epsilon U(x,t))e^{i(kx - \omega t)}, \quad x \in \mathbb{R}, \quad t > 0, \quad (2.29)$$

where $\epsilon$ is a very small positive number. $U$ is a superposition of ingoing and outgoing waves, that is:

$$U(x,t) = A_1 e^{i(Kx - \Omega t)} + \bar{A}_2 e^{-i(Kx - \bar{\Omega} t)}, \quad (2.30)$$

where,

- $K$ is a real number representing the wave number.
- $\Omega$ is a complex number representing the angular frequency.
- $A_1, A_2$ are complex numbers representing the amplitude of $U$.
- $\bar{A}_j; j = 1, 2$ denote the conjugate of $A_j$.

Next, we derive the dispersion relation so that the perturbed wave $\psi$ given by Eq. (2.29) is a solution of the Schrödinger equation given by Eq. (2.26). Note that the obtained dispersion relation is a first order approximation with respect to $\epsilon$.

**Theorem 2.** The perturbed wave $\psi$ given by Eq. (2.29) is a solution of the Schrödinger equation given by Eq. (2.26) if, in the first order approximation with respect to $\epsilon$, the following dispersion relation is satisfied:

$$\Omega^2 + a\Omega + b = 0, \quad (2.31)$$

where $a$ and $b$ are two complex numbers given by:

$$a = v_1 - \bar{u}_2,$$
$$b = \bar{u}_1 v_2 - \bar{u}_2 v_1, \quad (2.32)$$

and
By the properties of complex numbers:

\[ u_1 = M^2(k - K\alpha_1 - K\alpha_2 + 2kM^2\alpha_3 - 2KM^2\alpha_3 - 2KM^2\alpha_4 + \gamma_1 + 2M^2\gamma_2), \]

\[ u_2 = kM^2 - KM^2\alpha_1 - KM^2\alpha_2 + 2kM^4\alpha_3 - 3KM^4\alpha_3 - 2KM^4\alpha_4 + 2kK\beta_2 - K^2\beta_2 + 3k^2K\beta_3 - 3kk^2\beta_3 + K^3\beta_3 - 4k^3K\beta_4 + 6k^2K^2\beta_4 - 4kK^3\beta_4 + K^4\beta_4 + M^2\gamma_1 + 2M^4\gamma_2, \]

\[ v_1 = kM^2 + KM^2\alpha_1 + KM^2\alpha_2 + 2kM^4\alpha_3 + 3KM^4\alpha_3 + 2KM^4\alpha_4 - 2kK\beta_2 - K^2\beta_2 - 3k^2K\beta_3 - 3kk^2\beta_3 - K^3\beta_3 + 4k^3K\beta_4 + 6k^2K^2\beta_4 + 4kK^3\beta_4 + K^4\beta_4 + M^2\gamma_1 + 2M^4\gamma_2, \]

\[ v_2 = M^2(k + K\alpha_1 + K\alpha_2 + 2kM^2\alpha_3 + 2KM^2\alpha_3 + 2KM^2\alpha_4 + \gamma_1 + 2M^2\gamma_2). \]  

(2.33)

**Proof.** We substitute the perturbed wave \( \psi \) given by Eq. (2.29) into the Schrödinger equation given by Eq. (2.26), and solve it step by step to obtain the dispersion relation. First we obtain an expansion with respect to \( \varepsilon \) of the form:

\[ \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \varepsilon^3 \psi_3 + \varepsilon^4 \psi_4 + \varepsilon^5 \psi_5 = 0. \]  

(2.34)

Note that, \( \psi_0 = 0 \) by Lemma 4. Furthermore, we neglect the higher orders of \( \varepsilon \) and obtain that:

\[ \psi_1 = 0, \]  

(2.35)

where,

\[ \psi_1 = iM^2(k + 2kM^2\alpha_3 + \gamma_1 + 2M^2\gamma_2)U + iM^2(k + 2kM^2\alpha_3 + \gamma_1 + 2M^2\gamma_2)\tilde{U} - U_t + (M^2 + M^2\alpha_1 + M^2\alpha_2 + 3M^4\alpha_3 + 2M^4\alpha_4 - 2k\beta_2 - 3k^2\beta_3 + 4k^3\beta_4)U_x + M^2(\alpha_1 + \alpha_2 + 2M^2\alpha_3 + 2M^2\alpha_4)\tilde{U}_x + i(\beta_2 + 3k\beta_3 - 6k^2\beta_4)U_{xx} + \beta_3U_{xxx} - 4k\beta_4U_{xxx} + i\beta_4U_{xxxx}. \]  

(2.36)

Note that, we have not included the expressions for \( \psi_2, \psi_3, \psi_4, \) and \( \psi_5 \) because of their size, and more importantly, the corresponding terms are neglected. However, the interested reader can find these expressions in Appendix B.

Next, we substitute the expression of the perturbation \( U \) given by Eq. (2.30) into Eq. (2.36). Hence we obtain:

\[ e^{-i(Kx - \Omega t)}u + e^{i(Kx - \Omega t)}v = 0, \]  

(2.37)

where,

\[ u = u_1\tilde{A}_1 + (u_2 - \bar{\Omega})\tilde{A}_2, \]

\[ v = (v_1 + \Omega)A_1 + v_2A_2, \]  

(2.38)

and \( u_1, u_2, v_1, v_2 \) are given by Eq. (2.33).

By the properties of complex numbers:
\[ u = 0 \Rightarrow \bar{u} = 0, \quad \text{and} \quad v = 0 \Rightarrow \bar{v} = 0. \]

Hence, we consider the system of equations:

\[
\begin{align*}
\bar{u} &= 0, \\
v &= 0.
\end{align*}
\] (2.39)

That is:

\[
\begin{align*}
\bar{u}_1 A_1 + (\bar{u}_2 - \Omega) A_2 &= 0, \\
(v_1 + \Omega) A_1 + v_2 A_2 &= 0.
\end{align*}
\] (2.40)

The system given by Eq. (2.40) can be expressed in a matrix form as follows:

\[
\begin{pmatrix}
\bar{u}_1 & \bar{u}_2 - \Omega \\
v_1 + \Omega & v_2
\end{pmatrix}
\begin{pmatrix}
A_1 \\
A_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\] (2.41)

Nontrivial solutions of system (2.41) exist, iff the dispersion relation given by Eq. (2.31) holds.

\[ \square \]

**Remark 4.** Observe that the MI procedure transformed the study of the Schrödinger equation given by Eq. (2.26) to the study of the roots of a polynomial of the degree 2, given by Eq. (2.31).

**Remark 5.** The roots \( \Omega_+ \) and \( \Omega_- \) depend upon the values of the parameters in the coefficients of the dispersion relation. By Corollary 1, the signs of \( \text{Im}(\Omega_+) \) and \( \text{Im}(\Omega_-) \) determine the stability of the solution of the Schrödinger equation given by Eq. (2.26).

### 2.3.3 Numerical study

In this section, we investigate numerically the sensitivity of the stability of the perturbed wave \( \psi \) to the values of the parameters in the waves and the Schrödinger equation. To this end, we study numerically the dependence of the imaginary parts of roots of the dispersion relation given by Eq. (2.31). More specifically, we analyze numerically the dependence of \( \text{Im}(\Omega_+) \) and \( \text{Im}(\Omega_-) \) given by Eq. (2.25).

We have performed numerous experiments to analyze the effects of the parameters \( \beta_j, \gamma_j, \alpha_j \) on the stability of the perturbed wave \( \psi \) given by Eq. (2.29). For illustration purposes, we report here the obtained results of three experiments.
Experiment 1. In this experiment, we analyze the dependence of $Im(\Omega_+)$ given by Eq. (2.25) to the wave number $K$ and the dispersion parameter $\beta_j$. The results are reported in Figure 4. These results indicate the following:

- Figure 4(a)-4(b) indicate the dependence with respect to $\beta_3$, the third order dispersion parameter. The gain is visible in the form of two symmetric side lobes similar to the lobes obtained in the previous section. The peak of the gain spectrum is 400. Change in the sign of $k$ exhibits different patterns of the gain.

- Figure 4(a) shows the gain for $\beta_3 < 0$, and no gain for $\beta_3 > 0$.

- Figure 4(b) shows the gain for $\beta_3 > 0$, and no gain for $\beta_3 < 0$.

- Figure 4(c) depicts the dependence with respect to $\beta_4$, the fourth order dispersion parameter. Observe that there is a significant increase in the peak of the MI gain to 20,000, as the value of $\alpha_3$ increases.
Experiment 2. In this experiment, we analyze the dependence of $\text{Im}(\Omega_{\pm})$ given by Eq. (2.25) to the wave number $K$ and the dispersive parameters $\alpha_j$. The results are reported in Figure 5. These results indicate the following:

- Figure 5(a)-5(c) illustrate the gain in the form of two symmetric bumps.
- Figure 5(a) depicts the dependence with respect to the self steepening parameter $\alpha_1$. The gain can be observed for positive as well as negative values of $\alpha_1$.
- Figure 5(b) shows the sensitivity with respect to the Raman scattering term $\alpha_2$. The gain is visible for all values of $\alpha_2$. 

Figure 5: Sensitivity of the solution to the wave number $K$ and the dispersive coefficients $\alpha_j$.
• Figure 5(c) indicates the sensitivity with respect to the dispersive nonlinearity coefficient $\alpha_4$. The gain is in the shape of two parallel bands for all values of $\alpha_4$.

**Experiment 3.** In this experiment, we analyze the dependence of $Im(\Omega_+)$ given by Eq. (2.25) to the wave number $K$, the self modulation, and the quintic parameters $\gamma_j$. The results are reported in Figure 6. These results indicate the following:

• Figures 6(a)-6(b) exhibit the gain in the shape of two symmetric side bands.

• Figure 6(a) demonstrates the sensitivity with respect to $\gamma_1$, the cubic nonlinearity term. It shows widening gain as negative value of $\gamma_1$ increases.

• Figure 6(b) indicates the sensitivity with respect to $\gamma_2$, the quintic nonlinearity term. It shows gain the values of $\gamma_2 < 0$.

![Figure 6](image_url)

**Figure 6:** Sensitivity of the solution to the wave number $K$ and the dispersive coefficients $\gamma_j$.

**Case where:**

$M = \sqrt{5}, k = 5, \beta_2 = -1/2, \beta_3 = 1/3, \beta_4 = 1/4, \alpha_1 = -0.024, \alpha_2 = 0.037, \alpha_3 = -0.025, \alpha_4 = 0.031$. 
2.3.4 Summary and Conclusion

We have derived the dispersion relation of the Schrödinger equation given by Eq. (2.26), and analyzed numerically its stability. The numerical study reveals that:

• The MI gain spectra exhibits a limited variety of symmetric patterns of lobes, bumps, and bands.

• The self phase modulation parameter $g_1$ and quintic nonlinearity parameter $g_2$ together affect the magnitude of the MI gain.

• Higher order dispersive nonlinearity parameter $a_3$ has a major impact on the magnitude of the MI gain.
3.1 Introduction

In recent years the systems of coupled nonlinear Schrödinger equations have attracted a great deal of attention. Interactions of two or more wave packets of different carrier frequencies are governed by the coupled nonlinear Schrödinger equations. Examples include nonlinear light propagation in a birefringent optical fiber or a wavelength-division-multiplexed system [12], [31], [32] in optics. In nonlinear optics, the coupled nonlinear Schrödinger equations model pulse motions in multimode fibers, birefringent fibers, and directional couplers with adding or removing terms in the system given by (3.1) [35]. In hydrodynamics, the coupled nonlinear Schrödinger equations model the evolution of two surface wave packets in deep water [33], the evolution of wave packets in three layer stratified shear flow, the evolution of Rossby wave packet [34], etc.

A general form of the coupled nonlinear Schrödinger equations used in hydrodynamics can be written as follows:

\[
\begin{align*}
    iA_t + d_1 A_{xx} + x_1 |A|^2 A + s_1 |B|^2 A &= 0, \\
    iB_t + d_2 B_{xx} + x_2 |B|^2 B + s_2 |A|^2 B &= 0,
\end{align*}
\]

where \( A \) and \( B \) are the functions of slowly varying envelopes of waves. \( x \) and \( t \) are the space and time variables respectively. \( d_j \) are the dispersion coefficients, \( x_j \) are the self phase modulation parameters, \( s_j \) are the cross phase modulation parameters of the waves \( A \) and \( B \). All of these parameters are real numbers [35].

The coefficients of the nonlinear Schrödinger equations determine the stability region of the propagating wave. When two waves interact nonlinearly with each other, the nonlinearity provides a coupling between the waves. The coupling may lead to a change in the stability conditions [36].

In this study, we will investigate the stability of the system of coupled Schrödinger equations associated with polarization of light beams.

3.2 Analysis for the coupled Schrödinger equations

3.2.1 Problem Statement

We consider the following system of Schrödinger equations [14]:

\[
\begin{align*}
    iA_t + \delta A_{xx} - \gamma A + |A|^2 A + \sigma |B|^2 A + \lambda B^2 A &= 0, \\
    iB_t + \delta B_{xx} + \gamma B + |B|^2 B + \sigma |A|^2 B + \lambda A^2 B &= 0,
\end{align*}
\]

where,

- \( A(x,t) \) and \( B(x,t) \) are the functions representing slowly varying envelopes.
- The subscript \( x \) (resp. \( t \)) denotes the spatial (resp. temporal) partial derivative.

21
• \( i \) is an imaginary number. \( \gamma \) is a real constant.
• \( \sigma \) is a real number representing the cross phase modulation parameter.
• \( \lambda \) is a real number representing the four wave mixing parameter.

The cross phase modulation parameter \( \sigma \) affects the phase of another wave [12], [27]. The four wave mixing parameter \( \lambda \) incurs the interaction of three waves with different frequencies giving rise to the fourth wave [37]. The cross phase modulation and the four wave mixing are used in several applications in optics [12].

**Remark 1.** Note that the system of coupled Schrödinger equations given by Eq. (3.2) involve nonlinearities of the third order and spatial derivatives of the second order. It also includes two couplings induced by the cross phase modulation as well as the four wave mixing. The system cannot be solved analytically, and its numerical approximation is not a simple procedure. Hence, applying MI technique appears to be a feasible approach. We must mention, that the system given by Eq. (3.2) has been studied in [14]. We take this investigation further by extending the range of the parameter values.

### 3.2.2 Dispersion Relations

In the following section, we investigate the stability of the coupled Schrödinger equations given by Eq. (3.2). To this end, we apply the approach used in Chapter 2. However, unlike the scalar Schrödinger equation, we need to consider the following two plane waves:

\[
\begin{aligned}
\phi_1(x,t) &= Me^{i(k_1x-\omega_1t)}, \\
\phi_2(x,t) &= Pe^{i(k_2x-\omega_2t)},
\end{aligned}
\]  

(3.3)

where,

- \( M \) (resp. \( P \)) is a positive real number representing the amplitude of the plane wave \( \phi_1 \) (resp. \( \phi_2 \)).
- \( k_1, k_2 \) are real numbers representing the wave numbers.
- \( \omega_1, \omega_2 \) are real numbers representing the angular frequency.

Note that the waves traveling in the opposite directions are denoted by the opposite signs of the wave numbers (resp. angular frequencies).

The next lemma states the condition on the parameters of the plane waves \( \phi_1 \) and \( \phi_2 \) given by Eq. (3.3), such that they satisfy the system of coupled Schrödinger equations given by Eq. (3.2).

**Lemma 5.** The plane waves \( \phi_1 \) and \( \phi_2 \) are the solutions of the system of coupled Schrödinger equations given by Eq. (3.2) iff \( \omega_1 = \omega_2 = \omega \), and \( k_1 = k_2 = k \), where

\[
\omega = \delta k^2 - (\gamma + P^2 + M^2(\lambda + \sigma)),
\]

and

\[
P = \left( \frac{2\gamma + M^2(\lambda + \sigma - 1)}{\lambda + \sigma - 1} \right)^{1/2}.  
\]  

(3.4)
Proof. We substitute the plane waves \( \phi_1 \) and \( \phi_2 \) given by Eq. (3.3) into the system of the coupled Schrödinger equations given by Eq. (3.2), and solve step by step to obtain the dispersion relation. First we obtain the following equations:

\[
\begin{align*}
& e^{f_1} (P^2 \lambda M) + e^{f_2} (-\gamma + M^2 + P^2 \sigma - \delta k_1^2 + \omega_1) = 0, \\
& e^{f_3} (M^2 \lambda P) + e^{f_4} (\gamma + P^2 + M^2 \sigma - \delta k_2^2 + \omega_2) = 0. 
\end{align*}
\] (3.5)

where,

\[
\begin{align*}
& f_1 = x(2k_2 - k_1) + t(\omega_1 - 2\omega_2), \\
& f_2 = k_1 x - \omega_1 t, \\
& f_3 = x(2k_1 - k_2) + t(\omega_2 - 2\omega_1), \\
& f_4 = k_2 x - \omega_2 t. 
\end{align*}
\] (3.6)

The equations given by Eq. (3.5) hold iff \( f_1 = f_2 \) and \( f_3 = f_4 \). We substitute the values of \( f_1 \) and \( f_3 \) given by Eq. (3.6). We obtain that:

\[
\begin{align*}
& \omega_1 = \omega_2 = \omega, \\
& k_1 = k_2 = k. 
\end{align*}
\] (3.7)

Then, we substitute the values of \( \omega_1 \), \( \omega_2 \) and \( k_1 \), \( k_2 \) given by Eq. (3.7) in the equations given by Eq. (3.5), and obtain the dispersion relation:

\[
\omega = \delta k^2 - (\gamma + P^2 + M^2(\lambda + \sigma)), \\
P = \left( \frac{2\gamma + M^2(\lambda + \sigma - 1)}{\lambda + \sigma - 1} \right)^{1/2}. 
\]

Next, we examine the stability behavior of the plane waves \( \phi_1 \) and \( \phi_2 \) given by Eq. (3.3). Therefore, we perturb \( \phi_1 \) and \( \phi_2 \) as follows:

\[
\begin{align*}
\hat{\psi}(x,t) &= (M + \epsilon U(x,t)) e^{i(kx - \omega t)}, \\
\hat{\psi}(x,t) &= (P + \epsilon V(x,t)) e^{i(kx - \omega t)}, 
\end{align*}
\] (3.8)

where \( \epsilon \) is a very small positive number. \( \omega \) and \( k \) are defined in Lemma 5. \( U \) and \( V \) are superpositions of ingoing and outgoing waves, that is:

\[
\begin{align*}
U(x,t) &= A_1 e^{i(Kx - \Omega t)} + \bar{A}_2 e^{-i(Kx - \bar{\Omega} t)}, \\
V(x,t) &= B_1 e^{i(Kx - \bar{\Omega} t)} + \bar{B}_2 e^{-i(Kx - \Omega t)}, 
\end{align*}
\] (3.9)

where,

- \( K \) is a real number representing the wave number.
• $\Omega$ is a complex number representing the angular frequency.
• $A_j, B_j; j = 1, 2$ are complex numbers representing the amplitudes of $U$ and $V$.
• $\bar{A}_j, \bar{B}_j; j = 1, 2$ denote the conjugates of $A_j$ and $B_j$.

Next, we derive the dispersion relation so that the perturbed waves $\psi$, and $\hat{\psi}$ given by Eq. (3.8) are solutions of the system of coupled Schrödinger equations given by Eq. (3.2). Note that the obtained dispersion relation is a first order approximation with respect to $\varepsilon$.

**Theorem 3.** The perturbed waves $\psi$ and $\hat{\psi}$ given by Eq. (3.8) are the solutions of the system of coupled Schrödinger equations given by Eq. (3.2) if, in the first order approximation with respect to $\varepsilon$, the following dispersion relation is satisfied:

$$\Omega^4 + a\Omega^2 + b = 0,$$

(3.10)

where $a$ and $b$ are two complex numbers given by:

\[
a = c_1^2 - 2c_2^2 + 2c_3^2 + c_4^2 - d_1^2 - d_2^2,
\]

\[
b = c_2^4 - 2c_2^2 c_3^2 + c_3^4 - 2c_1c_2c_4 - 2c_1c_3^2 c_4 + c_1^2 c_4^2 + 4c_2 c_3 c_4 d_1 - c_4^2 d_1^2 + 4c_1 c_2 c_3 d_2 - 2c_2^2 d_1 d_2 - 2c_3^2 d_1 d_2 - c_1^2 d_2^2 + d_1^2 d_2^2, \tag{3.11}
\]

and

\[
d_1 = -2\gamma - K^2 \delta + P^2 (-1 + \sigma) - M^2 (-2 + \lambda + \sigma) + 2K \delta k,
\]

\[
d_2 = P^2 - M^2 \lambda + 2K \delta k,
\]

\[
c_1 = M^2 + P^2 \lambda,
\]

\[
c_2 = 2MP \lambda + MP\sigma,
\]

\[
c_3 = MP\sigma,
\]

\[
c_4 = P^2 + M^2 \lambda. \tag{3.12}
\]

**Proof.** We substitute the perturbed waves $\psi$ and $\hat{\psi}$ given by Eq. (3.8) into the system of coupled Schrödinger equations given by Eq. (3.2) and follow these steps to obtain the dispersion relation. First we obtain the expansions with respect to $\varepsilon$ of the form:

\[
\begin{align*}
\psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \varepsilon^3 \psi_3 &= 0, \\
\hat{\psi}_0 + \varepsilon \hat{\psi}_1 + \varepsilon^2 \hat{\psi}_2 + \varepsilon^3 \hat{\psi}_3 &= 0.
\end{align*}
\]

(3.13)

Note that, $\psi_0 = 0$ and $\hat{\psi}_0 = 0$ by Lemma 5. Furthermore, we neglect the higher orders of $\varepsilon$, and obtain that:

\[
\begin{align*}
\psi_1 &= 0, \\
\hat{\psi}_1 &= 0.
\end{align*}
\]

(3.14)

where,
\begin{align}
\psi_1 &= MP(2\lambda + \sigma)V - (2\gamma + P^2(1 - \sigma) + M^2(-2 + \lambda + \sigma))U + (M^2 + P^2\lambda)\tilde{U} + \\
&\quad MP\sigma\tilde{V} + iU + 2i\delta kU_x + \delta U_{xx},
\end{align}

(3.15)

\begin{align}
\hat{\psi}_1 &= (P^2 - M^2\lambda)V + MP(2\lambda + \sigma)U + MP\sigma\tilde{U} + (P^2 + M^2\lambda)\tilde{V} + iV + 2i\delta kV_x + \\
&\quad \delta V_{xx}.
\end{align}

Note that we have not included the expressions for \( \psi_2, \psi_3, \) and \( \hat{\psi}_2, \hat{\psi}_3 \) because of their size, and more importantly, the corresponding terms are neglected. However, the interested reader can find them in Appendix C.

Next, we substitute the expressions of the perturbations \( U \) and \( V \) given by Eq. (3.9) into Eq. (3.15). Hence we obtain:

\begin{align}
\left\{ \begin{array}{l}
\ e^{i(\kappa x - \tilde{\kappa}t)}u_1 + e^{i(\kappa x - \kappa t)}u_2 = 0, \\
\ e^{i(\kappa x - \tilde{\kappa}t)}v_1 + e^{i(\kappa x - \kappa t)}v_2 = 0,
\end{array}\right.
\end{align}

(3.16)

where,

\begin{align}
\tilde{u}_1 &= c_1A_1 + (d_1 - \Omega)A_2 + c_3B_1 + c_2B_2, \\
u_2 &= (d_1 + \Omega)A_1 + c_1A_2 + c_2B_1 + c_3B_2, \\
\tilde{v}_1 &= c_3A_1 + c_2A_2 + c_4B_1 + (d_2 - \Omega)B_2, \\
v_2 &= c_2A_1 + c_3A_2 + (d_2 + \Omega)B_1 + c_4B_2,
\end{align}

(3.17)

By the properties of complex numbers:

\begin{align*}
\ u_1 = 0 \Rightarrow \tilde{u}_1 = 0, \quad \text{and} \quad \nu_1 = 0 \Rightarrow \tilde{v}_1 = 0.
\end{align*}

We consider the system of equations:

\begin{align}
\left\{ \begin{array}{l}
\tilde{u}_1 = 0, \\
\ u_2 = 0, \\
\tilde{v}_1 = 0, \\
\nu_2 = 0.
\end{array}\right.
\end{align}

(3.18)

The system of equations given by Eq. (3.18) can be expressed in matrix form as:

\[
\begin{pmatrix}
d_1 - \Omega & c_1 & c_2 & c_3 \\
c_1 & d_1 + \Omega & c_3 & c_2 \\
c_2 & c_3 & d_2 - \Omega & c_4 \\
c_3 & c_2 & c_4 & d_2 + \Omega
\end{pmatrix}
\begin{pmatrix}
A_2 \\
A_1 \\
B_2 \\
B_1
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Nontrivial solutions of this system exist iff the dispersion relation given by Eq. (3.10) holds.
Remark 2. Observe that the MI procedure transformed the study of the system of coupled Schrödinger equations given by Eq. (3.2) to the study of the roots of a polynomial of the degree 4, given by Eq. (3.10). This polynomial can be transformed in a polynomial of degree 2 by substitution, similar to the dispersion relation obtained in Chapter 2 (see Theorem 1 and Theorem 2).

The roots of the polynomial given by Eq. (3.10) are determined explicitly. Their expressions are given as:

\[
\begin{align*}
\Omega_+ &= \left(\frac{-a + \sqrt{a^2 - 4b}}{2}\right)^{1/2}, \\
\Omega_- &= -\left(\frac{-a - \sqrt{a^2 - 4b}}{2}\right)^{1/2}, \\
\hat{\Omega}_+ &= \left(\frac{-a - \sqrt{a^2 - 4b}}{2}\right)^{1/2}, \\
\hat{\Omega}_- &= -\left(\frac{-a + \sqrt{a^2 - 4b}}{2}\right)^{1/2}.
\end{align*}
\]

(3.19)

Note that, the square root must be understood as the principal value of the complex square root. \(\Omega_+, \Omega_-, \hat{\Omega}_+,\) and \(\hat{\Omega}_-\) depend upon the values of the parameters in the coefficients of the dispersion relation. By Corollary 1 stated in section (2.2) of Chapter 2, the signs of \(\text{Im}(\Omega_+), \text{Im}(\Omega_-), \text{Im}(\hat{\Omega}_+),\) and \(\text{Im}(\hat{\Omega}_-)\) determine the stability of the solution of the system of coupled Schrödinger equations given by Eq. (3.2).

3.2.3 Numerical study

In this section, we investigate numerically the sensitivity of the stability of the perturbed waves \(\psi\) and \(\hat{\psi}\) given by Eq. (3.8) to the values of the parameters in the waves and the system of coupled Schrödinger equations given by Eq. (3.2). To this end, we study numerically the dependence of the imaginary parts of roots of the dispersion relation given by Eq. (3.10). More specifically, we numerically analyze the dependence of \(\text{Im}(\Omega_+), \text{Im}(\Omega_-), \text{Im}(\hat{\Omega}_+),\) and \(\text{Im}(\hat{\Omega}_-)\) given by Eq. (3.19) with respect to the parameter values \(a\) and \(b\).

Note that the regions of the instability, that are commonly called MI gain or MI gain spectrum, are the regions where \(\text{Im}(\Omega_+) > 0, \text{Im}(\Omega_-) > 0, \text{Im}(\hat{\Omega}_+) > 0, \text{Im}(\hat{\Omega}_-) > 0.\)

We have performed a number of experiments to analyze the effects of the parameters \(K\) and \(k\) on the stability of the perturbed waves \(\psi\) and \(\hat{\psi}\) given by Eq. (3.8). For illustration purposes, we report the obtained results of one experiment only. In the considered experiment, we analyze the dependence of \(\text{Im}(\Omega_+)\) to the amplitude \(M\) and the wave numbers \(K\) and \(k\).

The results are reported in Figure 7. These results indicate the following:
Figure 7: Sensitivity of the solution to the amplitude $M$ and the wave numbers $K$ and $k$.
Case where: $\delta = -0.5, \lambda = 0.5, \sigma = 1, \gamma = 0.5$.

- Figure 7 depicts the anomalous dispersion regime ($\delta < 0$).
- Figures 7(a)-7(b) exhibit the dependence with respect to $K$ (the wave number of the perturbation). The gain is in the shape of two separate symmetric side bands. $k > 0$ indicates the gain for the values of $K < 0$, and $k < 0$ shows it for the values of $K > 0$.
- Figures 7(c)-7(d) illustrate the sensitivity with respect to $k$ (the wave number of the plane waves). The gain is visible in the form of a peak in the center of the plots, and two side bands. $K > 0$ shows the gain for the values of $k < 0$, and $K < 0$ depicts it for the values of $k > 0$.
- Figure 7(b) is a reflection of Figure 7(a) and Figure 7(d) is a reflection of Figure 7(c).
3.3 Summary and Conclusion

We have derived the dispersion relation of the system of coupled nonlinear Schrödinger equations given by Eq. (3.2) and numerically analyzed its stability. The numerical study reveals that:

- The wave number influences the region of MI gain in the anomalous regime.
- The gain takes the form of symmetric bands.
Chapter 4

System of coupled complex Newell-Segel-Whitehead equations

4.1 Introduction

The Newell-Segel-Whitehead (NSW) equation is named after Newell, Segel, and Whitehead who derived it in 1969. The NSW equation has a wide applicability in mechanical and chemical engineering, ecology, biology and bio-engineering [38]. It almost exactly reproduces the stability criteria found directly from the Boussinesq equations that govern them [17]. It has been used for many years to describe the evolution of convection rolls in high Prandtl number fluids [17]. The NSW equation which appeared historically in the investigation of fluid mechanics is called the nonlinear real NSW equation [39] and can be written as:

\[ A_t = pA_{xx} + aA - bA^q, \]  

(4.1)

where \( A(x,t) \) is a function representing a slowly varying envelope, which can be thought of as the flow velocity of a fluid in an infinitely long pipe with a small diameter. \( x \) and \( t \) denote the normalized distance and time. The subscripts \( x \) and \( t \) denote the spatial and temporal derivatives. \( a, b, p \) are real numbers with \( p > 0 \). \( q \) is a positive integer [39].

The real NSW equation given by Eq. (4.1) is a well known equation to model the evolution of nearly one dimensional nonlinear patterns such as the Rayleigh-Benard convection [40]. It has been applied to a number of problems such as description of the disorder of singularities of the amplitude and phase envelope fields, etc. [18]. The dispersive NSW equation was derived to describe nearly one dimensional traveling wave patterns for solving simple problems including the transverse stability of plane waves and grain boundaries [41].

A general two dimensional equation for a complex order parameter \( A \) in an isotropic medium can be written as:

\[ A_t = A - (iA_t + \sqrt{\varepsilon A_{yy}})^2 - (1 + i\alpha)A - c_1|A|^2A + i\beta A_{xx} + i\gamma A_{yy}, \]  

(4.2)

where \( A(x,y,t) \) is a function representing a slowly varying envelope of a wave. \( x \) and \( y \) denote the location, and \( t \) denotes the time. \( \alpha \) is a nonlinear dispersion parameter. \( \beta \) is a longitudinal dispersion coefficient. \( \gamma \) is a transverse linear parameter. \( \varepsilon \) is a very small positive number. This is a simple model for the propagation of nearly one dimensional waves in a nonlinear medium. However, it admits only unidirectional waves. The waves traveling in both directions can be described by two coupled NSW equations [41]. Coupled NSW equations are applicable to isotropic systems near a subcritical oscillatory instability for which the rotational symmetry in the plane is broken. These equations are the generalizations of the NSW equation for stationary rolls to traveling rolls [16]. The goal in this chapter is to investigate the stability of these equations.

4.2 Analysis for a system of complex coupled Newell-Segel-Whitehead Equations

4.2.1 Problem Statement

We consider the following system of NSW equations [16]:
\[
A_t = \chi A + \gamma (A_x - \frac{i}{\kappa c} A_{yy})^2 + i s A_{xx} - \psi g (A_x - \frac{i}{\kappa c} A_{yy})^2 - \\
\beta |A|^2 A - \delta |A|^4 A - \xi |B|^2 A,
\]
\[
B_t = \chi B + \gamma (B_x - \frac{i}{\kappa c} B_{yy})^2 + i s B_{xx} - \psi g (B_x - \frac{i}{\kappa c} B_{yy})^2 - \\
\beta |B|^2 B - \delta |B|^4 B - \xi |A|^2 B,
\] (4.3)

where,
- \( A(x, y, t) \), \( B(x, y, t) \) are the functions of slowly varying envelopes.
- \( x, y \) are the spatial variables, \( t \) is the temporal variable.
- Subscripts \( x \) and \( y \) denote the spatial derivatives. Subscript \( t \) denotes the temporal derivative.
- \( \psi g \) is a real number representing the group velocity.
- \( s \) is a real number representing the group velocity dispersion.
- \( \kappa c \) is a non-zero real number.
- \( \beta \) is a complex number representing the cubic nonlinearity.
- \( \delta \) is a complex number representing the quintic nonlinearity.
- \( \xi \) is a complex number representing the coupling parameter.
- \( \gamma \) is a complex number.

From now on, we write a complex number \( z \) as: \( z = z_r + iz_i \), where \( z_r \) denotes the real part, and \( z_i \) denotes the imaginary part of \( z \).

**Remark 1.** Observe that the system of coupled NSW equations given by Eq. (4.3) involve nonlinearities of the fifth order and spatial derivatives up to the fourth order. The system is strongly coupled with complex coefficients. Solving it analytically is not possible, and approximating it numerically is not an easy option. Thus, employing the MI procedure to analyze the system appears to be a more viable alternative.

### 4.2.2 Dispersion Relations

In this section, we investigate the stability of the system of coupled NSW equations given by Eq. (4.3). To this end, we apply the approach used in Chapter 2. However, unlike the scalar equation, we need to consider here the following two plane waves:

\[
\begin{align*}
\phi_1(x, y, t) &= M e^{i(k_1 x + k_2 y - \omega_1 t)}, \\
\phi_2(x, y, t) &= P e^{i(k_3 x + k_4 y - \omega_2 t)},
\end{align*}
\] (4.4)

where,
- \( M \) (resp. \( P \)) is a positive real number representing the amplitude of the wave \( \phi_1(x, y, t) \) (resp. \( \phi_2(x, y, t) \)).
- \( k_j; j = 1, \ldots, 4 \) are real numbers representing the wave vectors.
- \( \omega_1, \omega_2 \) are real numbers representing the angular frequencies.

Note that waves traveling in the opposite directions are denoted by the opposite signs of
Therefore, we perturb the plane waves. Next, we examine the stability behavior of the plane waves. Hence, Eq. (4.5) follows readily from the second and the fourth equations in Eq. (4.6).

**Lemma 6.** The plane waves \( \phi_1 \) and \( \phi_2 \) are the solutions of the system of coupled NSW equations given Eq. (4.3) if the following dispersion relations are satisfied:

\[
\begin{align*}
\omega_1 &= \frac{1}{4\kappa^2} \left(4\kappa^2 k_1^2 s + 4\kappa e^2 k_1 v_s + 2\kappa^2 k_2^2 v_g + 4M^2 \kappa^2 \beta_i + 4\kappa^2 k_1^2 \gamma_r + 4\kappa c k_1 k_2^2 \gamma_l + k_2^4 \gamma_r + 4M^4 \kappa^2 \delta_i + 4P^2 \kappa^2 \xi_i \right), \\
\omega_2 &= \frac{1}{4\kappa^2} \left(4\kappa^2 k_3^2 s + 4\kappa^2 k_2 v_s + 2\kappa^2 k_3^2 v_g + 4P^2 \kappa^2 \beta_i + 4\kappa^2 k_2^2 \gamma_r + 4\kappa c k_3 k_4^2 \gamma_l + k_4^4 \gamma_r + 4P^4 \kappa^2 \delta_i + 4M^2 \kappa^2 \xi_i \right).
\end{align*}
\] (4.5)

**Proof.** We substitute the plane waves \( \phi_1 \) and \( \phi_2 \) given by Eq. (4.4) into the system of coupled NSW equations given by Eq. (4.3), and solve step by step to obtain the dispersion relations. First we obtain the following equations:

\[
\begin{align*}
4\chi \kappa^2 - 4M^2 \kappa^2 \beta_r - 4\kappa^2 k_1^2 \gamma_r - 4\kappa c k_1 k_2^2 \gamma_r - k_2^4 \gamma_r - 4M^4 \kappa^2 \delta_r - 4P^2 \kappa^2 \xi_r &= 0, \\
-k_1^4 \gamma_r - 4P^2 \kappa^2 \delta_r - 4M^2 \kappa^2 \xi_r &= 0, \\
-k_2^4 \gamma_r - 4P^2 \kappa^2 \delta_r - 4M^2 \kappa^2 \xi_r &= 0, \\
-k_4^4 \gamma_r - 4P^2 \kappa^2 \delta_r - 4M^2 \kappa^2 \xi_r &= 0.
\end{align*}
\] (4.6)

Hence, Eq. (4.5) follows readily from the second and the fourth equations in Eq. (4.6).

Next, we examine the stability behavior of the plane waves \( \phi_1 \) and \( \phi_2 \) given by Eq. (4.4). Therefore, we perturb the plane waves \( \phi_1 \) and \( \phi_2 \) given by Eq. (4.4) as follows:

\[
\begin{align*}
\psi(x, y, t) &= (M + \epsilon U(x, y, t)) e^{i(k_1 x + k_2 y - \omega_1 t)}, \\
\bar{\psi}(x, y, t) &= (P + \epsilon V(x, y, t)) e^{i(k_3 x + k_4 y - \omega_2 t)},
\end{align*}
\] (4.7)
where $\varepsilon$ is a very small positive number. $U$ and $V$ are superpositions of ingoing and outgoing waves given as:

\[
U(x,y,t) = A_1 e^{i(K_1 x + K_2 y - \Omega t)} + \bar{A}_2 e^{-i(K_1 x + K_2 y - \bar{\Omega} t)},
\]

\[
V(x,y,t) = B_1 e^{i(K_1 x + K_2 y - \Omega t)} + \bar{B}_2 e^{-i(K_1 x + K_2 y - \bar{\Omega} t)},
\]

(4.8)

where,

- $K_1, K_2$ are real numbers representing the wave vector.
- $\Omega$ is a complex number representing the angular frequency.
- $A_j, B_j; j = 1, 2$ are complex numbers representing the amplitudes of waves $U, V$.
- $\bar{A}_j, \bar{B}_j; j = 1, 2$ denote the conjugates of $A_j$ and $B_j$.

In the following, we derive the dispersion relation so that the perturbed waves $\psi, \hat{\psi}$ given by Eq. (4.7) are solutions of the system of coupled NSW equations given by Eq. (4.3). Note that the obtained dispersion relation is a first order approximation with respect to $\varepsilon$.

**Theorem 4.** The perturbed waves $\psi$ and $\hat{\psi}$ given by Eq. (4.7) are the solutions of the system of coupled NSW equations given by Eq. (4.3) if, in the first order approximation with respect to $\varepsilon$, the following dispersion relation is satisfied:

\[
256 \kappa_0^8 \varepsilon^4 + (w + ir)\Omega^3 + (g + il)\Omega^2 + (h + iq)\Omega + (f + id) = 0.
\]

(4.9)

where $w, r, g, l, h, q, f$, and $d$ are functions of the parameters of the system given by (4.3), and the waves given by (4.7).

We do not include the explicit expressions of these functions because they require hundreds of pages, and more importantly, such exposures do not add clarity to this statement.

**Proof.** We substitute perturbed waves $\psi$ and $\hat{\psi}$ given by Eq. (4.7) into the system of the coupled NSW equations given by Eq. (4.3), and follow these steps to obtain the dispersion relation. First we obtain the expansions with respect to $\varepsilon$ of the form:

\[
\left\{
\begin{array}{l}
\psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \varepsilon^3 \psi_3 + \varepsilon^4 \psi_4 + \varepsilon^5 \psi_5 = 0,
\hat{\psi}_0 + \varepsilon \hat{\psi}_1 + \varepsilon^2 \hat{\psi}_2 + \varepsilon^3 \hat{\psi}_3 + \varepsilon^4 \hat{\psi}_4 + \varepsilon^5 \hat{\psi}_5 = 0.
\end{array}
\right.
\]

(4.10)

Note that, $\psi_0 = 0$ and $\hat{\psi}_0 = 0$ by Lemma 6. Furthermore, we neglect the higher orders of $\varepsilon$, and obtain that:

\[
\left\{
\begin{array}{l}
\psi_1 = 0,
\hat{\psi}_1 = 0.
\end{array}
\right.
\]

(4.11)

where,
\[
\psi_1 = \begin{cases} 
(4\chi k_c^2 - 4iM^2k_c^2\beta - 8M^2k_c^2\beta, - 4k_c^2k_t^1\gamma, - 4k_c^2k_t^2\gamma, - k_c^2\gamma, - 8iM^4k_c^2\delta) - \\
12M^4k_c^2\delta, - 4P^2k_c^2\xi_r)U + (-4iMPk_c^2\xi_d - 4MPk_c^2\xi_r)V + (-4iM^2k_c^2\beta - 4M^2k_c^2\beta, - \\
8iM^4k_c^2\delta, - 4M^4k_c^2\delta)\dot{U} + (-4MPk_c^2\xi_d - 4MPk_c^2\xi_r)V - 4k_c^2U + \\
(-4k_c^2 k_2 \gamma - 8k_c k_1 k_2 \gamma + 8k_c k_1 k_2 \gamma - 4ik_2^2 \gamma)U + (2ik_c v_g + 4ik_c k_1 \gamma + \\
6ik_2^2 \gamma + 4k_c k_1 \gamma + 6ik_2^2 \gamma)U_{yy} + (4k_2 \gamma - 4ik_2 \gamma)U_{yy} + (-i\gamma, - \gamma)U_{yyy} + \\
(-8k_c^2 k_1 s - 4k_c^2 v_g - 8k_c^2 k_1 \gamma - 4k_c^2 \gamma + 8ik_2 k_1 \gamma + 4ik_c^2 \gamma)U + \\
(8ik_c k_2 \gamma + 8k_c k_2 \gamma)U_{xy} + (4k_c \gamma - 4ik_c \gamma)U_{xy} + (4ik_c^2 s + 4ik_c^2 \gamma + 4k_c^2 \gamma)U_{xx}, \\
\end{cases} 
\]

(4.12)

Note that, we have not included the expressions for \(\psi_2, \psi_3, \psi_4, \psi_5\) and \(\tilde{\psi}_2, \tilde{\psi}_3, \tilde{\psi}_4, \tilde{\psi}_5\) because of their size, and more importantly, the corresponding terms are neglected. However, the interested reader can find these expressions in Appendix D.

Next, we substitute the expressions of the perturbations \(U\) and \(V\) given by Eq. (4.8) in Eq. (4.12). Hence we obtain:

\[
\begin{align*}
&\left\{ e^{-i(K_1 x + K_2 y - \Omega t)}u_1 + e^{i(K_1 x + K_2 y - \Omega t)}u_2 = 0, \\
e^{-i(K_1 x + K_2 y - \Omega t)}v_1 + e^{i(K_1 x + K_2 y - \Omega t)}v_2 = 0. 
\right.
\end{align*}
\]

(4.13)

where,

\[
\begin{align*}
\bar{u}_1 & = (4i\Omega \kappa_c^2 + d_1)A_2 + c_1 A_1 + c_2 B_2 + c_2 B_1, \\
u_2 & = c_3 A_2 + (4i\Omega \kappa_c^2 + d_2)A_1 + c_4 B_2 + c_4 B_1, \\
\bar{v}_1 & = c_2 A_2 + c_2 A_1 + (4i\Omega \kappa_c^2 + d_3)B_2 + c_5 B_1, \\
v_2 & = c_4 A_2 + c_4 A_1 + c_6 B_2 + (4i\Omega \kappa_c^2 + d_4)B_1,
\end{align*}
\]

(4.14)

where,
\[d_1 = i(-4k^2_3 K_2(\gamma + i\gamma) + 6k^2_2 K^2_2(\gamma + i\gamma) - 4k_2 K^3_2(\gamma + i\gamma) + \\
K^2_2(\gamma + i\gamma) + ik^4_4 \gamma + kc(-4k_2K_2(v_g + 2(k_1 - K_1)(\gamma + i\gamma)) + \\
2K^2_2(v_g + 2(k_1 - K_1)(\gamma + i\gamma)) - 4k_2^2(K_1(\gamma + i\gamma) - ik_1 \gamma)) + \\
4k^2_4(-i\chi - K_1v_g + M^2\beta_i + 2im^2\beta_r - 2k_1K_1(s + \gamma + i\gamma) + \\
K^2_1(s + \gamma + i\gamma) + ik^2_1 \gamma + 2M^4\delta_i + 3im^4\delta_r + iP^2 \xi_r),\]

\[d_2 = 4\chi k^2_2 - 8ik^2_2 k_1 K_1 s - 4ik^2_2 K^2_1 s - 4ik^2_2 K_1 v_g - 4ik_2 K_2 v_g - 2ik_2 K^2_2 v_g - \\
4im^2 k^2_2 \beta - 8M^2 k^2_2 \beta - 8ik_2 k_1 K_1 \gamma - 4ik_2 K^2_2 K_1 \gamma - \\
8ik_2 k_1 K_2 \gamma - 4ik^3_2 K_1 \gamma - 8ik_2 K_2 K_1 \gamma - 4ik_2 k_2 K^2_2 \gamma - 6ik^2_2 K^2_2 \gamma - \\
4ik_2 K_1 K^2_2 \gamma - 4ik_2 K^2_2 \gamma - iK^2_1 \gamma - 4k^2_4 k^2_2 \gamma - 4k_2 k^2_1 \gamma - k^2_2 \gamma - \\
8k^2_4 k_1 K_1 \gamma - 4k_2 k^2_2 k^2_1 \gamma - 4k^2_4 k^2_2 \gamma - 8k_2 k_1 k_2 K_1 \gamma - 4k^3_2 K_2 \gamma - \\
8k_2 k_1 K_2 \gamma - 4k_2 k^2_2 K_1 \gamma - 6k^2_2 K^2_2 \gamma - 4k_2 K^2_2 \gamma - 4k^2_2 \gamma - \\
K^2_2 \gamma - 8iM^4 k^2_2 \delta_i - 12M^4 k^2_2 \delta_r - 4P^2 k^2_2 \xi,\]

\[d_3 = 6ik^2_4 K^2_2(\gamma + i\gamma) + iK^4_1(\gamma + i\gamma) - k^4_1 \gamma + 4k^3_4 K^2_2(-i\gamma + \gamma) + \\
4k_4 k_2 \chi - iK_1 v_g + iP^2 \beta_i - 2P^2 \beta_r - 2ik_3 K_1 
(s + \gamma + i\gamma) + iK^2_1(s + \gamma + i\gamma) - k^2_2 \gamma + 2iP^4 \delta_i - 3P^4 \delta_r - M^2 \xi,\]

\[d_4 = 4\chi k^2_2 - 4ik^2_2 k^2_3 s - 8ik^2_2 k_3 K_1 s - 4ik^2_2 K^2_1 s - 4ik^2_2 k_3 v_g - 2ik_2 k^2_3 v_g - 4ik^2_2 K_1 v_g - \\
4ik_2 k_4 K_2 v_g - 2ik_2 K^2_2 v_g - 8iP^2 k^2_2 \beta - 8P^2 k^2_2 \beta - 4ik^2_2 k^2_3 \gamma - 4ik_2 k_3 k^2_4 \gamma - iK^4_1 \gamma - \\
8ik^2_3 k_1 K_1 \gamma - 4ik^2_2 K^2_1 \gamma - 8ic^4_3 k_3 K_2 \gamma - 4ik^3_4 K^2_2 \gamma - \\
8ik_4 k_1 K_2 \gamma - 4ik_4 k_2 K^2_2 \gamma - 6ik^2_3 K^2_2 \gamma - 4ik_4 K_1 K^2_2 \gamma - 4ik^3_4 K^2_2 \gamma - \\
k^2_4 \gamma - 4k_4 k^2_4 \gamma - 4k^2_4 \gamma - 8k_2 k_3 K_1 \gamma - 4k_2 k^2_4 K_1 \gamma - 4k^2_4 k^2_1 \gamma - \\
4k^3_4 K^2_2 \gamma - 8k_4 k_3 K^2_1 \gamma - 4k_4 k^2_3 \gamma - 6k^2_3 K^2_2 \gamma - 4k_4 K_1 K^2_2 \gamma - \\
k^2_4 \gamma - 12iP^4 k^2_2 \delta_i - 12P^4 k^2_2 \delta_r - 4iM^2 k^2_3 \xi_i + i(4k^2_4 k^2_3 s + 4k^2_4 k^2_3 v_g + \\
2k_4 k^2_3 v_g + 4P^2 k^2_2 \beta + 4k^2_4 k^2_3 \gamma + 4k_2 k_3 k^2_4 \gamma + k^4_4 \gamma + 4P^4 k^2_2 \delta_i + 4M^2 k^2_2 \xi_i) - 4M^2 k^2_2 \xi,\]

\[c_1 = 4im^2 k^2_2 \beta - 4M^2 k^2_2 \beta - 8im^4 k^2_2 \delta_i - 8M^4 k^2_2 \delta_r,\]

\[c_2 = 4imP^2 k^2_2 \xi_i - 4MP^2 k^2_2 \xi,\]

\[c_3 = -4iM^2 k^2_2 \beta - 4M^2 k^2_2 \beta - 8im^4 k^2_2 \delta_i - 8M^4 k^2_2 \delta_r,\]

\[c_4 = -4iMP^2 k^2_2 \xi_i - 4MP^2 k^2_2 \xi,\]

\[c_5 = 4iP^2 k^2_2 \beta - 4P^2 k^2_2 \beta - 8iP^4 k^2_2 \delta_i - 8P^4 k^2_2 \delta_r,\]

\[c_6 = -4iP^2 k^2_2 \beta - 4P^2 k^2_2 \beta - 8iP^4 k^2_2 \delta_i - 8P^4 k^2_2 \delta_r.\]
By the properties of complex numbers:

\[ u_1 = 0 \Rightarrow \bar{u}_1 = 0, \quad \text{and} \quad v_1 = 0 \Rightarrow \bar{v}_1 = 0. \]

Hence, we consider the system of equations:

\[
\begin{cases}
\bar{u}_1 = 0, \\
u_2 = 0, \\
\bar{v}_1 = 0, \\
v_2 = 0.
\end{cases}
\] (4.15)

The system of equations given by (4.15) can be expressed in matrix form as:

\[
\begin{pmatrix}
d_1 + 4i\Omega\kappa_c^2 & c_1 & c_2 & c_2 \\
c_3 & d_2 + 4i\Omega\kappa_c^2 & c_4 & c_4 \\
c_2 & c_2 & d_3 + 4i\Omega\kappa_c^2 & c_5 \\
c_4 & c_4 & c_6 & d_4 + 4i\Omega\kappa_c^2
\end{pmatrix}
\begin{pmatrix}
A_2 \\
A_1 \\
B_2 \\
B_1
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\] (4.16)

Nontrivial solutions of the system given by (4.16) exist iff the dispersion relation, given by Eq. (4.9) holds.

\[ \square \]

**Remark 1.** Observe that the MI procedure transformed the study of the system of coupled NSW equations given by Eq. (4.3) to the study of a polynomial of degree 4, given by Eq. (4.9)

### 4.2.3 Numerical study

In this section, we investigate the numerical sensitivity of the stability of the perturbed waves \( \psi \) and \( \hat{\psi} \) to the values of the parameters in the waves given by Eq. (4.7) and the system of coupled NSW equations given by Eq. (4.3).

**Remark 2.** The dispersion relation given by Eq. (4.9) is very complicated because of its dependence on a large number of parameters. Hence, we analyze the behavior of its roots for three different sets of parameters.

**Case 1:** Assume that:

\[
\begin{cases}
w + ir = 0, \\
h + iq = 0.
\end{cases}
\] (4.17)

It follows from the first and third equation of Eq. (4.6) that:

\[ 256\kappa_c^8\Omega^4 + (g + il)\Omega^2 + (f + id) = 0, \] (4.18)
This polynomial can be transformed in a polynomial of degree 2 by substitution, similar to Chapter 2 (see Theorem 1 and Theorem 2). Therefore, the roots of the polynomial given by Eq. (4.18) are determined explicitly. Their expressions are given as:

$$
\Omega_+ = \frac{\left(\frac{-(g+il)+\sqrt{(g+il)^2-1024(f+id)\kappa^8}}{\kappa^8}\right)^{1/2}}{16\sqrt{2}},
$$

$$
\Omega_- = -\frac{\left(\frac{-(g+il)-\sqrt{(g+il)^2-1024(f+id)\kappa^8}}{\kappa^8}\right)^{1/2}}{16\sqrt{2}},
$$

$$
\hat{\Omega}_+ = \frac{\left(\frac{-(g+il)+\sqrt{(g+il)^2-1024(f+id)\kappa^8}}{\kappa^8}\right)^{1/2}}{16\sqrt{2}},
$$

$$
\hat{\Omega}_- = -\frac{\left(\frac{-(g+il)-\sqrt{(g+il)^2-1024(f+id)\kappa^8}}{\kappa^8}\right)^{1/2}}{16\sqrt{2}}.
$$

(4.19)

Note that, the square root must be understood as the principal value of the complex square root. \(\Omega_+, \Omega_-, \hat{\Omega}_+, \) and \(\hat{\Omega}_-\) depend on the values of the parameters in the coefficients of the dispersion relation. By Corollary 1 stated in section (2.2) of Chapter 2, the signs of \(\text{Im}(\Omega_+), \text{Im}(\Omega_-), \text{Im}(\hat{\Omega}_+), \) and \(\text{Im}(\hat{\Omega}_-)\) determine the stability of the solution of the system of coupled NSW equations given by Eq. (4.3).

We study numerically the dependence of the imaginary parts of roots of the dispersion relation given by Eq. (4.9). More specifically, we numerically analyze the dependence of \(\text{Im}(\Omega_+), \text{Im}(\Omega_-), \text{Im}(\hat{\Omega}_+), \) and \(\text{Im}(\hat{\Omega}_-)\) given by Eq. (4.19) with respect to the parameter values \(g, l, f, \) and \(d.\) Recall that, the regions of the instability, that are commonly called MI gain or MI gain spectrum, are the regions where \(\text{Im}(\Omega_+) > 0, \text{Im}(\Omega_-) > 0, \text{Im}(\hat{\Omega}_+) > 0, \text{Im}(\hat{\Omega}_-) > 0.\)

It follows from (4.17) that:

$$w = r = h = q = 0. \quad (4.20)$$

In addition, it follows from the first and third equation of Eq. (4.6) that:
Next, we substitute $\gamma_r = 0$ in the equations given by Eq. (4.21). Then, we combine the last two equations and obtain:

$$4(M - P)(M + P)\kappa_c^2(\beta_r + (M^2 + P^2)\delta_r - \xi_r) = 0$$

(4.22)

Since $\kappa_c \neq 0$, $M > 0$, and $P > 0$, then:

$$M - P = 0$$

or

$$\beta_r + (M^2 + P^2)\delta_r - \xi_r = 0 \quad \text{and} \quad M \neq P.$$

**Case 1.1: $M = P$.**

It follows from (4.21) and Eq. (4.20) that:
\[
\omega_1 = \frac{1}{4\kappa_c^2} \left( 4\kappa_c^2 k_1 s + 4\kappa_c^2 k_1 v_g + 2\kappa_c k_2^2 v_g + 4M^2 \kappa_c^2 \beta_i + 4\kappa_c^2 k_1^4 + 4\kappa_c k_1^2 k_2^2 + k_2^4 + 4M^4 \kappa_c^2 \delta_i + 4P^2 \kappa_c^2 \xi_i \right),
\]
\[
\omega_2 = \frac{1}{4\kappa_c^2} \left( 4\kappa_c^2 k_3^2 s + 4\kappa_c^2 k_2 v_g + 2\kappa_c k_3^2 v + 4P^2 \kappa_c^2 \beta_i + 4\kappa_c^2 k_3^2 \gamma_i + 4\kappa_c k_3^2 k_2^2 + k_3^4 \gamma_i + 4P^4 \kappa_c^2 \delta_i + 4M^2 \kappa_c^2 \xi_i \right),
\]
\[
\gamma_i = 0,
\]
\[
M = P,
\]
\[
P = \left( -\beta_r \right)^{1/2} /
\]
\[
\chi = \frac{\beta_r (\beta_r + 2 \xi_r)}{4\delta_r},
\]
\[
k_3 = \frac{-1}{2\kappa_c (k_4 K_2 \gamma_i + \kappa_c K_1 (s + \gamma_i))} (K_2 (k_3^3 + k_1^3 + k_2 k_2^2 + k_4 k_2^2) \gamma_i + 2\kappa_c K_1 (v_g + k_1 (s + \gamma_i)) + \kappa_c (k_4 K_2 v_g + k_2^2 K_1 \gamma_i + k_4^2 K_1 \gamma_i + 2K_1 k_2^2 \gamma_i + k_2 K_2 (v_g + k_1 \gamma_i)),
\]
\[
k_1 = \frac{1}{2\kappa_c (k_2 - k_4) K_2^2} \gamma_i^2 \left( -k_2^3 (k_2^3 - 3k_2^2 k_4 - 2k_4^3 + k_2 k_2^2) \gamma_i^2 + 4\kappa_c^3 k_1^3 (s + \gamma_i)^2 + 2\kappa_c^2 K_1 K_2 (K_2 s v_g + 2(k_2 + 2k_4) K_1 \gamma_i (s + \gamma_i) + \kappa_c k_2^2 \gamma_i (k_4 K_2 v_g + K_1 k_2^2 (s - \gamma_i) + 3k_4^2 K_1 (s + 2 \gamma_i) + k_4^2 K_1 (3s + 2 \gamma_i) + k_2 (-K_2 v_g + 4k_4 K_1 \gamma_i)) \right),
\]
\[
\delta_i = \frac{\beta_i \delta_r}{\beta_r}.
\]

We have performed numerous experiments to analyze the effects of the parameters on the stability of the perturbed waves \( \Psi \) and \( \dot{\Psi} \) given by Eq. (4.7). For illustration purposes, we report here the obtained results of three experiments.
**Experiment 1.** In this experiment, we analyze the dependence of $\text{Im}(\Omega_+)$ and $\text{Im}(\Omega_-)$ to the wave numbers $K_1$ and $k_2$. The results are reported in Figure 8. These results indicate the following:

- Figure 8(a) shows the gain of $\text{Im}(\Omega_+)$ in the form of a collection of similar looking pairs of pulses.

- Figure 8(b) depicts the gain of $\text{Im}(\Omega_-)$ in the shape of a big hump of several nonlinear waves interacting with each other.

- The magnitude of the gain of Figure 8(b) is larger than that of Figure 8(a).

\begin{itemize}
  \item (a) $K_2 = -2, k_4 = -1, \xi_i = 0.2$
  \item (b) $K_2 = 1, k_4 = -1, \xi_i = 0.2$
\end{itemize}

**Figure 8:** Sensitivity of the solution to the wave numbers $K_1$ and $k_2$.  
**Case where:** $\gamma_r = 0, \nu_g = 0.4, \gamma_l = 1, s = 0, \beta_r = -3, \beta_l = -1, \delta_r = 2.75, \kappa_r = 3.14, \xi_r = 0.2.$}

39
Figure 9: Sensitivity of the solution to the wave numbers $K_1$ and $K_2$.
Case where: $g_r = 0, v_g = 0.4, g_i = 1, s = 0, \beta_r = -3, \beta_i = -1, \delta_r = 2.75, \kappa_c = 3.14, \xi_r = 0.2$.

**Experiment 2.** In this experiment, we analyze the dependence of $\text{Im}(\Omega_+)$ to the wave numbers $K_1$ and $K_2$. The results are reported in Figure 9. These results indicate the following:

- The peak of the gain is the same for Figure 9(a)-9(b).
- Figure 9(a) exhibits two pairs of counter propagating pulses.
- Figure 9(b) shows a pair of packets of several small pulses formed on either side of the interval.
Experiment 3. In this experiment, we analyze the dependence of $\text{Im}(\Omega_+)$ and $\text{Im}(\Omega_-)$ to the wave numbers $K_1, k_2,$ and $k_4$. The results are reported in Figure 10. These results indicate the following:

- Figure 10(a) depicts the gain of $\text{Im}(\Omega_+)$ in terms of $k_2$ and $k_4$. The peak of the gain is 0.15.

- Figure 10(b) shows the gain of $\text{Im}(\Omega_-)$ with respect to $K_1$ and $k_4$. The peak of the gain is 0.2.

- Figures 10(a)-10(b) show a formation of a pair of counter propagating pulses. In Figure 10(a), the pulses subside before penetrating each other. In Figure 10(b), they collide with each other.
Case 1.2: $M \neq P$ and $\beta_r + (M^2 + P^2) \delta_r - \xi_r = 0$.

It follows from (4.21) and Eq. (4.20) that:

\[
\omega_1 = \frac{1}{4\kappa_c^2} \left( 4\kappa_c^2 k_1^2 s + 4\kappa_c^2 k_1 v_g + 2\kappa_c k_1^2 v_g + 4M^2 \kappa_c^2 \beta_r + 4\kappa_c^2 \beta_r + 4M^4 \kappa_c^2 \delta_r + 4P^2 \kappa_c^2 \xi_r \right),
\]

\[
\omega_2 = \frac{1}{4\kappa_c^2} \left( 4\kappa_c^2 k_3^2 s + 4\kappa_c^2 k_2 v_g + 2\kappa_c k_3^2 v_g + 4P^2 \kappa_c^2 \beta_r + 4\kappa_c^2 \beta_r + 4M^4 \kappa_c^2 \delta_r + 4M^2 \kappa_c^2 \xi_r \right),
\]

\[
\gamma_r = 0,
\]

\[
\chi = -\frac{\beta_r (\beta_r + \xi_r) + 2\xi_r^2}{4\delta_r},
\]

\[
P = \frac{-\beta_r^2 + \frac{\xi_r}{\delta_r} + \frac{\sqrt{\beta_r^2 + 4\chi \delta_r + 2\beta_r \xi_r - 3\xi_r^2}}{\delta_r}}{\sqrt{2}},
\]

\[
M = \frac{-\beta_r^2 + \frac{\xi_r}{\delta_r} - \frac{\sqrt{\beta_r^2 + 4\chi \delta_r + 2\beta_r \xi_r - 3\xi_r^2}}{\delta_r}}{\sqrt{2}},
\]

\[
k_1 = \frac{1}{2\kappa_c (k_2 K_2 \gamma + \kappa_c K_1 (s + \gamma))} \left( -(k_2^3 + k_4^3 + k_2 k_3^2 + k_2 k_4^2) \gamma_i \right.
\]

\[
+ 2\kappa_c^2 K_1 (v_g + k_3 (s + \gamma)) + \kappa_c (k_2 K_2 v_g + k_2^2 K_1 \gamma + k_2^2 K_1 \gamma i)
\]

\[
+ 2K_1 k_2^2 \gamma + k_4 K_2 (v_g + 2k_3 \gamma)) )
\]

We have performed numerous experiments to analyze the effects of the parameters on the stability of the perturbed waves $\psi$ and $\psi'$ given by Eq. (4.7). For illustration purposes, we report here the obtained results of one experiment only. In the considered experiment, we analyze the dependence of $\text{Im}(\Omega_+)$ and $\text{Im}(\Omega_-)$ to the wave numbers $K_1$ and $K_2$. The results are reported in Figure 11. These results indicate the following:

- In Figure 11(a), the gain of $\text{Im}(\Omega_-)$ looks like two symmetric sidebands parallel in the middle and going away from each other as the magnitude of $K_2$ increases. The peak of the MI gain is 6.

- In Figure 11(b), counter propagating pulses form at the either side of the interval. The peak of the MI gain is 1.

42
• Figure 11(c) illustrates the dependence of $Im(\Omega_-)$. A head-on collision of two waves, and a bump can be observed. The peak of the MI gain is 4.

• Figure 11(d) depicts the sensitivity of $Im(\Omega_+)$. Two pairs of counter propagating waves begin and subside before colliding with each other. The peak of the MI gain is 1.5.

• The gain of $Im(\Omega_-)$ takes a form of two waves or bands. On the other hand, the gain of $Im(\Omega_+)$ is exhibited in the form of four waves.

• $Im(\Omega_-)$ depicts the gain in the form of two pulses. $Im(\Omega_+)$ illustrates the gain in the form of four pulses.

Figure 11: Sensitivity of the solution to the wave numbers $K_1$ and $K_2$.
Case where: $\gamma = 0, \phi = 1, \beta_r = -3, \beta_i = -1, \delta_r = 2.75, \delta_i = -1$
$\nu_r = 0.4, s = 0, \kappa = 3.14, k_4 = -1, \xi_r = -0.2, \xi_i = 0.2$. 
Case 2: Assume that:
\begin{align*}
  \begin{cases}
    w + ir = 0, \\
    h + iq \neq 0.
  \end{cases}
\end{align*}
\tag{4.23}

It follows from the first and third equation of Eq. (4.6) that:
\[ 256 \kappa_e^8 \Omega^4 + (g + il) \Omega^2 + (h + iq) \Omega + (f + id) = 0, \]
\tag{4.24}

The roots of the polynomial given by Eq. (4.24) are given as:
\[ \Omega_1 = \frac{1}{2\sqrt{6}} (b_1 + \sqrt{b_2 - b_3}), \]
\[ \Omega_2 = -\frac{1}{2\sqrt{6}} (b_1 - \sqrt{b_2 + b_3}), \]
\[ \Omega_3 = \frac{1}{2\sqrt{6}} (b_1 - \sqrt{b_2 - b_3}), \]
\[ \Omega_4 = -\frac{1}{2\sqrt{6}} (b_1 + \sqrt{b_2 + b_3}), \]
\tag{4.25}

where,
\[ b_1 = (-4m_2 + \frac{a_1}{a_2} + 2^{2/3} a_2)^{1/2}, \]
\[ b_2 = (-8m_2 - \frac{a_1}{a_2} - 2^{2/3} a_2), \]
\[ b_3 = \frac{12\sqrt{6}m_1}{a_1}, \]
\[ a_1 = 2 \times 2^{1/3} (12m + m^2), \]
\[ a_2 = \left( -72mm_2 + 2m_2^3 + 27m_1^2 + \sqrt{-4a_1^3 + (-72mm_2 + 2m_2^3 + 27m_1^2)^2} \right)^{1/3}, \]
\[ m_2 = \frac{g + il}{256 \kappa_e^8}, \]
\[ m_1 = \frac{h + iq}{256 \kappa_e^8}, \]
\[ m = \frac{f + id}{256 \kappa_e^8}. \]

Note that, the square root must be understood as the principal value of the complex square root. \( \Omega_1, \Omega_2, \Omega_3, \) and \( \Omega_4 \) depend on the values of the parameters in the coefficients of the dispersion relation. By Corollary 1 stated in section (2.2) of Chapter 2, the signs of \( \text{Im}(\Omega_1), \text{Im}(\Omega_2), \text{Im}(\Omega_3), \) and \( \text{Im}(\Omega_4) \) determine the stability of the solution of the system of coupled NSW equations given by Eq. (4.3).

We study numerically the dependence of the imaginary parts of the roots of the dispersion relation given by Eq. (4.9). More specifically, we analyze numerically the dependence of
\(Im(\Omega_1), Im(\Omega_2), Im(\Omega_3), \) and \(Im(\Omega_4)\) given by Eq. (4.25) with respect to the parameter values \(h, q, g, l, f, \) and \(d.\)

Note that, the regions of the instability, that are commonly called MI gain or MI gain spectrum, are the regions where \(Im(\Omega_1) > 0, Im(\Omega_2) > 0, Im(\Omega_3) > 0, Im(\Omega_4) > 0.\)

It follows from (4.23) that:

\[
w = r = 0. \tag{4.26}
\]

In addition, it follows from the first and the third equation of Eq. (4.6) that:

\[
\begin{align*}
 w &= 512\, \kappa_c^6 (2\, \kappa_c^2 k_1 K_1 s - 2\, \kappa_c^2 k_3 K_1 s - 2\, \kappa_c^2 K_1 v_g - \kappa_c k_3 K_2 v_g - \\
& \quad \kappa_c k_4 K_2 v_g - 2\, \kappa_c^2 k_1 K_1 \gamma_l - 2\, \kappa_c^2 k_3 K_1 \gamma_l - \kappa_c k_2 K_1 \gamma_l - \\
& \quad \kappa_c k_4 K_1 \gamma_l - 2\, \kappa_c k_1 k_2 K_2 \gamma_l - k_2^3 K_2 \gamma_l - 2\, \kappa_c k_3 k_4 K_2 \gamma_l - \\
& \quad k_3^4 K_2 \gamma_l - 2\, \kappa_c K_1 k_2^3 \gamma_l - k_2^3 \gamma_l - k_4 k_2^3 \gamma_l) = 0, \\
& \quad r = -128\, \kappa_c^6 (8\, \chi \, \kappa_c^2 - 8\, M^2 \, \kappa_c^2 \beta_r - 8\, P^2 \, \kappa_c^2 \beta_r - 4\, \kappa_c^2 k_3^2 \gamma_l - \\
& \quad 4\, \kappa_c k_1 k_2^2 \gamma_l - k_2^4 \gamma_l - 4\, \kappa_c k_3 k_2^3 \gamma_l - k_2^4 \gamma_l - 8\, \kappa_c^2 k_1^2 \gamma_l - \\
& \quad 8\, \kappa_c k_1 k_2 K_2 \gamma_l - 4\, \kappa_c k_1 k_2^3 \gamma_l - 4\, \kappa_c k_3 k_2^3 \gamma_l - 6k_2^2 k_2^3 \gamma_l - \\
& \quad 6k_4^2 K_2 \gamma_l - 2k_2^4 \gamma_l - 12\, M^4 \, \kappa_c^2 \delta_r - 12\, P^2 \, \kappa_c^2 \delta_r - \\
& \quad 4\, M^2 \, \kappa_c^2 \xi_r - 4\, P^2 \, \kappa_c^2 \xi_r) = 0.4\, \kappa_c^2 - 4\, M^2 \, \kappa_c^2 \beta_r - 4\, \kappa_c^2 k_1^2 \gamma_l - 4\, \kappa_c k_1 k_2^3 \gamma_l - k_2^4 \gamma_l - \\
& \quad -4\, M^4 \, \kappa_c^2 \delta_r - 4\, P^2 \, \kappa_c^2 \xi_r = 0, \\
& \quad 4\, \chi \, \kappa_c^2 - 4\, P^2 \, \kappa_c^2 \beta_r - 4\, \kappa_c^2 k_3^2 \gamma_l - 4\, \kappa_c k_3 k_3^2 \gamma_l - k_4^4 \gamma_l - \\
& \quad -4\, P^4 \, \kappa_c^2 \delta_r - 4\, P^2 \, \kappa_c^2 \xi_r = 0. \tag{4.27}
\end{align*}
\]

Next, we substitute \(\gamma_l = 0\) in the equations given by Eq. (4.27). Then, we combine the last two equations, and obtain that:

\[
4(M - P) \, (M + P) \, \kappa_c^2 (\beta_r + (M^2 + P^2) \, \delta_r - \xi_r) = 0 \tag{4.28}
\]

Since \(\kappa_c \neq 0, M > 0, \) and \(P > 0,\) then,

\[
M - P = 0 \quad \text{or} \quad \beta_r + (M^2 + P^2) \, \delta_r - \xi_r = 0 \quad \text{and} \quad M \neq P.
\]

Since we have performed numerous experiments to analyze the effects of the parameters on the stability of the perturbed waves \(\psi\) and \(\tilde{\psi}\) given by Eq. (4.7). For illustration purposes, we report here the obtained results of one experiment only. In this experiment, we analyze the dependence of \(Im(\Omega_1), Im(\Omega_2), Im(\Omega_3), \) and \(Im(\Omega_4)\) to the wave numbers \(K_1\) and \(K_2.\) The results are reported in Figure 12. These results indicate the following:
• Figure 12(a) shows a much smaller gain as compared to Figures 12(b)-12(d).

• A head-on collision of three waves can be observed in Figure 12(b).

• Figures 12(c)-12(d) indicate the same pattern of MI gain in different directions. Several nonlinear waves are interacting with each other.

• Higher values of $\beta_r, \beta_i, \delta_i$ are contributing to the increase of the magnitude of the gain.

![Figure 12: Sensitivity of the solution to the wave numbers $K_1$ and $K_2$. Case where: $\nu_g = 0.4, \gamma_r = 0, \kappa_r = 3.14, \xi_r = -0.2, k_0 = 0.5.$](image-url)
Case 3: This is a general case. We do not assume any constraints on the coefficients of the \( \Omega^m_i \) (\( n_i = 1, 2, 3, 4 \)). We assume that all of the coefficients of the polynomial given by Eq. (4.9) are non-zero.

The roots of this polynomial are given by:

\[
\begin{align*}
\Omega_1 &= -3m_3 + \sqrt[3]{-8m_2 + 3m_3^2 + 4p_1} + \sqrt{6} \sqrt{-8m_2 + 3m_3^2 + 2p_3 - p_4}, \\
\Omega_2 &= -3m_3 - \sqrt[3]{-8m_2 + 3m_3^2 + 4p_1} - \sqrt{6} \sqrt{-8m_2 + 3m_3^2 + 2p_3 + p_4}, \\
\Omega_3 &= -3m_3 - \sqrt[3]{-8m_2 + 3m_3^2 + 4p_1} + \sqrt{6} \sqrt{-8m_2 + 3m_3^2 + 2p_3 + p_4}, \\
\Omega_4 &= -3m_3 + \sqrt[3]{-8m_2 + 3m_3^2 + 4p_1} - \sqrt{6} \sqrt{-8m_2 + 3m_3^2 + 2p_3 - p_4}.
\end{align*}
\]  

(4.29)

where,

\[
\begin{align*}
m &= \frac{f + id}{256 \kappa^8}, \\
m_1 &= \frac{h + iq}{256 \kappa^8}, \\
m_2 &= \frac{g + il}{256 \kappa^8}, \\
m_3 &= \frac{w + ir}{256 \kappa^8}, \\
p_1 &= 2^{1/3}(12m + m_2^2 - 3m_1m_3), \\
p_2 &= 27m_1^2 - 72m_1m_2 + 2m_2^3 - 9m_1m_2m_3 + 27m_3^3, \\
p_3 &= \left( p_2 + \sqrt{-4(12m + m_2^2 - 3m_1m_3)^3 + p_2^2} \right)^{1/3}, \\
p_4 &= -8m_2 + 3m_3^2 + \frac{4p_1}{p_2} + 2(2^{2/3})p_2.
\end{align*}
\]

Note that, the square root must be understood as the principal value of the complex square root. \( \Omega_1, \Omega_2, \Omega_3, \) and \( \Omega_4 \) depend upon the values of the parameters in the coefficients of the dispersion relation. By Corollary 1 stated in section (2.2) of Chapter 2, the signs of \( Im(\Omega_1), Im(\Omega_2), Im(\Omega_3), \) and \( Im(\Omega_4) \) determine the stability of the solution of the system of coupled NSW equations given by Eq. (4.3).

We numerically study the dependence of the imaginary parts of roots of the dispersion relation given by Eq. (4.9). More specifically, we analyze numerically the dependence of \( Im(\Omega_1), Im(\Omega_2), Im(\Omega_3), \) and \( Im(\Omega_4) \) given by Eq. (4.25) with respect to the parameter values \( h, q, l, f, \) and \( d. \)

Note that the regions of the instability, that are commonly called MI gain or MI gain spectrum, are the regions where \( Im(\Omega_1) > 0, Im(\Omega_2) > 0, Im(\Omega_3) > 0, Im(\Omega_4) > 0. \)

We have performed numerous experiments to analyze the effects of the parameters on the stability of the perturbed waves \( \psi \) and \( \hat{\psi} \) given by Eq. (4.7). For illustration purposes, we report here the obtained results of two experiments.
Figure 13: Sensitivity of the solution to the wave numbers $K_1$ and $K_2$.

Case where: $v_x = 0.4, \kappa = 3.14, s = 0.01, \xi_r = 0.2, \bar{\xi}_r = -0.2, \chi = -2, \beta_r = 1, \gamma_r = 1$.

Experiment 1. In this experiment, we analyze the dependence of $\text{Im}(\Omega_2), \text{Im}(\Omega_3)$, and $\text{Im}(\Omega_4)$ to the wave numbers $K_1$ and $K_2$. The results are reported in Figure 13. These results indicate the following:

- Figure 13(a) indicates the gain of $\text{Im}(\Omega_3)$ for the negative values of $K_1$.

- The picture is reversed in Figure 13(b), showing the gain of $\text{Im}(\Omega_4)$ for the positive values of $K_1$.

- Figures 13(c)-13(d) illustrating the gain of $\text{Im}(\Omega_2)$ and $\text{Im}(\Omega_4)$ respectively are upside down images of each other. They exhibit an enormous increase in the magnitude of the MI gain as compared to the Figures 13(a)-13(b).

- Imaginary parts of distinct roots result in a reflection of the same pattern.
Experiment 2. In this experiment, we analyze the dependence of $\text{Im}(W_1), \text{Im}(W_2)$, and $\text{Im}(W_3)$ to the wave numbers $K_1$ and $K_2$. The results are reported in Figure 14. These results indicate the following:

- Figure 14(a) demonstrates the sensitivity of $\text{Im}(\Omega_1)$. The gain is observed for $K_1 > -5$.

- Figure 14(b) shows $\text{Im}(\Omega_3)$. A wave is formed for $K_1 < 0$, and it moves towards negative infinity. Another wave is formed for $K_1 > 0.5$ moving towards positive infinity.

- In Figure 14(c), three counter propagating waves are formed, and the head-on collision of two of them is seen. The gain is present for almost all of the interval.

- Two counter propagating pulses in Figure 14(d) subside before penetrating each other. The peak of the gain is 40. Figures 14(c)-14(d) show the gain of $\text{Im}(\Omega_2)$.

- A variation in $\delta_r$ has resulted in a significant rise in the peak of the MI gain.
4.3 Summary and Conclusion

We have derived the dispersion relation of the system of coupled NSW equations and numerically analyzed its stability. The numerical study reveals that:

- The magnitude of the gain is sensitive to the value of $k_4$.

- The real part of the cubic and quintic nonlinearity parameters, $\delta_r, \beta_r$ affect the pattern of the waves, as expected.

- Positive $\delta_r, \beta_r$ helped the formation of counter propagating waves.

- $\chi = -2$ contributes to a higher MI gain.

- $\gamma_i$, the imaginary part of $\gamma$ affects the magnitude of the gain.

- Distinct roots show different gain patterns.
General Conclusion

We have applied the Modulation Instability procedure (MI) to analyze the stability of the solutions of the generalized nonlinear Schrödinger equation with cubic and quintic terms, as well as with higher order nonlinear and dispersion terms. We have also investigated the stability of a system of nonlinear Schrödinger equations and a system of Newell-Segel-Whitehead equations.

MI procedure appears to be an easy method to understand and relatively simple to implement. It can run on a personal computer. The main feature of this method is that it transforms the study of the solution of the considered PDE to the study of the roots of a polynomial. The sensitivity of the sign of the imaginary parts of roots of the equation parameters leads to the identification of the zones of the instability behavior. Hence, MI provides an insight on the local behavior of the PDE without knowing the expression of its solution.

The proposed investigation suggests that the effect of MI on a scalar PDE differs from its effect on a vector PDE. The results obtained shed light on the influential factors of the instability and on how they dictate the instability gain. More specifically, we observed that the distinct roots of the dispersion relation give rise to a variety of instability gain patterns. In some cases, they cause totally distinct patterns, whereas in other cases, they exhibit just a reflection of the same pattern of the gain. The values of the nonlinearity parameters and the dispersion parameters affect significantly the instability patterns as well as their magnitudes. Positive sign and larger magnitudes of these parameters demonstrate higher peak of the gain as compared to the parameters with negative sign and smaller magnitudes. Higher nonlinearity parameter values cause more instability as compared to the lower nonlinearity parameter values. Variation in the values of these parameters result in a variation in the instability gain pattern. In spite of the presence of an array of factors for instabilities, there is a limited variety of instability gain patterns. These patterns exhibit symmetry with respect to the wave numbers, indicating that the waves traveling in opposite directions showed the same pattern of the gain.

This investigation is an important step toward understanding the behavior of the solution of this class of PDEs. Theoretical analysis needs to be conducted to determine rigorously the sensitivity to the parameters, or at least to some of them. Such an accomplishment will provide engineers and applied mathematicians with the prerequisite knowledge and practical guidelines for selecting the discretization parameters of the numerical techniques to be employed for efficiently solving this class of equations.
Bibliography


[23] Ping AO, David J. Thouless, X-M Zhu *Nonlinear Schrödinger equation for superconductors*.


Appendix

We provide in this section the explicit expressions of the terms involved in the dispersion relation in this study.

A. Expression of $\psi_j; j = 2, 3, 4, 5$ given by Eq. (2.17).

$$
\psi_2 = M(U_x(\bar{U} + U \alpha_1 + \bar{U} \alpha_1) + 2U \alpha_1 \bar{U}_x + i(kU^2 + 2k|U|^2 + U^2 \gamma_1 \\
+ 2|U|^2 \gamma_1 + 3M^2 U^2 \gamma_2 + 6M^2 |U|^2 \gamma_2 + M^2 U \bar{U} + U U_x)),
$$

$$
\psi_3 = U(\bar{U}U_x + \bar{U}U_x \alpha_1 + U \bar{U}_x \alpha_1 + i(k|U|^2 + |U|^2 \gamma_1 + M^2 U^2 \gamma_2 + 6M^2 |U|^2 \gamma_2 + 3M^2 U \bar{U} + U U_x)),
$$

$$
\psi_4 = iM|U|^2 U(2U + 3\bar{U}) \gamma_2,
$$

$$
\psi_5 = \gamma_2 |U|^2 U.
$$

B. Expression of $\psi_j; j = 2, 3, 4, 5$ given by Eq. (2.34).

$$
\psi_2 = M(i(k + \gamma_1 + 3M^2(k\alpha_3 + \gamma_2))U^2 + i(k + \gamma_1 + 3M^2(k\alpha_3 + \gamma_2))|U|^2 + iM^2(k\alpha_3 + \gamma_2)\bar{U}^2 + \\
(1 + \alpha_1 + \alpha_2 + 2M^2(3\alpha_3 + 2\alpha_4))(U + \bar{U})U_x + (2(\alpha_1 + \alpha_2) + 6M^2(\alpha_3 + \alpha_4))U \bar{U}_x + \\
(2M^2(\alpha_3 + \alpha_4)\bar{U} \bar{U}_x. 
$$

$$
\psi_3 = (iM^2(k\alpha_3 + \gamma_2))U^3 + i(k + 6kM^2\alpha_3 + \gamma_1 + 6M^2 \gamma_2)U^2 \bar{U} + 3iM^2(k\alpha_3 + \gamma_2)|U|^2 \bar{U} + \\
M^2(3\alpha_3 + 2\alpha_4)U^2 U_x + (1 + \alpha_1 + \alpha_2 + 12M^2 \alpha_3 + 8M^2 \alpha_4)|U|^2 U_x + \\
M^2(3\alpha_3 + 2\alpha_4)\bar{U}^2 U_x(\alpha_1 + \alpha_2 + 6M^2(\alpha_3 + \alpha_4))U^2 \bar{U}_x + 6M^2(\alpha_3 + \alpha_4)|U|^2 \bar{U}_x,
$$

$$
\psi_4 = MU((2i\alpha_3 + 3i \gamma_2)|U|^2 U + (3i k \alpha_3 + 3i \gamma_2)|U|^2 \bar{U} + (6\alpha_3 + 4\alpha_4)|U|^2 U_x + \\
(6\alpha_3 + 4\alpha_4)\bar{U}^2 U_x + (2\alpha_3 + 2\alpha_4)U^2 \bar{U}_x + (6\alpha_3 + 6\alpha_4)U \bar{U} \bar{U}_x),
$$

$$
\psi_5 = |U|^2 U (i \alpha_3 |U|^2 + i \gamma_2 |U|^2 + 3\alpha_3 \bar{U} U_x + 2\alpha_4 U \bar{U}_x + 2\alpha_3 U \bar{U}_x + 2\alpha_4 \bar{U}_x).
$$

C. Expression of $\psi_j; j = 2, 3$ given by Eq. (3.13).

$$
\psi_2 = M U^2 + P \sigma U V + M \lambda V^2 + 2M |U|^2 + 2P \lambda U \bar{U} + P \sigma U \bar{V} + M \sigma |V|^2,
$$

$$
\psi_3 = |U|^2 U + \lambda V^2 \bar{U} + \sigma U |V|^2.
$$

$$
\dot{\psi}_2 = P U^2 + M \sigma U V + P \lambda U^2 + 2P |V|^2 + 2M \lambda U \bar{V} + M \sigma V \bar{U} + P \sigma |V|^2,
$$

$$
\dot{\psi}_3 = |V|^2 V + \lambda U^2 \bar{V} + \sigma U |V|^2.
$$
D. Expression of $\psi_j; j = 2, 3, 4, 5$ given by Eq. (4.10).

$$\begin{align*}
\psi_2 &= M^3 \bar{U}^2 (\bar{\delta_r} - i \delta_i) + 6M^3 U \bar{U} (\bar{\delta_r} - i \delta_i) + 2MU \bar{U} (\bar{\beta_r} - i \beta_i) - MV \bar{V} (\bar{\xi_r} + i \xi_i) - PU \bar{V} (\bar{\xi_r} + i \xi_i) + 3M^3U^2 (\bar{\delta_r} - i \delta_i) + MU^2 (\bar{\beta_r} - i \beta_i) - PUV (\bar{\xi_r} + i \xi_i), \\
\psi_3 &= 6M^2 U^2 \bar{U} (\bar{\delta_r} - i \delta_i) + 3M^2 U \bar{U}^2 (\bar{\delta_r} - i \delta_i) + U^2 \bar{U} (\bar{\beta_r} - i \beta_i) + UV \bar{V} (\bar{\xi_r} - i \xi_i) + M^2 U^3 (\bar{\delta_r} - i \delta_i), \\
\psi_4 &= 2MU^3 \bar{U} (\bar{\delta_r} - i \delta_i) + 3MU^2 \bar{U}^2 (\bar{\delta_r} - i \delta_i), \\
\psi_5 &= U^3 \bar{U}^2 (\bar{\delta_r} - i \delta_i).
\end{align*}$$