TWO-PERSON ZERO-SUM GAME THEORY AND LINEAR-PROGRAMMING

A thesis submitted in partial satisfaction of the requirements for the degree of Master of Science in Mathematics

by

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ABSTRACT

TWO-PERSON ZERO-SUM GAME THEORY AND LINEAR-PROGRAMMING

by

Edward Harold Bellin

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This paper is a study of the related mathematical subjects of two-person zero-sum game theory and linear-programming. It is discussed how these subjects apply to practical fields such as business and economics. In Section 1, the basic concepts of a two-person zero-sum game are given. The conservative method of playing a game is illustrated by use of an example. The graphical method of solving a certain type of game is also illustrated by this example. An example is given of how games can be applied to the field of business.

In Section 2, the conservative method is discussed in general and the minimax theorem is stated which relates the maximum values that the two players of any game can guarantee themselves.

In Section 3, two methods of simplifying a game, finding a saddlepoint and eliminating dominated rows of columns, are given.

In Section 4, it is shown how a game can be reduced to a linear-programming problem. In the next section, a method to solve linear-programming problems (called the simplex method) is discussed.
In Section 6, the minimax theorem is proved by use of an adaptation of the simplex method for solving games.

In Section 7, a theorem is proved which relates linear-programming problems to their duals, and an economic interpretation of the dual of a linear-programming problem involving economics is given.

Lastly, in Section 8, a mathematical model describing the economics of price equilibrium based on the theorem of Section 7 and linear-programming is given.
SECTION 1

THE FUNDAMENTAL ASPECTS OF A TWO-PERSON ZERO-SUM GAME AND THE GRAPHICAL SOLUTION OF THE 2xn GAME

To illustrate a two-person zero-sum game, I will first examine an ancient Italian game called two-finger morra. This game is played by two people, each of whom shows one or two fingers and, at the same time, calls his guess as to how many fingers the other person will show. If only one player is correct, he wins an amount equal to the sum of the fingers shown by himself and the other person. Otherwise, no one pays anything. It is possible to describe all the possible results by use of a table. Such a table (called a payoff matrix), indicating my gains under the different conditions, would look like the following:

<table>
<thead>
<tr>
<th></th>
<th>1(1,1)</th>
<th>2(1,2)</th>
<th>3(2,1)</th>
<th>4(2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1(1,1)</td>
<td>0</td>
<td>2</td>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>2(1,2)</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3(2,1)</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>4(2,2)</td>
<td>0</td>
<td>-3</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

The first element of each ordered pair represents the number of fingers that I show and the second represents my guess of the number my opponent shows.¹

To find the result of my choosing strategy \( i \) (\( i = 1,2,3,4 \)) and my opponent's choosing strategy \( j \) (\( j = 1,2,3,4 \)), find the entry that is in the \( i \)th row and in the \( j \)th column. For example, if I choose strategy 3 (I show two fingers and guess one finger) and my opponent chooses strategy 4 (he shows two fingers and guesses two fingers), the payoff to me is \(-4\), i.e., I pay him four dollars since he guessed correctly and I did not, and four fingers were shown in total. This type of game is called a two-person zero-sum game because two persons play it and money is exchanged and not added or withdrawn.\(^1\) I shall proceed to discuss the question of what is the best way to play this kind of game.

Let us look at a game for which there is a similar payoff matrix. This game will consist of two strategies available to me and three available to my opponent. The payoff matrix is the following:

\[
\begin{array}{c|ccc}
\text{His Strategy} & 1 & 2 & 3 \\
\hline
\text{My Strategy} & \text{1} & 2 & 3 & 5 \\
\hline
\text{2} & 7 & 5 & 2 \\
\end{array}
\]

I might be optimistic and be after the largest possible gain which is the seven dollars shown in the lower left-hand corner in the matrix. Thus, I should play my strategy 2. However, as time goes on, my opponent may detect my pattern of using strategy 2 and counteract my strategy by using his strategy 3. He will then be able to limit my gain to two dollars.\(^2\)

\(^1\text{Ibid., pp. 256-257.}\) \(^2\text{Ibid.}\)
If I play only one of my strategies, this is called pure strategy. Maybe I will be better off by using more than one strategy. Let us assume that my opponent is using his strategy 3 and that I use my two strategies 50 percent of the time on a random basis. My gain will be \((.50)5 + (.50)2 = 2.50 + 1.00 = 3.50\) dollars. In general, if I play on a random basis my strategy 1 a fraction \(x\) of the time and my strategy 2 the rest of the time, my gain when my opponent uses his strategies 1, 2 and 3 are as follows:

\[
\begin{align*}
  z &= 2x + 7(1-x) = 7 - 5x \\
  z &= 3x + 5(1-x) = 5 - 2x \\
  z &= 5x + 2(1-x) = 2 + 3x
\end{align*}
\]

In such games where I have only two strategies available, the game can be represented as the graphs of linear functions. The horizontal axis \(X\) represents the fraction of the time I play my strategy 1. The gain is shown by the vertical axis \(Z\). My gains when my opponent uses each of his strategies can be graphed as lines (see figure 1).\(^2\)

We have seen that being an optimist is not necessarily a rational way of playing the game. We may try to use the "principle of insufficient reason." That is, I reason that I have no idea how my opponent is going to play and therefore I arbitrarily assume that one-third of the time he is going to play his strategy 1, one-third his strategy 2, and one-third of the time his strategy 3. If I play my strategy 1 a fraction \(x\) of the time, my gain under these conditions is

\[
z = \frac{1}{3}(7-5x) + \frac{1}{3}(5-2x) + \frac{1}{3}(2+3x) = \frac{14}{3} - \frac{4}{3}x
\]

\(^1\)Ibid., p. 257. \(^2\)Ibid., pp. 257-258. \(^3\)Ibid., pp. 258-259.
In figure 1, this equation is represented by the dotted line. In order to maximize my gain, I should take x such that the corresponding z on the dotted line is the biggest. In this case, at x = 0, z has its maximum for the dotted line (z = 14/3). So I should use a pure strategy of playing only my strategy 2 if I use the principle of insufficient reason. However, if my opponent detects my pattern of playing my strategy 2, he can counteract by playing his strategy 3 all the time and limiting me to two dollars instead of the 14/3 dollars which I had hoped for.

I would like to find another method to play the game. I will call the method the conservative method, which I will now discuss. I know that, whatever number of times I play my strategy 1, i.e., for any value of x, I am certain to get at least the gain given by \( z = \min\{7-5x, 5-2x, 2+3x\} \). I want to select x so that this gain that I am certain to get is as large as possible. We may examine the graphs of the three solid lines and note the portions of each line that are below both of the other lines for some given interval. We may darken these portions. These darkened segments representing minimums for all choices of x form a rooflike structure (see figure 1). At the top of this "roof" is the point (call it D) at which my certain gain becomes the largest, i.e., it is the largest minimum. We can determine point D by computing the intersection of the two lines it lies on. The value of z for point D must be the same in both equations, or we have 5-2x = 2+3x, so x = .6. This is the fraction of the time I should play my strategy 1 on a random basis. The remaining time, i.e., .4 of the time, 

\[ \text{Ibid., p. 259.} \]
I should play my strategy 2. My expected gain is 5-2(.6) = 3.8 or 3.80 dollars. This gain is called the value of the game and is denoted by \( V \). The value of our game is given by:

\[
V = \max_{0 \leq x \leq 1} \min \{7-5x, 5-2x, 2+3x\}.
\]

In conclusion, if I play my strategy 1 .6 of the time and my strategy 2 .4 of the time, then whatever my opponent does, I can expect to get an average of at least 3.80 dollars in the long run.\(^1\)

I will now state the setup of this problem using a different mathematical notation. Denote \( x_1 \) as the fraction of the time I am to play my strategy 1 and \( x_2 \) as the fraction of the time I am to play my strategy 2. So we must have:

\[
x_1 + x_2 = 1.
\]

(1)

My gain will be given by one of the three expressions \((2x_1 + 7x_2), (3x_1 + 5x_2), (5x_1 + 2x_2)\), determined by whether my opponent plays his strategy 1, 2 or 3. I want to find the numbers \( x_1 \) and \( x_2 \) such that, when they are substituted into the three above expressions, the smallest of the resulting three numbers becomes a maximum. The value of the game is:

\[
V = \max_{x_1 + x_2 = 1} \min \{2x_1 + 7x_2, 3x_1 + 5x_2, 5x_1 + 2x_2\}.
\]

This problem can also be written in the following manner. Let \( v \) be a number such that:

\[
2x_1 + 7x_2 \geq v
\]

\[
3x_1 + 5x_2 \geq v
\]

\[
5x_1 + 2x_2 \geq v.
\]

\(^1\)Ibid., pp. 259-260.
If \( v \) denotes the number of dollars, then this number of dollars is smaller than or equal to any of the three numbers shown on the left-hand side of the above inequalities. If I mix my strategies 1 and 2 in proportions \( x_1 \) and \( x_2 \), I can expect to get at least \( v \) dollars, no matter what strategy my opponent plays. We can reformulate our original problem as the problem of finding the variables \( x_1, x_2 \) and \( v \) such that equation (1) and the inequalities of equation (2) hold true and \( v \) takes the largest possible value.\(^1\)

Let us now continue to assume that I play my strategy 1 a fraction \( x_1 \) of the time, and my strategy 2 a fraction \( x_2 \) of the time. Then \( x_1 + x_2 = 1 \). Furthermore, let us assume that my opponent uses his strategy 1 a fraction \( y_1 \) of the time and his strategy 2 a fraction \( y_2 \) of the time and his strategy 3 a fraction \( y_3 \) of the time. We must have \( y_1 + y_2 + y_3 = 1 \). For any value of \( x_1, x_2, y_1, y_2, \) or \( y_3 \), my expected gain is given by:

\[
z = 2x_1y_1 + 3x_1y_2 + 5x_1y_3 + 7x_2y_1 + 5x_2y_2 + 2x_2y_3.
\]

The notation for the expected gain is given by \( E(x_1, x_2; y_1, y_2, y_3) \).

With this notation, we can state:

\[
E(.6, .4; y_1, y_2, y_3) \geq 3.8. \quad \text{\(^2\)}
\]

Let us now discuss the strategy of my opponent. His problem is to prevent me from gaining too much money. If he plays his pure strategy 1, all he can be certain of is keeping me from getting more than seven dollars. If he plays pure strategy 2 or pure strategy 3, he can guarantee that I will not exceed five dollars. It will be seen

\(^1\)Ibid., p. 261. \(^2\)Ibid., pp. 262-263.
that he should mix strategies. Let us see what happens when he mixes strategies 1 and 3. What is the best way to mix these two strategies? Let C be the point of intersection of the lines which are the graphic representations of the strategies 1 and 3. Any line representing a mix of strategies must contain this point C also. Thus, no such line will be completely below the z-component of this point C. To insure that such a line is not above this z-component at any point, he can pick a strategy corresponding to a horizontal line through C.

Now we note that the intersection of the graphic representations of strategies 2 and 3 is at point D which is below point C, i.e., the z-component of point D is less than the z-component of point C. My opponent can use the mix as represented by the line through point D which is parallel to the x-axis as his best strategy. He would have rejected a mix of strategies 1 and 2 since the line representing such a mix of strategies would be at least partially above point D. Take for example the left endpoint of a line representing a mix of strategies 1 and 2. The left endpoint of the line representing strategy 1 has the value of its z-component equal to 7 and the left endpoint of the line representing strategy 2 has the value of its z-component equal to 5. So a line representing a mix of strategies 1 and 2 must have a left endpoint with a z-component the value of which is between 5 and 7. Since the value of the z-component of point D is less than 5, any line representing a mix of strategies 1 and 2 lies at least partially above point D. Player 1 can choose an x-component where the value of the z-component is greater than the value of the z-component of point D.

1 Ibid., pp. 263-264.
(e.g., player 1 can choose the x-component as \( x = 0 \)). However, if player 2 chooses the strategy represented by the horizontal line through point D, the value of the z-component is limited to the value of the z-component at point D, no matter which x-component is chosen by player 1.

So we wish to find the z-component of point D. We have already computed this figure while determining the best strategy for me. It was \( z = 3.8 \). We have used \( y_2 \) to represent the portion of the time that my opponent uses strategy 2 and \( y_3 \) to represent the portion of the time that my opponent uses strategy 3. We then have \( z = y_2(5-2x) + y_3(2+3x) \) and, in particular, the equation for the horizontal line through point D is \( 3.8 = y_2(5-2x) + y_3(2+3x) \) for all \( x \). Since my opponent is only using strategies 2 and 3, the equation \( y_1 + y_2 + y_3 = 1 \) becomes \( y_2 + y_3 = 1 \). Choosing \( x = 0 \), we have two equations in two unknowns:

\[
\begin{align*}
3.8 &= 5y_2 + 2y_3 \\
y_2 + y_3 &= 1.
\end{align*}
\]

The unique solution is \((y_2, y_3) = (0.6, 0.4)\). If my opponent plays his strategy \((0, 0.6, 0.4)\), he can be certain my expected gain is not going to be more than 3.80 dollars. We can write this as:

\[ E(0.6, 0.4, 0, 0.6, 0.4) = 3.8. \]

We can look at this same problem in another way. If I play my pure strategy 1, my gain is given by \( z = 2y_1 + 3y_2 + 5y_3 \). If I play my pure strategy 2, my gain is given by \( z = 7y_1 + 5y_2 + 2y_3 \). Assuming that he does not know which of these pure strategies I play (or in

\[ ^1 \text{Ibid., p. 265.} \]
which way I mix them), all he can expect to do is to limit me to the larger of the two numbers, or that he can keep me down to the gain of:

\[ z = \max_{y_1+y_2+y_3=1} \{2y_1 + 3y_2 + 5y_3, 7y_1 + 5y_2 + 2y_3\} \]

Since he wants to keep my gain down, he should select the numbers \(y_1, y_2, y_3\) so that my gain as shown by this equation becomes the smallest. That is, his problem is to select \(y_1, y_2\) and \(y_3\) by the equation:

\[ z = \min_{y_1+y_2+y_3=1} \max_{y_1+y_2+y_3=1} \{2y_1 + 3y_2 + 5y_3, 7y_1 + 5y_2 + 2y_3\} \]

Once more, let us take another view of my opponent's problem. Let \(w\) be a number such that \(2y_1 + 3y_2 + 5y_3 \leq w\) and \(7y_1 + 5y_2 + 2y_3 \leq w\). My opponent can expect to keep me down to, or below, \(w\) dollars. His problem is then to select \(y_1, y_2, y_3\) such that \(w\) becomes the smallest possible gain.\(^2\)

Sometimes, a problem involving how much a business firm should produce in the face of competition can be set up as a two-person zero-sum game. Consider the following situation: The Doe Company and the Roe Company become licensed to produce a toy train with a new type of engine. The Roe Company can make the train in either plastic or metal and in either "O" or "HO" scale. The Doe Company has no facilities to make plastic trains but can make either the "O" or "HO" model in metal. The licenses are received so close to the Christmas sales season, however, that neither can get more than one model into production this year. The management of the Doe Company studies the situation and estimates the amount by which the sales (in tens of thousands of

\(^1\text{Ibid., pp. 265-266.} \quad ^2\text{Ibid., p. 266.}\)
dollars) of the Doe Company will exceed the sales of the Roe Company if each makes only one model. The effects of other products in the lines of each company were considered in setting up the matrix. The matrix is biased in favor of the larger company (the Doe Company). The game is represented by the following matrix: 1

<table>
<thead>
<tr>
<th>Doe Company</th>
<th>Roe Company</th>
</tr>
</thead>
<tbody>
<tr>
<td>M-O</td>
<td>M-O 45</td>
</tr>
<tr>
<td></td>
<td>M-HO 35</td>
</tr>
<tr>
<td></td>
<td>P-O 70</td>
</tr>
<tr>
<td></td>
<td>P-HO 50</td>
</tr>
<tr>
<td>M-HO</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>30</td>
</tr>
</tbody>
</table>

This is a 2x4 game so it is possible to solve it by the graphical method described in this section. More commonly, both companies would have more than two strategies available to them. I will describe a method which can be used to solve such games in Section 5.

In the next section, I will describe the mixed strategy of the conservative method for the general two-person zero-sum game. A theorem will also be stated indicating the maximum gain I can guarantee I will receive is equal to the value of the gain my opponent can guarantee I do not exceed.

SECTION 2

GENERAL NOTATION AND STATEMENT OF THE MINIMAX THEOREM

In this section, the notation given in the example illustrated in Section 1 is generalized. Also the minimax theorem is stated. The general two-person zero-sum game with finite pure strategy sets can be expressed by the following:

1. There are two players, player 1 and player 2.
2. Player 1 has a set \( A = \{a_1, a_2, \ldots, a_m\} \) of \( m \) pure strategies.
3. Player 2 has a set \( B = \{\beta_1, \beta_2, \ldots, \beta_n\} \) of \( n \) pure strategies.
4. Associated to each pair of strategies \((a_i, \beta_j)\) is a payoff of \( M(a_i, \beta_j) \) units from player 2 to player 1. \( M(a_i, \beta_j) \) is written as \( a_{ij} \). Thus, the values to player 1 and player 2 of the strategy pair \((a_i, \beta_j)\) are \( a_{ij} \) and \(-a_{ij}\) units, respectively. Because these values sum to zero for every \((a_i, \beta_j)\) pair, the game is called zero-sum.
5. Player 1 may use a mixed strategy by employing \( a_1 \) with probability \( x_1 \), \( a_2 \) with probability \( x_2 \), \ldots, \( a_m \) with probability \( x_m \), where \( \sum_{i=1}^{m} x_i = 1 \) and \( x_i \geq 0 \). Such a strategy is represented by \( \vec{x} = (x_1, x_2, \ldots, x_m) \). The strategy \((0, 0, \ldots, 1, \ldots, 0)\), where 1 is the \( i \)th component of this vector, which places all the weight on \( a_i \), is considered to be the same as the pure strategy \( a_i \). The set of all mixed strategies for player 1 is designated \( X_m \) (where \( m \) indicates the
number of pure strategies available to player 1).

6. The mixed strategy for player 2 is denoted by \( \mathbf{y} = (y_1, y_2, \ldots, y_n) \), where \( \sum_{j=1}^{n} y_j = 1 \) and \( y_j \geq 0 \). The pure strategy \( \beta_j \) is considered to be the same as the mixed strategy \( (0, 0, \ldots, 1, \ldots, 0) \) where 1 is the \( j \)-th component of this vector. The set of all mixed strategies for player 2 is denoted by \( \mathcal{Y}_n \).

7. For each mixed strategy pair \((x, \mathbf{y})\) the payoff \( M(x, \mathbf{y}) \) to player 1 is defined to be:

\[
M(x, \mathbf{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j = \sum_{j=1}^{n} y_j \left( \sum_{i=1}^{m} x_i a_{ij} \right) = \sum_{i=1}^{m} x_i \left( \sum_{j=1}^{n} a_{ij} y_j \right).
\]

The payoff to player 2 is \(-M(x, \mathbf{y})\).

The symbol \( M(\alpha_i, \mathbf{y}) = \sum_{j=1}^{n} a_{ij} y_j \) means the payoff to player 1 when player 1 uses the pure strategy \( \alpha_i \) and player 2 uses \( \mathbf{y} \). Analogously, when player 1 uses \( \mathbf{x} \) and player 2 uses \( \beta_j \), the payoff is

\[
M(\mathbf{x}, \beta_j) = \sum_{i=1}^{m} a_{ij} x_i. \quad \text{Of course, } M(\alpha_i, \beta_j) = a_{ij}.
\]

8. We may denote the whole pure strategy game by the triplet \((\mathcal{A}, \mathcal{B}, M)\). The extension of \((\mathcal{A}, \mathcal{B}, M)\) to mixed strategies is the triplet \((\mathcal{X}_m, \mathcal{Y}_n, M)\).

9. Player 1's goal is to choose a mixed strategy \( \mathbf{x} \) from \( \mathcal{X}_m \) so as to maximize his return or, equivalently, to minimize player 2's return. The actual outcome of the game depends upon the players' joint actions. We are given the number \( M(\mathbf{x}, \mathbf{y}) \) for each pair \((\mathbf{x}, \mathbf{y})\) and player 1 tries to maximize \( M(\mathbf{x}, \mathbf{y}) \) by choosing \( \mathbf{x} \) and, at the same time,
player 2 tries to minimize \( M(\vec{x}, \vec{y}) \) by choosing \( \vec{y} \). The rules of the game require that each player choose his strategy in complete ignorance of his opponent's selection.

10. For each \( \vec{x} \) belonging to \( X \), player 1's security level is defined to be \( v_1(\vec{x}) = \min_{\vec{y}} M(\vec{x}, \vec{y}) \). Since \( M(\vec{x}, \vec{y}) = \sum_{j=1}^{n} y_j \left( \sum_{i=1}^{m} x_i a_{ij} \right) \), \( \sum_{j=1}^{n} y_j M(\vec{x}, \vec{y}) \) is a weighted average of the \( n \) payoffs \( M(x, \beta_j), j = 1, 2, \ldots, n \), it is minimized when all the weight is assigned to the least of these, i.e., \( v_1(\vec{x}) = \min \{ M(\vec{x}, \beta_1), M(\vec{x}, \beta_2), \ldots, M(\vec{x}, \beta_n) \} \). We may think of \( v_1(\vec{x}) \) as the return to player 1 if he tells player 2 that \( \vec{x} \) is his choice and if player 2 is allowed to choose his best response to \( \vec{x} \). If player 1 wishes to maximize his security level, he must choose a strategy \( \vec{x}(0) \) such that \( v_1(\vec{x}(0)) \geq v_1(\vec{x}) \), for all \( \vec{x} \) of \( X \). If we let \( v_1(\vec{x}(0)) = v_1 \), then \( v_1 = v_1(\vec{x}(0)) = \max_{\vec{x}} v_1(\vec{x}) = \max_{\vec{x}} \min_{\vec{y}} M(\vec{x}, \vec{y}) \).

\( v_1(\vec{x}(0)) = v_1 \) gives us \( M(\vec{x}(0), \vec{y}) \geq v_1 \), for all \( \vec{y} \); therefore, \( \vec{x}(0) \) guarantees to player 1 a return of no less than \( v_1 \). A strategy \( \vec{x}(0) \) which maximizes player 1's security level is called a maximin strategy for player 1. Maximin strategies always exist, but they do not have to be unique. We let \( \theta_1 \) designate the set of all maximin strategies for player 1. If \( \vec{x}^* \) belongs to \( \theta_1 \), then \( \vec{x}^* \) has a security level of \( v_1 \). If \( \vec{x}' \) does not belong to \( \theta_1 \), then \( \vec{x}' \) has a security level of less than \( v_1 \).

11. We may express player 2's goal as the minimization of player 1's return rather than the maximization of his own. If player 2 uses \( \vec{y} \), player 1 cannot get a return bigger than \( v_2(\vec{y}) = \max_{\vec{x}} M(\vec{x}, \vec{y}) \). Player 2 tries to minimize \( v_2(\vec{y}) \). Let \( \vec{y}(0) \) be such that \( v_2 = v_2(\vec{y}(0)) \leq v_2(\vec{y}) \), for all \( \vec{y} \) of \( Y_n \). Then \( v_2 = v_2(\vec{y}(0)) = \min_{\vec{y}} \max_{\vec{x}} M(\vec{x}, \vec{y}) \), and
\( M(\tilde{x}, \tilde{y}(0)) \leq v_2 \), for all \( \tilde{x} \). The strategy \( \tilde{y}(0) \) is called a minimax strategy for player 2. We let \( \theta_2 \) denote the set of all minimax strategies for player 2. If \( \tilde{y}^* \) belongs to \( \theta_2 \), then player 1 can be held down to at most \( v_2 \) by using \( \tilde{y}^* \). If a \( \tilde{y}' \) is used which does not belong to \( \theta_2 \), then it is possible for player 1 to get more than \( v_2 \).

12. If player 1 uses a maximin strategy, he assures himself a return of at least \( v_1 \) units. If player 2 uses a minimax strategy, he guarantees that player 1 cannot receive more than \( v_2 \) units. Therefore, \( v_1 \leq v_2 \).

13. A pair \((\tilde{x}', \tilde{y}')\) is said to be in equilibrium if \( \tilde{x}' \) is good against \( \tilde{y}' \) [i.e., \( M(\tilde{x}, \tilde{y}') \leq M(\tilde{x}', \tilde{y}') \) for all \( \tilde{x} \)] and if \( \tilde{y}' \) is good against \( \tilde{x}' \) [i.e., \( M(\tilde{x}', \tilde{y}) \leq M(\tilde{x}', \tilde{y}') \) for all \( \tilde{y} \)]. This may be written simply as \( M(\tilde{x}, \tilde{y}') \leq M(\tilde{x}', \tilde{y}') \leq M(\tilde{x}', \tilde{y}) \) for all \( \tilde{x} \) and \( \tilde{y} \), or equivalently, as \( \max_{\tilde{x}} M(\tilde{x}, \tilde{y}') = M(\tilde{x}', \tilde{y}') = \min_{\tilde{y}} M(\tilde{x}', \tilde{y}) \).

Using this notation, the minimax theorem is given below.

For any arbitrary finite strategy game, the following hold:

**Proposition 1:** An equilibrium pair exists.

**Proposition 2:** \( v_1 = v_2 \).

**Proposition 3:** There exists a triplet \((v, x(0), y(0))\) satisfying \( \sum_{i=1}^{m} a_{ij}x_i(0) \geq v \), for \( j = 1, 2, \ldots, n \) and \( \sum_{j=1}^{n} a_{ij}y_j(0) \leq v \), for \( i = 1, 2, \ldots, m \).

The significance of the minimax theorem is that, if player 1 uses his best mixed strategy, the value he can guarantee himself is equal to the value that player 2 can limit him to, when player 2 uses

---


2 Ibid., p. 390.
his own best mixed strategy, i.e., by computing the value of the gain for player 1 which he guarantees himself when he uses a maximin strategy, we obtain the value that player 1 is limited to when player 2 uses a minimax strategy.
SECTION 3

SADDLE POINTS AND ROW OR COLUMN DOMINATION

In this section, two methods which simplify finding the solution to a two-person zero-sum game will be examined.

If an element of a two-person zero-sum game matrix is the smallest in its row and largest in its column, it is called a saddle point. If \( a_{ij} \) is a saddle point of some game, then the \( i \)th pure strategy for player 1 is an optimal strategy for player 1, the \( j \)th pure strategy for player 2 is an optimal strategy for player 2, and \( a_{ij} \) is the value of the game. Let us look at an example of a game matrix:

\[
\begin{bmatrix}
-13 & -2 & 17 \\
10 & 8 & 14 \\
14 & 0 & 6
\end{bmatrix}
\]

\( a_{22} \) is the smallest element in row 2 and is the largest element in column 2. So \( a_{22} = 8 \) is a saddle point of this game. If player 1 uses his strategy 2, he can assure himself a gain of 8. If player 2 uses his strategy 2, he can keep player 1 from getting more than a gain of 8. In general, if we call a saddle point \( s \), player 1 can guarantee himself a value of \( s \), and player 2 can limit player 1 to a value of \( s \). So the value of the game must be \( s \).

In games that do not have a saddle point, we can apply the rule of dominance to simplify the game. This rule has two forms: 1) strict dominance and 2) nonstrict dominance (sometimes just called dominance).

\[\text{Llewellyn, pp. 298-299.}\]
Column r of a matrix game is said to strictly dominate column s if 
a_{ir} > a_{is} for every i. Row q strictly dominates row k if 
a_{kj} < a_{qj} for every j. If column r strictly dominates column s, column r can be 
eliminated from the matrix game without changing the set of optimal 
strategies for player 2. If row q strictly dominates row k, row k can 
be eliminated from the matrix game without changing the set of optimal 
strategies for player 1.¹

Column r is said to dominate (nonstrict sense) column s if 
a_{ir} > a_{is} for every i and a_{ir} > a_{is} for at least one i. Row q dominates 
row k if a_{kj} < a_{qj} for every j and a_{kj} < a_{qj} for at least one j. If 
column r dominates column s, column r may be eliminated since player 2 
can do just as good or better by playing column s instead of column r. 
If row q dominates row k, row k may be eliminated. But some optimal 
strategies may be lost in such cases.² Consider the matrix game:

\[
\begin{array}{ccc}
3 & 2 & 3 \\
3 & 2 & 1
\end{array}
\]

Row 1 dominates row s, so we may eliminate row 2. However, if we solve 
this 2x3 matrix game by the graphical method discussed in Section 1, 
we will obtain an optimal strategy for player 1 where strategy 1 should 
be used 50 percent of the time and strategy 2 should be used 50 percent 
of the time. If we eliminate row 2, we obviously lose this optimal 
strategy as a possibility.


SECTION 4

REDUCTION OF A GAME TO A LINEAR-PROGRAMMING PROBLEM

In this section, it will be seen how to reduce a two-person zero-sum game to a linear-programming problem.

Let us take a specific two-person zero-sum game with A, B and M defined as in Section 2. We want \( M(a_i, b_j) = a_{ij} > 0 \) for all i and j. We can guarantee this by adding a sufficiently large positive number \( a \) to each payoff. This changes the problem in an insignificant way as can be seen in the following: Let \( M' \) be the payoff of the game obtained by adding \( a \) to every entry of the payoff matrix \( M \). Also, let \( \tilde{x}' \) and \( \tilde{y}' \) be the corresponding maximum strategies for players 1 and 2 for the game with payoff \( M' \). By the minimax theorem, there are mixed strategies \( \tilde{x}' \) and \( \tilde{y}' \) such that \( M'(\tilde{x}', b_j) \geq v \geq M'(a_i, \tilde{y}') \) for all \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \). Also for all \( j = 1, \ldots, n \):

\[
M'(\tilde{x}', b_j) = \sum_i (a_{ij} + a)x'_i = \sum_i a_{ij}x'_i + \sum_i ax'_i = M(\tilde{x}', b_j) + a.
\]

We thus have \( M'(\tilde{x}', b_j) = M(\tilde{x}', b_j) + a \). By subtracting \( a \) from both sides of the equation, we obtain \( M(\tilde{x}', b_j) = M'(\tilde{x}', b_j) - a \). Similarly, we can obtain \( M(a_i, \tilde{y}') = M'(a_i, \tilde{y}') - a \). We can now write:

\[
M(\tilde{x}', b_j) = M'(\tilde{x}', b_j) - a \geq v - a \geq M'(a_i, \tilde{y}') - a
\]

\[
= M(a_i, \tilde{y}').
\]
If we write \((v-a)\) as \(v'\), our final result is:
\[
M(x', \beta_j) \geq v' \geq M(a_i, y')
\]
for all \(i = 1, \ldots, m\) and \(j = 1, \ldots, n\). This means that a solution of the original matrix game is the strategies \(x'\) and \(y'\) which is the same as the solution to the new matrix game. Also, any optimal strategy of the original game is an optimal strategy of the new game. Lastly, we note that the value of the original game is a less than the value of the new one.\(^1\)

We can now proceed to discuss the game with a payoff matrix with all positive entries. Player 1 can guarantee himself at least \(v^* (v^* > 0)\), if there exists an \(x^* = (x_1, x_2, \ldots, x_m)\) where \(x_i \geq 0\) and
\[
\sum_{i=1}^{m} x_i = 1 \text{ such that } M(x^*, \beta_j) \geq v^* \text{ for } j = 1, 2, \ldots, n
\]
which is equivalent to
\[
\sum_{i=1}^{m} a_{ij} x_i \geq v^*, \text{ for } j = 1, 2, \ldots, n.
\]
By dividing both sides of this inequality by \(v^*\) and writing \(x_i / v^* = u_i\), we see that player 1 can get at least \(v^*\) if there is a \(u^* = (u_1, u_2, \ldots, u_m)\) where \(u_i \geq 0\), for \(i = 1, 2, \ldots, m\) and \(\sum_{i=1}^{m} u_i = 1 / v^*\) such that
\[
\sum_{i=1}^{m} a_{ij} u_i \geq v^*, \text{ for } j = 1, 2, \ldots, n.
\]
we have player 1's problem as the following. Let \(U\) be the set of all \(m\)-tuples \(u^* = (u_1, u_2, \ldots, u_m)\) such that \(u_i \geq 0\), for \(i = 1, 2, \ldots, m\) and
\[
\sum_{i=1}^{m} a_{ij} u_i \geq 1, \text{ for } j = 1, 2, \ldots, n.
\]
Find those \(u^*\) belonging to \(U\) such that \(\sum_{i=1}^{m} u_i\) is a minimum.\(^2\)

---


\(^2\)Luce and Raiffa, p. 409.
If \( \mathbf{u} = (u_1, u_2, \ldots, u_m) \) belongs to \( U \), we have seen that player 1 can guarantee himself at least \( 1/\sum_i u_i \). Player 1 should attempt to find \( \mathbf{u} \) in \( U \) which maximizes \( 1/\sum_i u_i \) or minimizes \( \sum_i u_i \). The problem of minimizing a linear form such as \( \sum_i u_i \) (or more generally, \( \sum_i c_i u_i \)) subject to restrictions involving linear inequalities such as \( \sum_i a_{ij} u_i \geq 1 \) for \( j = 1, 2, \ldots, n \) (or more generally, \( \sum_i a_{ij} u_i \geq b_j \), for \( j = 1, 2, \ldots, n \)), where \( u_i \geq 0 \), \( i = 1, 2, \ldots, m \) is called a linear-programming problem (of the minimizing type).\(^1\)

In a similar fashion, by letting \( y_j/v^* = w_j \), we find that player 2's problem is to minimize \( 1/\sum_j w_j \), or equivalently, to maximize \( \sum_j w_j \) subject to \( w_j \geq 0 \), for \( j = 1, 2, \ldots, n \) and \( \sum_j a_{ij} w_j \leq 1 \) for \( i = 1, 2, \ldots, m \). As in the case for player 1's problem, player 2's problem is thus a linear-programming problem. The only difference is that it is of the maximizing type.\(^2\)

Consider the following two-person zero-sum game.\(^3\) Player R has two cards, one black and one red. He selects one. Without showing it to his opponent, he lays it down on the table. Player C then calls it. The card is turned over. If the card is called, R pays C a penny, otherwise he loses a penny. The payoff matrix is the following:

\[
\begin{array}{c|cc}
& \text{Call Black} & \text{Call Red} \\
\hline
\text{Choose Black} & -1 & 1 \\
\text{Choose Red} & 1 & -1 \\
\end{array}
\]

\(^1\text{Ibid., pp. 409-410.}\) \(^2\text{Ibid., pp. 410-411.}\) \(^3\text{George B. Dantzig, Linear Programming and Extensions (Princeton, New Jersey: Princeton University Press, 1963), p. 278.}\)
In order to convert this game to a linear-programming problem, it is first necessary to add a number $a$ to each entry so that we obtain a matrix with all positive entries. Let $a = 2$. The new matrix is:

$$
\begin{pmatrix}
1 & 3 \\
3 & 1 \\
\end{pmatrix}
$$

The value $v^*$ of this new matrix game will equal the sum of the value $v$ of the original matrix game and $a$ (which we have chosen to be equal to 2). Player 1's problem is to maximize $v^*$ subject to:

$$
\begin{align*}
& x_1 + 3x_2 \geq v^* \\
& 3x_1 + x_2 \geq v^* \\
& x_1 + x_2 = 1 \\
& x_1 \geq 0, \ x_2 \geq 0.
\end{align*}
$$

This is reduced to a linear programming problem by dividing both sides of (1), (2), (3), and (4) by $v^*$ and setting $u_1 = x_1/v^*$ and $u_2 = x_2/v^*$. We obtain:

$$
\begin{align*}
& \frac{x_1}{v^*} + 3\frac{x_2}{v^*} \geq 1 \\
& 3\frac{x_1}{v^*} + \frac{x_2}{v^*} \geq 1 \\
& \frac{x_1}{v^*} + \frac{x_2}{v^*} = \frac{1}{v^*} \\
& \frac{x_1}{v^*} \geq 0, \ \frac{x_2}{v^*} \geq 0.
\end{align*}
$$

Since we have defined $u_1 = \frac{x_1}{v^*}$ and $u_2 = \frac{x_2}{v^*}$, we now get:

$$
\begin{align*}
& u_1 + 3u_2 \geq 1 \\
& 3u_1 + u_2 \geq 1 \\
& u_1 + u_2 = \frac{1}{v^*} \\
& u_1 \geq 0, \ u_2 \geq 0.
\end{align*}
$$

Instead of maximizing $v^*$, we may minimize $1/v^* = z$. So our linear-programming problem is to minimize $z = u_1 + u_2$ subject to:
In the next section, a method for solving linear-programming problems will be discussed. Thus, we may solve two-person zero-sum games by converting them to linear-programming problems and then using a special method for solving linear-programming problems.

\begin{align*}
    u_1 + 3u_2 & \geq 1 \\
    3u_1 + u_2 & \geq 1 \\
    u_i & \geq 0 \quad (i = 1, 2)
\end{align*}
SECTION 5

THE SIMPLEX METHOD

In this section, a procedure for solving linear-programming problems (of the minimizing type) called the simplex method will be discussed. The problem is to find values for $u_1, u_2, \ldots, u_n$ satisfying the system of inequalities:

\[
\begin{align*}
    a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n & \geq b_1 \\
    a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n & \geq b_2 \\
    \vdots & \vdots \vdots \vdots \\
    a_{m1}u_1 + a_{m2}u_2 + \cdots + a_{mn}u_n & \geq b_m
\end{align*}
\]

and minimizing:

\[
c_1u_1 + c_2u_2 + \cdots + c_{n}u_{n} = z
\]

where the $u_j$ are restricted to be non-negative:

\[
u_j \geq 0 \quad (j = 1, 2, \ldots, n).
\]

Any set of such values is called a feasible solution. \(^1\)

The system of inequalities must be reduced to a system of equalities by subtracting what are called slack variables from the side of the inequalities in which the variables appear. The first slack variable is $u_{n+1}$ and the last one is $u_{n+m}$. Next, the system of equations must be arranged so that all constant terms are positive or zero. This is done by changing, where necessary, the signs on both sides of any of the equations. Some problems may have a feasible solution

\(^1\)Ibid., pp. 100-101.
already available. For those problems where this is not the case, a
basic set of variables (which will be called artificial variables),
\( u_{n+m+1} \geq 0, \ldots, u_{n+2m} \geq 0 \), is introduced into the system of equations
so that we obtain (the \( i \text{th} \) basic variable is defined to have a unit
coefficient in the \( i \text{th} \) equation and have zero coefficients elsewhere):
\[
\begin{align*}
\tilde{a}_{11} u_1 + \tilde{a}_{12} u_2 + \ldots + \tilde{a}_{1n} u_n + \tilde{a}_{1,n+1} u_{n+1} + u_{n+m+1} &= \tilde{b}_1 \\
\tilde{a}_{21} u_1 + \tilde{a}_{22} u_2 + \ldots + \tilde{a}_{2n} u_n + \tilde{a}_{2,n+2} u_{n+2} + u_{n+m+2} &= \tilde{b}_2 \\
& \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\tilde{a}_{m1} u_1 + \tilde{a}_{m2} u_2 + \ldots + \tilde{a}_{mn} u_n + \tilde{a}_{m,n+m+1} u_{n+m+1} + u_{n+2m} &= \tilde{b}_m \\
c_1 u_1 + c_2 u_2 + \ldots + c_n u_n + (-z) &= 0
\end{align*}
\]
where \( z \) is to be minimized, and where:
\[
\tilde{a}_{ij} = a_{ij} \text{ or } -a_{ij} \quad \text{for } 1 \leq j \leq n
\]
\[
\tilde{a}_{1,n+1} = 1 \text{ or } -1
\]
\[
\tilde{a}_{2,n+2} = 1 \text{ or } -1
\]
\[
\quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\tilde{a}_{m,n+m} = 1 \text{ or } -1
\]
\[
\tilde{b}_i = b_i \text{ or } -b_i \quad \text{for } 1 \leq i \leq m
\]
\[
u_j \geq 0 \quad \text{for } 1 \leq j \leq n+2m.
\]
We will use the simplex algorithm (which will be discussed later) to
find a solution to (1) and (2) which minimizes the sum of the artificial
variables denoted by \( t \) (this process is called Phase I of the algo-
rithm):
\[
u_{n+m+1} + u_{n+m+2} + \ldots + u_{n+2m} = t.
\]
We select as basic variables $u_{n+m+1}$, $u_{n+m+2}$, ..., $u_{n+2m}$, $(-z)$, $(-t)$ and eliminate these variables (except 5) from the t form by subtracting the sum of the first $m$ equations of (1) from (3). We obtain:

$$d_1u_1 + d_2u_2 + \ldots + d_{n+m}u_{n+m} - t = -t_0$$

(4)

where $d_j = -(a_{1j} + a_{2j} + \ldots + a_{mj})$ for $j = 1, 2, \ldots, n+m$ and $-t_0 = -(b_1 + b_2 + \ldots + b_m)$. Now we write the coefficients of the systems of equations made up by combining the equations of (1) and equation (4) in the form of a table that we call the initial tableau.¹

The next step is to apply Phase I of the simplex algorithm to the initial tableau. If there exists a feasible solution to the system of equalities (as it was before the artificial variables were introduced), then this solution also satisfies (1) and (2) with the artificial variables set equal to 0; thus, $t = 0$ in this case. From (3), the smallest possible value for $t$ is 0 since $t$ is the sum of nonnegative variables. Therefore, if feasible solutions exist, the minimum value of $t$ will be $t = 0$; conversely, if there is a solution for (1) and (2) with $t = 0$, all $u_{n+m+1} = 0$ and the values of $u_j$ for $j \leq n+m$ make up a feasible solution to the system of equalities (as it was before the artificial variables were introduced). On the other hand, if $\min t > 0$, then no feasible solutions to this system of equalities exist. Phase I is thus a procedure where a basic feasible solution (if one exists) is found.²

Denote $u_{\bar{i}}$ as the $i$th basic variable. A bar over a letter will indicate that it is the entry in some cycle $k$. Denote $\bar{z}_0$ as the constant term of the equation containing the variable $z$ in some cycle $k$.

¹Ibid., pp. 102-103. ²Ibid., p. 103.
We then use the following rules to apply the algorithm to the initial tableau and the additional resulting tableaux (at the end of Phase I, if \( \min t = 0 \), the algorithm is applied again to obtain a solution which minimizes the value of \( z \). This is called Phase II of the algorithm):

Step I:

(i) If all entries \( \overline{d}_j > 0 \) (in Phase I) or \( c_j > 0 \) (in Phase II), then for

(a) Phase I with \( \overline{t}_0 > 0 \): Stop--no feasible solution exists.

(b) Phase I with \( \overline{t}_0 = 0 \): Start Phase II by

(1) dropping all variables \( u_j \) with \( \overline{d}_j > 0 \),
(2) dropping the \( t \) row of tableau, and
(3) restarting cycle (Step I) using Phase II rules.

(c) Phase II: Stop--an optimal solution is

\[ u_{j_1} = b_1, \quad u_j = 0, \quad z = \overline{z}_0 \quad (j \neq j_1, \ i = 1, 2, \ldots, m). \]

(ii) If some entry \( \overline{d}_j < 0 \) (Phase I) or \( \overline{c}_j < 0 \) (Phase II), choose \( u_s \) as the variable to enter the basic set in the next cycle in place of the \( r \)th basic variable (\( r \) to be determined in Step II), such that

\[
\text{Phase I: } \overline{d}_s = \text{Min } \overline{d}_j < 0 \\
\text{Phase II: } \overline{c}_s = \text{Min } \overline{c}_j < 0
\]

Step II:

(i) If all entries \( \overline{a}_{is} < 0 \), terminate; the class of solutions

\[
\begin{align*}
\overline{u}_s & \geq 0 \text{ arbitrary} \\
\overline{u}_j & = \overline{b}_1 - \overline{a}_{is} \overline{u}_s \\
\overline{u}_j & = 0
\end{align*}
\]
(where $u_j$ represents a basic variable and $u^\dagger$ represents a non-basic variable for $j \neq s$) satisfies the original system and has the property $z = z_0 + \bar{c}_s u_s \to -\infty$ as $u_s \to +\infty$.

(ii) If some $\bar{a}_{is} > 0$, choose the $r$th basic variable to drop in the next cycle, where

$$\frac{\bar{b}_r}{\bar{a}_{rs}} = \text{Min} \ \frac{\bar{b}_i}{\bar{a}_{is}}$$

and $i$ and $r$ are restricted to those $i$ such that $\bar{a}_{is} > 0$. In case of ties, choose $r$ at random (with equal probability) from those $i$ which are tied.

Step III:

To obtain entries in the tableau for the next cycle from the current cycle, multiply each entry in the selected row $r$ by the reciprocal of the term $\bar{a}_{rs}$ (which is called the pivot term) and record the products in row $r$ of the next cycle. Enter the $r$th basic variable as $u$ in place of $u^\dagger$ of the current cycle. To get row $i$, column $j$ entry of the next cycle, subtract from the corresponding entry of the current cycle the product of the entry in row $i$, column $s$ of the current cycle and the entry in row $r$, column $j$ of the next cycle.¹

Consider the two-person zero-sum game that was converted to a linear-programming problem in the last section. We can now solve this game or any other game by solving its linear-programming problem through

¹Ibid., pp. 105, 107, 108.
use of the simplex method. The linear-programming problem that we had obtained for this game was:

\[
\text{Minimize } z = u_1 + u_2 \text{ subject to }
\]
\[
u_1 + 3u_2 \geq 1
\]
\[
3u_1 + u_2 \geq 1
\]
\[
u_i \geq 0 \quad (i = 1, 2)
\]

By using slack variables \(u_3 \geq 0\) and \(u_4 \geq 0\), we obtain the equations

\[
u_1 + 3u_2 - u_3 = 1
\]
\[
3u_1 + u_2 - u_4 = 1
\]

in place of inequalities (5) and (6). Next, the artificial variables \(u_5 \geq 0\) and \(u_6 \geq 0\) are introduced, giving us:

\[
u_1 + 3u_2 - u_3 + u_5 = 1
\]
\[
3u_1 + u_2 - u_4 + u_6 = 1
\]
\[
u_1 + u_2 - z = 0
\]
\[
u_i \geq 0 \quad \text{for } i = 1, 2, 3, 4, 5, 6
\]
\[
u_5 + u_6 = t
\]

Next, we subtract the sum of equations (7) and (8) from equation (10), giving us \(-4u_1 + u_3 + u_4 - t = -2\). It is now possible to write the initial tableau as the following:

Cycle 0

\[
\begin{array}{cccccccc}
& u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & -z & -t & \text{constant} \\
\hline
u_5 & 1 & 3 & -1 & 1 & & & 1 & \\
u_6 & 3 & 1 & -1 & 1 & & & 1 & \\
-z & 1 & 1 & & & & & 1 & \\
-t & -4 & -4 & 1 & 1 & & & 1 & -2
\end{array}
\]
We have $d_s = \min d_j = d_1 = -4$, so $s = 1$. We choose the $r$th basic variable to drop in the next cycle where $b_r/\bar{a}_r = \min b_i/\bar{a}_{is} = b_2/\bar{a}_{21} = 1/3$ for $\bar{a}_{is} > 0$. We multiply each entry in row $r = 2$ by the reciprocal of $\bar{a}_{rs}$ and record these products in row 2 of the next cycle. Next, we enter the $r$th basic variable as $u_s = u_1$ in place of $u_j = u_2 = u_6$. To obtain the other entries of the new cycle, we use the rule given in Step III of the simplex algorithm. For example, the entry in row 1 and column 2 of the new cycle (cycle 1) equals the difference of the corresponding entry in cycle 0 (i.e., 3) and the product of the entry in row $i = 1$ and column $s = 1$ of cycle 0 (i.e., 1) and the entry in row $r = 2$ and column $j = 1$ of cycle 1 (i.e., 8/3). So we have

$$3 - (1)(8/3) = 1/3.$$ We obtain cycle 1 as follows:

<table>
<thead>
<tr>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$u_5$</th>
<th>$u_6$</th>
<th>$-z$</th>
<th>$-t$</th>
<th>constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8/3</td>
<td>-1</td>
<td>1/3</td>
</tr>
<tr>
<td>$u_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>1/3</td>
<td>-1/3</td>
</tr>
<tr>
<td>$-z$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2/3</td>
<td>1/3</td>
<td>-1/3</td>
</tr>
<tr>
<td>$-t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-8/3</td>
<td>1</td>
<td>-1/3</td>
</tr>
</tbody>
</table>

Continue the process of Phase I for this tableau of cycle 1. Upon obtaining the tableau of cycle 2 as follows:

<table>
<thead>
<tr>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$u_5$</th>
<th>$u_6$</th>
<th>$-z$</th>
<th>$-t$</th>
<th>constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>-3/8</td>
<td>1/8</td>
</tr>
<tr>
<td>$u_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>1/8</td>
<td>-9/24</td>
</tr>
<tr>
<td>$-z$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6/24</td>
<td>6/24</td>
<td>-6/24</td>
</tr>
<tr>
<td>$-t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
we note that all \( \overline{d}_j \geq 0 \) and \( \overline{t}_0 = 0 \). So we initiate Phase II by dropping the \( t \) row of the tableau and restarting with Step I (of the algorithm) using Phase II rules. Since all \( \overline{c}_j \geq 0 \) for cycle 2, we see that we should terminate Phase II at this point and our optimal solution is \( u_{j_1} = u_2 = \overline{b}_1 = 2/8 \) and \( u_{j_2} = u_1 = \overline{b}_2 = 6/24 \), \( z = z_0 = 12/24 \). Lastly, going back to the original setup of the game illustrated in the last section, we get \( v^\star = 1/z = 1/1/2 = 2 \) \( (v = v^\star - a = 2 - 2 = 0) \) and \( x_1 = u_1 v^\star = 1/4(2) = 1/2 \) and \( x_2 = 1/4(2) = 1/2 \). Thus, the value for the original matrix game is 0 and the optimal strategy for player 1 is to play strategy 1 one-half of the time and to play strategy 2 one-half of the time.
SECTION 6

PROOF OF THE MINIMAX THEOREM BY USE
OF THE SIMPLEX METHOD

In this section, a proof of the minimax theorem (stated in
Section 2) based upon a special adaptation for games of the simplex
method will be discussed.

In our proof, it will be necessary to compare some vectors
"lexicographically." This term implies some type of ordering. A is
ordered after B if vector A is "greater" than vector B (written as
A > B). Specifically, A > B if A-B > 0, i.e., if A-B has non-zero
components, the first of which is positive.\footnote{1}

We know from statement (7) of Section 2 that, if the minimizing
player plays pure strategy j, then player 1's expected payoff becomes
\[ \sum_{i} a_{i}x_{i} \] where \( x_{i} \) is the probability of playing strategy i. Player 1
wishes to choose his \( x_{i} \) so that the smallest such sum (which will be
denoted by \( x_{0} \)) is a maximum. On the other hand, player 2 chooses a
mixed strategy \( y_{1}, y_{2}, \ldots, y_{n} \) such that the largest sum \( \sum_{j} a_{ij}y_{j} \) (which
will be denoted by \( y_{0} \)) is a minimum. The minimax theorem states that
there exists a choice for player 1 of \( \hat{x}_{i} = x_{i} \) and a choice for player 2
of \( y_{j} = \hat{y}_{j} \) such that the corresponding \( x_{0} = \hat{x}_{0} \) is the maximum value for
\( x_{0} \) and the corresponding \( y_{0} = \hat{y}_{0} \) is the minimum value for \( y_{0} \). Moreover,
\( \hat{x}_{0} = \hat{y}_{0} \). The common value of \( \hat{x}_{0} \) and \( \hat{y}_{0} \) is the value of the game.\footnote{2}


\footnote{2}Ibid.
Let \( x \) and \( y \) satisfy the following system of relations:

\[
\begin{align*}
    x_i &\geq 0 \quad (i = 1, \ldots, m) \\
    \sum_{i=1}^{m} x_i &= 1 \\
    x_0 &\leq \sum_{i=1}^{m} x_i a_{ij} \quad (j = 1, \ldots, n) \\
    y_j &\geq 0 \quad (j = 1, \ldots, n) \\
    \sum_{j=1}^{n} y_j &= 1 \\
    \sum_{j} a_{ij} y_j &\leq y_0 \quad (i = 1, \ldots, m).
\end{align*}
\]

If we multiply (3) through by any \( y_j \) satisfying (4), (5) and (6) and sum with respect to \( j \); similarly, multiply through (6) by any \( x_i \) satisfying (1), (2), (3) and sum with respect to \( i \), thus obtaining:

\[
    x_0 = x_0 \sum_{j} y_j \leq \sum_{i} \sum_{j} x_i a_{ij} y_j \leq y_0 \sum_{i} x_i = y
\]

so that the lower bounds \( x_0 \) never are greater than the upper bounds \( y_0 \).

A solution \( x_i = \hat{x}_i \) and \( y_j = \hat{y}_j \) can be constructed with the property that \( \hat{x}_0 = \hat{y}_0 \). For such a solution, (7) holds for \( \hat{y}_0 \) and any \( x_0 \) and also for \( \hat{x}_0 \) and any \( y_0 \). It follows therefore that:

\[
\begin{align*}
    x_0 &\leq y_0 = \hat{x}_0 \leq y_0 \\
    \hat{x}_0 &= \max x_0 \\
    \hat{y}_0 &= \min y_0
\end{align*}
\]

and the minimax theorem would be demonstrated.\(^1\)

---

\(^1\)Ibid., p. 27.
\(^2\)Ibid.
Consider the following matrix (call it matrix 1):

\[
\begin{bmatrix}
0 & 1 & \ldots & 1 & 0 & \ldots & 0 \\
-1 & a_{11} & \ldots & a_{1n} & 1 \\
& \vdots & \ddots & \vdots \\
-1 & a_{m1} & \ldots & a_{mn} & 0 & \ldots & 1
\end{bmatrix}
\]

The columns of this matrix will be denoted \( P_0, P_1, \ldots, P_n, P_{n+1} = U_1, \ldots, P_{n+m} = U_m \), where \( U_i \) are unit vectors with 1 as the \((i+1)st\) component. It will be helpful to arrange the rows of the matrix such that

\[
a_{ml} = \max_i a_{il}.
\]

Let \( B \) be a matrix formed by a subset of \( m+1 \) columns of matrix 1 including \( P_0 \) as the first column. \( B \) will be called a basis if it has an inverse such that each row of the inverse has its first non-zero component positive. Denote the inverse by \( B^{-1} \) and its rows by \( \beta_i \) \((i = 0, 1, \ldots, m)\). Thus, for \( B \) to be a basis, we must have \( \beta_i > 0 \) (in the lexicographic sense). For example, we may choose \( B = B_0 \) as consisting of the first two columns of the matrix and the unit vectors \( U_1, \ldots, U_{m-1} \). The matrix \( B_0 = [P_0, P_1, U_1, \ldots, U_{m-1}] = [P_0, P_1, P_{n+1}, \ldots, P_{n+m-1}] \) is non-singular and possesses the following inverse (call this inverse matrix 2):

\[
\begin{bmatrix}
a_{ml} & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 \\
b_1 & 1 & 0 & \ldots & -1 \\
b_2 & 0 & 1 & \ldots \\
& \vdots & \ddots & \ddots \\
b_{m-1} & 0 & 0 & 1 & -1
\end{bmatrix}
\]

\(^1Ibid.\)
Because of (9), it follows that \( b_i > 0 \) and thus \( \beta_i > 0.1 \).

Let the columns of a general basis be denoted by:

\[
B = [P_0, P_{j_1}, \ldots, P_{j_m}]
\]

and note that the conditions \( \beta_k P_{j_i} = 0 \) for \( i \neq k \) and \( \beta_i P_{j_i} = 1 \) for \( i, k = 0, 1, \ldots, m \) (\( j_0 = 0 \)) must hold between \( B \) and its inverse. We shall now prove the following theorem.

**Theorem:** If for all \( j = 1, 2, \ldots, n+m \) we have:

\[
\beta_0 p_j \leq 0,
\]

then the components of the 0-row and 0-column of \( B^{-1} \) yield the required optimal strategies.

**Proof:** Denote the components of the 0-row of \( B^{-1} \) by

\[
(\hat{x}_0, -\hat{x}_1, \ldots, -\hat{x}_m)
\]

and the components of the 0-column of \( B^{-1} \) by

\[
(\hat{y}_0, \hat{y}_{j_1}, \ldots, \hat{y}_{j_m}).
\]

We shall now demonstrate that an optimum mixed strategy for player 1 is obtained by setting \( x_i = \hat{x}_i \) for \( i = 1, 2, \ldots, m \); and one for player 2 by setting \( y_{j_i} = \hat{y}_{j_i} \) for \( j_i \leq n \) and \( y_j = \hat{y}_{j} = 0 \) for all other \( j \leq n \). The value of the game is \( \hat{x}_0 = \hat{y}_0 \). We already have the condition \( \beta_0 p_0 = 1 \).

The left side of this equality is obtained by finding the product

\[
(\hat{x}_0, -\hat{x}_1, \ldots, -\hat{x}_m)
\]

and the vector \((0, -1, \ldots, -1)\) which has \( m+1 \) components. It is easy to see that we obtain \( x_1 + x_2 + \ldots + x_m \). So it is easy to see that \( \beta_0 p_0 = 1 \) is the same as (2). Moreover, \( \beta_0 p_j \leq 0 \) for \( 1 \leq j \leq n \) are the same as (3), while for \( n+1 \leq j \leq n+m \), they are the same as (1). \( \beta_i > 0 \) for \( i = 1, \ldots, m \) implies that the first component

\[\text{Ibid.}, \text{ p. 28.}\]
of $\beta_1$ (which by definition is $\hat{y}_{j_1}$) is non-negative; thus (4) is satisfied. Multiplying $B$ on the right by 0-column of $B^{-1}$ gives us $m+1$ linear expressions in $(\hat{y}_0, \hat{y}_{j_1}, \ldots, \hat{y}_{j_m})$ which may be equated to the unit vector $u_0$.\footnote{Ibid., pp. 28-29.}

The first of these $m+1$ linear equations gives us (5), since the first components of $P_{j_1}$ are equal to 1 for $1 \leq j \leq n$ and zero otherwise, i.e., for our equation, the coefficient of $\hat{y}_{j_1}$ where $1 \leq j_1 \leq n$ is 1. Also, we already know that the value of all other $\hat{y}_j$, where $1 \leq j \leq n$, is zero. Lastly, the coefficients of any remaining $\hat{y}_{j_1}$'s are zero. So we have $\hat{y}_1 + \hat{y}_2 + \ldots + \hat{y}_n = 1$ which is equivalent to (5).

The remaining $m$ equations give us the inequalities (6) if the terms involving $j_1 > n$ are dropped (the latter are non-negative because $\hat{y}_{j_1} > 0$ and their coefficients are the components of the unit vectors $P_{n+1}$). The proof is completed by noting that $\hat{x}_0 = \hat{y}_0$ since both are defined as being the element which is in the first row and the first column of $B^{-1}$ \footnote{Ibid., p. 29.}.

We can see that our problem is to construct a basis $B$ with the property that $\beta_0 P_j < 0$ for $j = 1, 2, \ldots, n+m$ since this yields an optimal mixed strategy for each player. We shall now show that, if some basis $B$, such as $B_0$, does not satisfy (10), then it is easy to construct from $B$ a new basis $B^*$ which differs from $B$ by only one column where 0-row of $[B^*]^{-1}$ (which we denote by $\beta_0^*$) has the property that:

$$\beta_0 > \beta_0^*, \quad (11)$$

that is, the first non-zero component of $\beta_0 - \beta_0^*$ is positive. If the new basis $B^*$ does not satisfy (10), then the algorithm just outlined
for \( B \) is iterated, with \( B \) replaced by \( B^* \). This process generates a sequence of bases which terminates when a basis is obtained that satisfies the necessary property. This must occur in a finite number of steps, since the condition (11) is a strict inequality which insures that no basis can be repeated and the number of different bases cannot exceed the number of ways of choosing \( m \) columns out of \( n+m \) columns from matrix \( 1 \). The 0-column of successive bases of the iterative process may be interpreted as a succession of improved mixed strategies for player 2. For these strategies, his expected loss \( y_0 \), if his opponent is playing optimally, is decreasing to a minimum. The components of the first column of any basis satisfy (4) and (5) independent of (10), while \( y_0 \), the first component of \( \beta_0 \), is non-increasing from basis to basis due to (11).¹

To construct \( B^* \) from \( B \), let \( P_s \) denote the column of matrix \( 1 \) which replaces the \( r \)th column after \( P_0 \) of \( B \) where \( P_s \) and \( P_r \) are determined by the following rules. Choose \( P_s \) so that:

\[
\beta_0 P_s = \max \beta_0 P_j \text{ for all } j = 1, \ldots, n+m \quad (12)
\]

such that \( \beta_0 P_s > 0 \). Find the column vector \( V = \{v_0, v_1, \ldots, v_m\} \) satisfying \( BV = P_s \). By multiplying both sides of this equality by \( B^{-1} \) on the left, we obtain \( V = B^{-1} P_s \). It can be seen that:

\[
v_i = \beta_i P_s \text{ for } i = 0, 1, \ldots, m \quad (13)
\]

where, in particular, \( v_0 = \beta_0 P_s > 0 \) from (12). We now choose to drop from \( B \) that column \( P_r \) such that the lexicographic minimum of the vectors \( (1/v_i)\beta_i \) for \( v_i > 0 \) is attained for \( i = r \). Thus,

\[
(1/v_r)\beta_r = \min (1/v_i)\beta_i \quad (v_r > 0, v_i > 0) \quad (14)
\]

¹Ibid.
where \( i, r \neq 0 \) and where it is assumed for the moment that there is at least one \( v_i > 0 \). The minimizing vector is easily obtained in practice by finding the vector whose first component is the least; if there is a tie, then one passes to the second components of the tying vectors and selects the least, etc.\(^1\)

By (14), \((1/v_r)\beta_r < (1/v_i)\beta_i\) for \( i \neq r \). So \((v_i/v_r)\beta_r = v_i(1/v_r)\beta_r < v_i(1/v_i)\beta_i = \beta_i\). I.e., \((v_i/v_r)\beta_r < \beta_i\) which gives us:

\[
\beta_i - (v_i/v_r)\beta_r > 0 \quad (v_i > 0).
\]

we will use this relation later.\(^2\)

It is easy to see from the structure of matrix \( P \) that the first column \( P_0 \) cannot be formed as a positive linear combination of the other columns \( P_j \). However, if we assume, contrary to the assumptions of (14), that all \( v_i \leq 0 \), \( (i \neq 0) \) and write:

\[
P_s = BV = v_0P_0 + \sum v_iP_{ji},
\]
then by transposing to the left all terms other than \( v_0P_0 \), we obtain a positive linear combination of columns \( P_s \) and \( P_j \) that yields \( v_0P_0 \), where we know \( v_0 > 0 \); which yields a contradiction.\(^3\)

Up to this point, it has been shown that the iterative process is finite and must end. However, it has not been shown that this implies that a solution must be obtained through the iterative process. Assume that all possible iterations have been performed. If we have not reached a solution, then (10) has not been satisfied and we have \( \beta_0P_j > 0 \) for some \( j \). This means that it is possible to find a \( P_s \) so that \( \beta_0P_s = \max \beta_0P_j \) for all \( j \) such that \( \beta_0P_j > 0 \). Using this \( P_s \) and (13) and (14), we can determine a \( P_{jr} \) giving us a new basis. This

\(^1\)Ibid., pp. 29-30. \(^2\)Ibid., p. 30. \(^3\)Ibid.
would mean that the iterative process had not been completed which yields a contradiction to our assumption. Therefore, a solution must be obtained by the end of the iterative process.

We only have left to show that $B^*$ satisfies $\beta_* > 0$ and (11).

We construct $[B^*]^{-1}$ from $B^{-1}$ using the relations

$$
\begin{align*}
\beta^*_{ik} &= \beta_{ik} - (v_i / v_r)\beta_{rk} \quad (i \neq r) \\
\beta^*_{ri} &= (1 / v_r)\beta_{rk} \\
\end{align*}
$$

(16)

where $\beta^*_{ik}$ is the $i$th row of $[B^*]^{-1}$. To verify that (16) is the inverse of $B^*$, we note from (16) that, for $i \neq r$, the values $\beta^*_{ikj}$ are the same as $\beta_{ikj} = 0$ (or 1 if $i = k$). To see this is true, let us first look at $\beta^*_{ikj}$ for $k \neq i$. We then have $\beta^*_{ikj} = [\beta_{ik} - (v_i / v_r)\beta_{rk}]_{jk} = \beta_{ikj} - (v_i / v_r)\beta_{rjk}$. We already know that $\beta_{ikj} = 0$ for $k \neq i$. This also gives us $\beta_{rjk} = 0$, since we are given that $r \neq i$. So $\beta^*_{ikj} = 0$ for $k \neq i$. When $k = i$, we have $\beta^*_{ikj} = \beta^*_{i1j} = [\beta_{i1} - (v_i / v_r)\beta_{r1}]_{1j} = \beta_{i1j} - (v_i / v_r)\beta_{r1j}$. We know from before that $\beta_{i1j} = 1$ and we have just seen that $\beta_{r1j} = 0$. So $\beta^*_{i1j} = 1$ for $i = k$. Also, we have $\beta^*_{irs} = 1$ and $\beta^*_{irs} = 0$ (for $i \neq r$). These two equalities follow by using the definition of $v_i$ given in (13).

$$
\begin{align*}
\beta^*_{irs} &= (1 / v_r)\beta_{irs} = (1 / \beta_{irs})\beta_{irs} = 1 \\
\end{align*}
$$

and

$$
\begin{align*}
\beta^*_{irs} &= [\beta_{i1} - (v_i / v_r)\beta_{r1}]_{i1} = \beta_{i1} - (v_i / v_r)\beta_{r1} \\
&= \beta_{i1} - v_i \beta^*_{irs} = \beta_{i1} - v_i (1) = \beta_{i1} - v_i \\
&= \beta_{i1} - \beta_{i1} = 0 .
\end{align*}
$$

1Ibid., pp. 30-31.
The properties necessary of $\beta^*_i$ are immediately seen: The first non-zero component of $\beta^*_r$ is positive because $\beta_r$ has this property and $v_r > 0$. Next, for all other $i = 1, 2, \ldots, m$ the property must hold if $v_i \leq 0$ since $\beta^*_i$ is the sum of two vectors with this property. If $v_i > 0$, then $\beta^*_i > 0$ by (15) and (16). Lastly, the relation $\beta_0 > \beta^*_0$ (and not $\beta_0 \geq \beta^*_0$) holds because $\beta_r$, a row of a non-singular matrix, possesses at least one non-zero component and $\beta^*_0$ is formed by subtracting from $\beta_0$ a vector $(v_0/v_r)\beta_r$ where $v_0 > 0$, $v_r > 0$; therefore, (11) holds and the proof is complete.

The following is an example of how to solve a game matrix by this simplex method. Solve the 3x6 game matrix $M$:

$$
M = \begin{bmatrix}
4 & 3 & 3 & 2 & 2 & 6 \\
6* & 0 & 4 & 2 & 6 & 2 \\
0 & 7 & 3 & 6 & 2 & 2 \\
\end{bmatrix}
$$

Element $a_{21}$ of $M$ has been starred. It will be noted that this is the maximal element in the first column. For convenience, the second and third rows have been interchanged so that this element appears in the bottom position of this column in forming the matrix $[P_0, \ldots, P_9]$ which follows:

$$
\begin{bmatrix}
P_0 & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\
-1 & 4 & 3 & 3 & 2 & 2 & 6 & 1 & & \\
-1 & 0 & 7 & 3 & 6 & 2 & 2 & 1 & & \\
-1 & 6 & 0 & 4 & 2 & 6 & 2 & 1 \end{bmatrix}
$$

1Ibid., p. 31. 2Ibid.
The initial basis, \( B = B_0 \), consists of \( P_0, P_1, P_7 = U_1 \), \( P_8 = U_2 \). The inverse of \( B_0 \) (given below) is determined by examining matrix 2. For example, the element labeled \( b_{m-1} \) in matrix 2 is found by finding \( a_{m1} - a_{(m-1)1} = 6 - 0 = 6 \). The element \( b_1 = a_{m1} - a_{11} = 6 - 4 = 2 \).

\[
B_0^{-1} = \begin{bmatrix}
\beta_0 \\
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix} = \begin{bmatrix}
6 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
2 & 1 & 0 & -1 \\
6 & 0 & 1 & -1
\end{bmatrix}
\]

Next, \( P_s = P_2 \) is determined by computing each \( \beta_0^P_j \) (\( j = 1, \ldots, 9 \)).

\[
\beta_0^P_1 = (6, 0, 0, -1) (1, 4, 0, 6)^T = 0
\]
\[
\beta_0^P_2 = (6, 0, 0, -1) (1, 3, 7, 0)^T = 6
\]
\[
\beta_0^P_3 = 2
\]
\[
\beta_0^P_4 = 4
\]
\[
\beta_0^P_5 = 0
\]
\[
\beta_0^P_6 = 4
\]
\[
\beta_0^P_7 = 0
\]
\[
\beta_0^P_8 = 0
\]
\[
\beta_0^P_9 = -6
\]

So \( \beta_0^P_2 = \beta_0^P_2 = \max_{j \neq 0} \beta_0^P_j = 6 > 0 \). The entries \( v_i = \beta_1^P_s \) can now be computed.

\[
v_0 = \beta_0^P_2 = (6, 0, 0, -1) (1, 3, 7, 0)^T = 6
\]
\[
v_1 = \beta_1^P_2 = (1, 0, 0, 0) (1, 3, 7, 0)^T = 1
\]
\[
v_2 = \beta_2^P_2 = 5
\]
\[
v_3 = \beta_3^P_2 = 13.
\]
The column \( r \) to be dropped from the basis is determined by forming the lexicographic minimum of the vectors \( (1/v_r)\beta_i \).

\[
(1/v_1)\beta_1 = 1/1 (1, 0, 0, 0) = (1, 0, 0, 0)
\]

\[
(1/v_2)\beta_2 = 1/5 (2, 1, 0, -1) = (2/5, 1/5, 0, -1/5)
\]

\[
(1/v_3)\beta_3 = 1/13(6, 0, 1, -1) = (6/13, 0, 1/13, -1/13)
\]

So \( (1/v_r)\beta_r = (1/5)\beta_2 = \min (\text{lexico}) (1/v_i). \) We should drop \( v_{i>0,i\neq0} \) column \( r = 2 \); that is, \( P_7 \).

We will now perform the first iteration. The next basis \( B^* - B_1 \) is \( [P_0, P_1, P_2, P_8] \). To obtain its inverse, we use:

\[
\beta'_i = \beta_i - (v_i/v_r), \ (i \neq r) \text{ and } \beta'_r = (1/v_r)\beta_r
\]

where \( r = 2 \), where the superscript (in place of \( * \)) refers to the basis \( B = B_k \). For example, \( \beta'_1 = \beta_1 - (v_1/v_2)\beta_2 = (1, 0, 0, 0) - 1/5(2, 1, 0, -1) = (3/5, -1/5, 0, 1/5) \).

\[
P_s = P_5 \] is determined by \( \beta'_0 P_s = \beta'_0 P_5 = \max_{j \neq 0} \beta'_0 P_j = 12/5 > 0 \). We can then compute \( v_0 = 12/5, v_1 = 7/5, v_2 = -2/5, v_3 = 36/5 \) and \( P_j = P_3 \).

\( P_8 \) is determined by \( (1/v_r)\beta'_r = (5/36)\beta'_3 = \min (\text{lexico}) (1/v_i)\beta'_i. \) We are now able to make the second (final) iteration.

---

\textsuperscript{1}Ibid., pp. 31-32.
\[ B_2 = \begin{bmatrix} P_0 & P_1 & P_2 & P_5 \\ 0 & 1 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ -1 & 0 & 7 & 2 \\ -1 & 6 & 0 & 6 \end{bmatrix} \]


where no \( P_s \) can be determined since \( \beta_0^2 P_j \leq 0 \) for \( j \geq 1 \). For example, \( \beta_0^2 P_1 = (50/15, -5/15, -5/15, -5/15) (1, 4, 0, 6) = 0 \). In fact, \( \beta_0^2 P_j = 0 \) for all \( j \geq 1 \). Thus, an optimal solution has been obtained (from top row), \( \bar{x}_1 = 5/15, \bar{x}_2 = 5/15, \bar{x}_3 = 5/15 \) and (from first column) \( \bar{y}_1 = 16/36, \bar{y}_2 = 16/36, \bar{y}_5 = 4/36 \), where all other \( \bar{y}_j = 0 \). The value of the game (from upper left corner) is \( \bar{x}_0 = \bar{y}_0 = 50/15 \).\(^1\)

\(^1\)Ibid., pp. 32-33.
SECTION 7

A DUALITY THEOREM

The following are various forms of linear-programming problems. Let $\vec{b}$ be a given n-tuple, $\vec{c}$ be a given m-tuple, and let $a_{ij}$ (for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$) denote the element in the $i$th row and the $j$th column of a given $m \times n$ matrix. We can write $\vec{b} = (b_1, b_2, \ldots, b_n)$ and $\vec{c} = (c_1, c_2, \ldots, c_m)$.

The Minimization Problem

Let $U$ be the set of all $m$-tuples $\vec{u} = (u_1, u_2, \ldots, u_m)$ such that:

$$u_i \geq 0, \quad \text{for } i = 1, 2, \ldots, m \quad (1)$$

$$\sum_{j=1}^{n} a_{ij} u_j \geq b_j \quad \text{for } j = 1, 2, \ldots, n \quad (2)$$

Find those $\vec{u}$ belonging to $U$ such that the linear form $c_1 u_1 + c_2 u_2 + \ldots + c_m u_m$ is a minimum.

The Maximization Problem

Let $W$ be the set of all $n$-tuples $\vec{w} = (w_1, w_2, \ldots, w_n)$ such that:

$$w_j \geq 0, \quad \text{for } j = 1, 2, \ldots, n \quad (3)$$

$$\sum_{i=1}^{m} a_{ij} w_i \leq c_i \quad \text{for } i = 1, 2, \ldots, m \quad (4)$$

Find those $\vec{w}$ belonging to $W$ such that the linear form $b_1 w_1 + b_2 w_2 + \ldots + b_n w_n$ is a maximum.

The Symmetric Problem

To find those pairs $(\vec{u}, \vec{w})$ where $\vec{u}$ belongs to $U$ and $\vec{w}$ belongs to $W$ such that:
The minimization and maximization problems are called duals of each other.\(^1\)

The following theorem deals with the above linear-programming problems. The principal theorem of linear-programming is:

1. If there exists a \(\mathbf{u}\) in \(U\) and a \(\mathbf{w}\) in \(W\), then for such \(\mathbf{u}\), \(\mathbf{w}\):

\[
c_1 u_1 + \ldots + c_m u_m \geq b_1 w_1 + \ldots + b_n w_n.
\]

2. If \(\left(\mathbf{u}^{(0)},\mathbf{w}^{(0)}\right)\) is a solution to the symmetric problem, then \(\mathbf{u}^{(0)}\) is a solution to the minimization problem and \(\mathbf{w}^{(0)}\) is a solution to the maximization problem.

3. If \(\mathbf{u}^{(0)}\) is a solution to the minimization problem and \(\mathbf{w}^{(0)}\) is a solution to the maximization problem, then \(c_1 u_1^{(0)} + \ldots + c_m u_m^{(0)} = b_1 w_1^{(0)} + \ldots + b_n w_n^{(0)}\), i.e., \(\left(\mathbf{u}^{(0)},\mathbf{w}^{(0)}\right)\) is a solution to the symmetric problem.

4. If the solution exists to one problem, then solutions exist to the other two problems.

5. If both \(U\) and \(W\) are non-empty, then all three problems have solutions.

**Proof of 1.**

We obtain:

\[
\sum_j b_j w_j \leq \sum_i \left(\sum_i a_{i,j}\right) u_j
\]

by multiplying the \(j\)th inequality of (2) by \(w_j\) and then summing over all \(j\).

---

\(^1\)Luce and Raiffa, pp. 412-413.
We obtain:
\[
\sum_j \left( \sum_i u_i a_{ij} \right) w_j = \sum_i \left( \sum_j a_{ij} w_j \right) u_i
\]
by simply changing the order of summation.

Lastly, we obtain:
\[
\sum_i \left( \sum_j a_{ij} w_j \right) u_i < \sum_i c_i u_i
\]
by multiplying the \textit{i}th inequality of (4) by \( u_i \) and then summing over all \( i \).

Proof of 2.

If \( (\tilde{u}^{(0)}, \tilde{w}^{(0)}) \) is a solution to the symmetric problem, then \( \tilde{u}^{(0)} \) belongs to \( U \) and \( \tilde{w}^{(0)} \) belongs to \( W \). By the inequality of (1.), the linear form for \( \tilde{u}^{(0)} \) must be a minimum and the linear form for \( \tilde{w}^{(0)} \) must be a maximum. Therefore, \( \tilde{u}^{(0)} \) and \( \tilde{w}^{(0)} \) are solutions of their respective problems (i.e., for \( c_1 u_1 + \ldots + c_m u_m \geq b_1 w_1 + \ldots + b_n w_n \), \( c_1 u_1 + \ldots + c_m u_m \) would be its smallest at \( c_1 u_1 + \ldots + c_m u_m = b_1 w_1 + \ldots + b_n w_n \) and \( b_1 w_1 + \ldots + b_n w_n \) would be the biggest possible under the same conditions).\(^1\)

To prove 3., 4., and 5., we will use the reduction of a linear-programming problem to a game. Consider the game matrix given on the following page.

This game matrix is skew-symmetric. The value of such games can be shown to be zero. Consequently, the mixed strategy:
\[
(z_1^{(0)} \beta_1, \ldots, z_j^{(0)} \beta_j, \ldots, z_n^{(0)} \beta_n, z_{n+1}^{(0)} \beta_{n+1}, \ldots, z_{n+m}^{(0)} \beta_{n+m}, \ldots, z_{n+m+1}^{(0)} \beta_{n+m+1})
\]

\(^1\)Ibid., pp. 413-414.
is minimax for player 2 if and only if it holds player 1 down to 0, that is, if and only if:

\[
[z^{(0)}_{n+1}a_{i} + \ldots + z^{(0)}_{n+1}a_{i} + \ldots + z^{(0)}_{n+m}a_{ij} + z^{(0)}_{n+m+1}b_{j}] \leq 0 \quad (5)
\]

\[
j = 1, \ldots, n
\]

\[
[z^{(0)}_{1}a_{11} + \ldots + z^{(0)}_{j}a_{ij} + \ldots + z^{(0)}_{n}a_{in}] - z^{(0)}_{n+m+1}c_{i} \leq 0 \quad (6)
\]

\[
i = 1, \ldots, m
\]

\[-[z^{(0)}_{1}b_{1} + \ldots + z^{(0)}_{n}b_{n}] + [z^{(0)}_{n+1}c_{1} + \ldots + z^{(0)}_{n+m}c_{m}] \leq 0 \quad (7)
\]

There exist two possibilities for this matrix game. By examining the two cases, we will see that this matrix game is the solution to the
linear-programming problems, assuming solutions exist at all.

Case 1: There exists a minimax strategy for player 2 with \( z_{n+m+1}^{(0)} > 0 \).

Dividing each inequality of (5), (6), and (7) by \( z_{n+m+1}^{(0)} \) and denoting

\[
rac{z_{n+1}^{(0)}}{z_{n+m+1}^{(0)}} \quad \text{by} \quad u_i^{(0)} \quad \text{for} \quad i = 1, 2, \ldots, m
\]

\[
rac{z_j^{(0)}}{z_{n+m+1}^{(0)}} \quad \text{by} \quad w_j^{(0)} \quad \text{for} \quad j = 1, 2, \ldots, n
\]

we find that

\[
\begin{aligned}
\mathbf{u}^{(0)} &= (u_1^{(0)}, u_2^{(0)}, \ldots, u_m^{(0)}) \quad \text{belongs to} \ U \\
\mathbf{w}^{(0)} &= (w_1^{(0)}, w_2^{(0)}, \ldots, w_n^{(0)}) \quad \text{belongs to} \ W \\
\begin{bmatrix} w_1^{(0)} b_1 + \cdots + w_n^{(0)} b_n \end{bmatrix} &\geq u_1^{(0)} c_1 + \cdots + u_m^{(0)} c_m
\end{aligned}
\]  

(8)

by making the substitutions using \( u \) and \( w \).

From 1. of the principal theorem of linear-programming and (8), we obtain

\[
c_1 u_1^{(0)} + \cdots + c_m u_m^{(0)} = b_1 w_1^{(0)} + \cdots + b_n w_n^{(0)},
\]

i.e., \( \mathbf{u}^{(0)}, \mathbf{w}^{(0)} \) is a solution of the symmetric problem.

By 2. of the theorem, \( \mathbf{u}^{(0)} \) and \( \mathbf{w}^{(0)} \) must then be solutions of the maximizing and minimizing problems, respectively.

Case 2: There does not exist a minimax strategy for player 2 with \( z_{n+m+1}^{(0)} > 0 \). For this case, we will show the following:

**Proposition 1:** Either \( U \) or \( W \) is empty.

**Proposition 2:** If \( W \) is non-empty, then the linear form:

\[
w_1 b_1 + w_2 b_2 + \cdots + w_n b_n,
\]

where \( \mathbf{w} \) is in \( W \),

\[^1\text{Ibid., p. 420.}\]
can be made arbitrarily large (i.e., the maximization problem has no solution).

**Proposition 3:** If U is non-empty, then the linear form:

\[ u_1c_1 + u_2b_2 + \ldots + u_mc_m, \text{ where } u \text{ is in } U, \]

can be made arbitrarily small.\(^1\)

Once we have shown that Propositions 1, 2 and 3 hold true, then we will have also shown that the solutions of our matrix game provide solutions to the linear-programming problems provided solutions exist at all. Also, the three remaining parts of the principal theorem follow easily. For, if U and W are both non-empty or if solutions exist to either the maximizing or minimizing problems, then Case 2 does not hold, but when Case 1 holds, there is a minimax solution of the game which yields solutions to all three versions of the linear-programming problem.\(^2\)

Before we establish Propositions 1, 2 and 3, we will prove three remarks which are valid for Case 2.

**Remark 1:** The proof depends upon the following claim about our game matrix. If all minimax strategies of player 2 result in a return of exactly zero against \(a_{n+m+1}\), then (since player 1 can gain at most zero if player 2 uses his minimax strategy) player 1 has a maximin strategy which puts positive weight on \(a_{n+m+1}\). However, by the symmetry of the problem, this would mean that player 2 has a minimax strategy which puts positive weight on \(b_{n+m+1}\) and, under the assumption of Case 2, this cannot be.

\(^1\text{Ibid., pp. 420-421.}\) \(^2\text{Ibid., p. 421.}\)
Thus, there is a minimax strategy $\hat{x}^{(0)}$ for player 2 which gives player 1 a return less than zero against $a_{n+m+1}$. This means

$$-\sum_{j=1}^{n} z_{j}^{(0)} b_{j} + \sum_{i=1}^{m} z_{i}^{(0)} c_{i} < 0,$$

i.e.,

$$\sum_{i=1}^{m} z_{i}^{(0)} c_{i} < \sum_{j=1}^{n} z_{j}^{(0)} b_{j}.$$

**Remark 2:** If there exists a $\hat{w}'$ in $W$ (i.e., $W$ is non-empty), then for $z^{(0)}$ of Remark 1, we show

$$\sum_{j=1}^{n} z_{j}^{(0)} b_{j} > 0.$$  

We first establish the following statements:

$$\sum_{j=1}^{n} z_{j}^{(0)} b_{j} > \sum_{i=1}^{m} z_{i}^{(0)} c_{i},$$

by Remark 1,

$$\sum_{i=1}^{m} z_{i}^{(0)} c_{i} \geq \sum_{j=1}^{n} z_{j}^{(0)} \left(\sum_{i=1}^{m} a_{ij} w'_j\right),$$

by the fact that $\hat{w}'$ is in $W$.

We now have:

$$\sum_{j=1}^{n} z_{j}^{(0)} b_{j} > \sum_{j=1}^{n} w'_j \left(\sum_{i=1}^{m} z_{i}^{(0)} a_{ij}\right) > 0.$$
This means that
\[ \sum_{j=1}^{n} z_j^{(0)} b_j > 0. \]

Remark 3: If there exists a \( u' \) in \( U \) (i.e., \( U \) is non-empty) then for \( z^{(0)} \) of Remark 1, we want to show that
\[ \sum_{j=1}^{n} z_j^{(0)} b_j \leq 0. \]

We obtain:
\[ \sum_{j=1}^{n} z_j^{(0)} b_j \leq \sum_{j=1}^{n} z_j^{(0)} \left( \sum_{i=1}^{m} a_{ij} u'_i \right) = \sum_{i=1}^{m} u'_i \left( \sum_{j=1}^{n} z_j^{(0)} a_{ij} \right) \]
where the inequality follows from the fact that \( u' \) is in \( U \) and the equality follows from a summation interchange. Since
\[ \sum_{j=1}^{n} z_j^{(0)} a_{ij} \leq 0 \quad \text{for all } i \]
by (6) and the fact that we have \( z_{m+n+1}^{(0)} = 0 \), and for \( u' \) in \( U \), \( u'_i \geq 0 \) for all \( i \), we then have:
\[ \sum_{i=1}^{m} u'_i \left( \sum_{j=1}^{n} z_j^{(0)} a_{ij} \right) \leq 0 \]
which gives us
\[ \sum_{j=1}^{n} z_j^{(0)} b_j \leq 0. \]

Now we return to Propositions 1, 2 and 3. Proposition 1 that \( U \) and \( W \) cannot both be non-empty is true since otherwise Remark 1 and Remark 2 would both hold true and Remarks 1 and 2 contradict each other.
Proposition 2, that even if \( W \) is non-empty no maximum exists, is proved as follows:

If \( \mathbf{w}' = (w'_1, \ldots, w'_j, \ldots, w'_n) \) lies in \( W \), then so does

\[
\left( w'_1 + \lambda z_1^{(0)}, \ldots, w'_j + \lambda z_j^{(0)}, \ldots, w'_n + \lambda z_n^{(0)} \right)
\]

for \( \lambda > 0 \), since

\[
\sum_{j=1}^{n} a_{ij}(w'_j + \lambda z_j^{(0)}) = \sum_{j=1}^{n} a_{ij}w'_j + \lambda \sum_{j=1}^{n} a_{ij}z_j^{(0)} \leq c_i + \lambda \cdot 0 = c_i \quad \text{for all } i.
\]

We get

\[
\sum_{j=1}^{n} a_{ij}w'_j \leq c_i
\]

by \( \mathbf{w}' \in W \) and we get

\[
\sum_{j=1}^{n} a_{ij}z_j^{(0)} < 0
\]

by \( z_n^{(0)} = 0 \) and (6). But the linear form for this point is:

\[
\sum_{j=1}^{n} b_j(w'_j + \lambda z_j^{(0)}) = \sum_{j=1}^{n} b_jw'_j + \lambda \left( \sum_{j=1}^{n} b_jz_j^{(0)} \right)
\]

and, by Remark 2 (i.e., \( \sum_{j=1}^{n} b_jz_j^{(0)} > 0 \)), the linear form for this point can be made arbitrarily large by making \( \lambda \) large enough. Proposition 3, that even if \( U \) is non-empty no minimum exists, is proved similarly. It requires the dual of Remark 2, which is:

If there exists a \( \mathbf{u}' \) in \( U \), then

\[
\sum_{i=1}^{n} z_i^{(0)} c_i < 0
\]

(we obtain this by using Remark 3, i.e., if there exists a \( \mathbf{u}' \) in \( U \), then
\[
\sum_{j=1}^{n} z^{(0)} b_j \leq 0.
\]

Going back to Remark 1, we get from all this:
\[
\sum_{i=1}^{m} z^{(0)} c_i < \sum_{j=1}^{n} z^{(0)} b_j \leq 0;
\]
so
\[
\sum_{i=1}^{m} z^{(0)} c_i < 0 \quad \text{which was what we wanted}. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
A nutrition pill manufacturer wishes to supply the entire dietary requirements by marketing in the drug stores a pure caloric pill and a pure vitamin pill at prices that will not only compete with similar "foods" 1 and 2 offered in the grocery store but will be a cheaper source of nutritional needs than any food on the market. What price should he charge in order to maximize his revenues? Let \( \pi_1 \) be the price he charges per calorie pill and \( \pi_2 \) be the price per vitamin pill (each pill = 100 units). Then the dual problem is the following:

\[
\begin{align*}
-\pi_1 & \leq 20 \\
-\pi_2 & \leq 20 \\
-\pi_1 - 2\pi_2 & \leq 31 \\
-\pi_1 - \pi_2 & \leq 11 \\
-2\pi_1 - \pi_2 & \leq 12 \\
\pi_1 & \leq 0 \\
\pi_2 & \leq 0 \\
-21\pi_1 - 12\pi_2 & = v \text{ (max)}.
\end{align*}
\]

It will be convenient to substitute \(-y_1\) for \(\pi_1\). We then obtain:

\[
\begin{align*}
y_1 & \leq 20 \\
y_2 & \leq 20 \\
y_1 + 2y_2 & \leq 31 \\
y_1 + y_2 & \leq 11 \\
2y_1 + y_2 & \leq 12 \\
y_1 & \geq 0
\end{align*}
\]
\[ y_2 \geq 0 \]
\[ 21y_1 + 12y_2 = v \text{ (max)}. \]

For any inequality of (9), such as \( y_1 + 2y_2 \leq 31 \), the sum of the terms on the left side of the inequality represents the cost to the housewife if she simulates the type of food in question by purchasing nutrition pills with equal amounts of nutritional elements. Also, for any such inequality, the quantity to the right (i.e., the constant term) represents the cost to her if, instead, she buys the food. In each case, the inequality indicates that it is required that it cost no more to buy the simulated food. Thus, it is seen that the economic problem of the housewife has a dual problem which is the economic problem for the pill manufacturer.\(^1\)

\(^1\)Ibid., p. 263.
SECTION 8

PRICE EQUILIBRIUM

In this section, a mathematical model of price equilibrium is discussed. A theorem relating to price equilibrium is now introduced.

Equilibrium Theorem

The feasible solutions \( w_1, \ldots, w_n \) and \( u_1, \ldots, u_m \) of (2) and (4) of Section 7 (for the minimum and maximum linear-programming problems, respectively) are optimal solutions if and only if:

\[
u_i = 0 \quad \text{whenever} \quad \sum_{j=1}^{n} w_j a_{ij} < c_i \quad (1)\]

\[
w_j = 0 \quad \text{whenever} \quad \sum_{i=1}^{m} u_i a_{ij} > b_j \quad (2)\]

Proof: Assume (1) and (2) hold. Multiplying the \( i \)th inequality of (4) of Section 7 by \( u_i \) and summing on \( i \), and using (1), gives:

\[
\sum_{i=1}^{m} u_i c_i = \sum_{i=1}^{m} u_i \sum_{j=1}^{n} w_j a_{ij} = \sum_{i,j} w_j u_i a_{ij} \quad (3)
\]

Similarly, from (2) of Section 7 and (2) of this section, we get:

\[
\sum_{j=1}^{n} w_j b_j = \sum_{j=1}^{n} w_j \sum_{i=1}^{m} u_i a_{ij} = \sum_{i,j} w_j u_i a_{ij} \quad (4)
\]

From (3) and (4), we have:

\[
\sum_{j=1}^{n} w_j b_j = \sum_{i=1}^{m} u_i c_i
\]
By 2. of the principal theorem of linear-programming (given in Section 7), we have \( \mathbf{u}^+ = (u_1, \ldots, u_m) \) and \( (w_1, \ldots, w_n) = \mathbf{w}^+ \) as solutions to

the minimization problem and maximization problem, respectively. That is, \((u_1, \ldots, u_m)\) and \((w_1, \ldots, w_n)\) are optimal solutions. Conversely, if \( \mathbf{w}^+ \) and \( \mathbf{u}^+ \) are optimal solutions, then from 3. and 1. of the principal theorem of linear-programming, we obtain:

\[
\sum_{j=1}^{n} w_j b_j = \sum_{i,j} w_j u_{ij} a_{ij} = \sum_{i=1}^{m} u_i c_i.
\]

(5)

From the first equation of (5), we have

\[
\sum_{j=1}^{n} w_j (b_j - \sum_{i=1}^{m} u_{ij} a_{ij}) = 0
\]

but since the numbers \( u_{ij} \) are feasible, i.e.:

\[
\sum_{i=1}^{m} u_{ij} a_{ij} \geq b_j,
\]

it follows that the terms

\[
(b_j - \sum_{i=1}^{m} u_{ij} a_{ij})
\]

are non-positive and therefore (since \( w_j \geq 0 \)) for each \( j \):

\[
w_j (b_j - \sum_{i=1}^{m} u_{ij} a_{ij}) = 0;
\]

so if

\[
\sum_{i=1}^{m} u_{ij} a_{ij} > b_j
\]

we have

\[
b_j - \sum_{i=1}^{m} u_{ij} a_{ij} < 0
\]

and therefore, \( w_j = 0 \). Thus, (2) is satisfied.
We can use a symmetrical argument to prove (1).  

A linear production model is defined to be a model $P$ involving $n$ goods if it consists of a set of activities $P_1, \ldots, P_n$. Such a model is completely described by an array of $mn$ numbers $a_{ij}$ where $a_{ij}$ is the amount of goods $G_j$ produced (or consumed, if $a_{ij}$ is negative) when $P_j$ is operated at unit level. This array of numbers is called the production matrix of the model.

\[
\begin{array}{cccc}
& P_1 & \ldots & P_n \\
G_1 & a_{11} & \ldots & a_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
G_m & a_{m1} & \ldots & a_{mn}
\end{array}
\]

The activity $P_j$ is being operated at the level or intensity $w_j$ if its inputs and outputs are given by the numbers $w_j a_{ij}, \ldots, w_j a_{mj}$. A production schedule for $P$ is defined to be a set of non-negative intensities $w_1, \ldots, w_n$ for the activities $P_j$. Given these numbers $w_j$, we see that the total amount of $G_1$ produced is the sum of the amounts produced by each of the activities and is given by the expressions $w_1 a_{11} + w_2 a_{12} + \ldots + w_n a_{1n}$, where this quantity may be negative, which simply means that the $i$th good is being consumed rather than produced.  

Consider a linear production model where we let $b_j \geq 0$ be the rate of return or income associated with the activity $P_j$. Assume there is a given fixed supply $s_i$ of the $i$th good. The problem is now to find

\[1\]Gale, pp. 19-20. \[2\]Ibid., pp. 5-6.
a production schedule $w_1, \ldots, w_n$ which will maximize the total income without exceeding the given supplies. Symbolizing this, we want to maximize:

$$
\max \sum_{j=1}^{n} w_j b_j
$$

subject to the conditions

$$
- \sum_{j=1}^{n} w_j a_{ij} \leq c_i
$$

where the negative sign is due to the fact that we are taking supplies as positive and thus the amount $c_i$ consumed in production is not to exceed $c_i$.\(^1\)

Think of the maximizing linear-programming problem as the production problem discussed above. We shall now interpret the equilibrium theorem economically and justify the use of the word "equilibrium." It is natural to interpret the dual variables $u_i$ as prices, and have the feasibility conditions of the minimizing linear-programming problem correspond to the requirement that no activity makes a positive profit. (1) has the following interpretation. It says that, if the cost of an activity exceeds the income derived from it, then it will not be used, i.e., it will be operated at level zero. Equation (4) of Section 7 and (2) of this section together may be thought of as stability conditions in the following sense. If the model is operating at activity levels $w_1, \ldots, w_n$ and these conditions are satisfied, then there will be no incentive to change the activity levels since there is no way of increasing income. Looked at the other way, if either of these conditions

\(^1\)Ibid., pp. 7-8.
failed to hold, then activity levels would be unstable, for the producer could increase his income by changing the production levels.\textsuperscript{1}

As to (4) of Section 7 and (1) of this section, the first is simply the technological requirement that the available supply must not be exceeded. Equation (1) states that, if there are goods of which there is a surplus, that is, whose supply is not exhausted, then the price of these goods must be zero. This is also a stability condition, this time on prices rather than on activity levels. According to the classical "law of supply and demand," if the supply of a good exceeds the demand for it, then its price will drop. On the other hand, prices cannot drop below zero and therefore a good which is oversupplied, even when income is being maximized, must become a free good.\textsuperscript{2}

\textsuperscript{1}Ibid., pp. 20-21. \textsuperscript{2}Ibid., p. 21.
BIBLIOGRAPHY


