RELIABILITY PREDICTION, REDUNDANCY AND DEMONSTRATION

A graduate project submitted in partial satisfaction of the requirements for the degree of Master of Science in Engineering

by

Orville Lloyd Dopps

January, 1975
The graduate project of Orville Lloyd Dopps is approved:

Committee Chairman

California State University, Northridge
January, 1975
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section/Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approval Page</td>
<td>ii</td>
</tr>
<tr>
<td>Abstract</td>
<td>v</td>
</tr>
<tr>
<td><strong>Section I - Statistical Distributions for Reliability Prediction</strong></td>
<td></td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>General Reliability Concepts</td>
<td>1</td>
</tr>
<tr>
<td>Definitions</td>
<td>1</td>
</tr>
<tr>
<td>Mathematical Relationships</td>
<td>2</td>
</tr>
<tr>
<td>Confidence Limits</td>
<td>4</td>
</tr>
<tr>
<td>The Exponential Distribution</td>
<td>5</td>
</tr>
<tr>
<td>Properties of the Exponential Distribution</td>
<td>5</td>
</tr>
<tr>
<td>Parameter Estimation for the Exponential Distribution</td>
<td>6</td>
</tr>
<tr>
<td>Confidence Limits for the Exponential Distribution</td>
<td>7</td>
</tr>
<tr>
<td>The Weibull Distribution</td>
<td>8</td>
</tr>
<tr>
<td>Properties of the Weibull Distribution</td>
<td>8</td>
</tr>
<tr>
<td>Parameter Estimation for the Weibull Distribution</td>
<td>10</td>
</tr>
<tr>
<td>Confidence Limits for the Weibull Distribution</td>
<td>11</td>
</tr>
<tr>
<td>The Normal Distribution</td>
<td>12</td>
</tr>
<tr>
<td>Applicability of Distributions</td>
<td>14</td>
</tr>
<tr>
<td><strong>Section II - Reliability Redundancy Considerations</strong></td>
<td>18</td>
</tr>
<tr>
<td><strong>Section III - A Bayesian/Classical Approach to Reliability Demonstration</strong></td>
<td>28</td>
</tr>
<tr>
<td>The Bayesian Method in Reliability Demonstration</td>
<td>28</td>
</tr>
<tr>
<td>The Reliability Prediction Used as Prior Information</td>
<td>28</td>
</tr>
<tr>
<td>The Bayesian/Classical Reliability Demonstration</td>
<td>30</td>
</tr>
<tr>
<td>Case Histories</td>
<td>33</td>
</tr>
<tr>
<td>Conclusions</td>
<td>40</td>
</tr>
<tr>
<td>References</td>
<td>42</td>
</tr>
<tr>
<td>Appendix A</td>
<td>44</td>
</tr>
<tr>
<td>Appendix B</td>
<td>47</td>
</tr>
</tbody>
</table>

iii
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Comparison of Distribution Characteristics</td>
<td>17</td>
</tr>
<tr>
<td>2</td>
<td>Single Three-Fold Voting</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>Multiple Voting with Single Voters</td>
<td>19</td>
</tr>
<tr>
<td>4</td>
<td>Multiple Voting with Duplicated Voters</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>Probability of Success for N-Cross-Strapped Elements</td>
<td>22</td>
</tr>
<tr>
<td>6</td>
<td>Probability of Success for N-Cross-Strapped Elements, $T = 5$ years</td>
<td>23</td>
</tr>
<tr>
<td>7</td>
<td>Expansion of Figure 6</td>
<td>24</td>
</tr>
<tr>
<td>8</td>
<td>Probability of Success for N-Cross-Strapped Elements, $T = 7$ years</td>
<td>25</td>
</tr>
<tr>
<td>9</td>
<td>Probability of Success for N-Cross-Strapped Elements, $T = 10$ years</td>
<td>26</td>
</tr>
<tr>
<td>10</td>
<td>Expansion of Figure 9</td>
<td>27</td>
</tr>
<tr>
<td>11</td>
<td>Test Plan - Example 1</td>
<td>33</td>
</tr>
<tr>
<td>12</td>
<td>Test Plan - Case History 1</td>
<td>34</td>
</tr>
<tr>
<td>13</td>
<td>Test Plan - Case History 2</td>
<td>34</td>
</tr>
<tr>
<td>14</td>
<td>Test Plan - Case History 3</td>
<td>37</td>
</tr>
</tbody>
</table>
ABSTRACT

RELIABILITY PREDICTION, REDUNDANCY AND DEMONSTRATION

by

Orville Lloyd Dopps

Master of Science in Engineering

January, 1975

The purpose of this report, is to provide some insight into the so called "Black Magic" world of Reliability Engineering. How does an engineer predict the reliability of a system? Can a systems reliability be improved? How many hours of test are required to demonstrate this reliability? Some of the methods used to answer these questions are presented in the three sections of this report.

The properties and application of the most common statistical distributions used to predict reliability are discussed in Section I.

Section II discusses one type of redundancy to consider for increasing reliability of a unit composed of several elements.

Description of a reliability demonstration method which requires the use of a reliability prediction with its measure of uncertainty, the coefficient of variation, is presented in Section III. When used in conjunction with a classical test plan, the Bayesian/Classical (B/C) test reduces the number of test hours required to reach a decision.

Three case histories, which represent actual reliability demonstration testing, compare results using the Bayesian/Classical test with those of the original demonstration test. This report also describes the establishment of accept/reject criteria for use in applying the Bayesian/Classical test method in situations where such decision criteria have not been preselected.
SECTION I - STATISTICAL DISTRIBUTIONS FOR RELIABILITY PREDICTION

INTRODUCTION

If a sample of like components is selected and placed on test until all have failed, the data obtained will be a set of positive numbers in the time domain representing the life-lengths of the components tested. If these components are all operated under the same environmental and stress conditions, the time to failure for an arbitrarily selected component will be a random variable. This is true because all components will not fail at the same time due to variations in materials, tolerances, manufacturing processes, and other factors which cannot be uniformly controlled.

A distribution of the times to failure for a sample of items can be obtained by plotting the times to failure in the order of their magnitude as a function of their respective probabilities of failure. A smooth curve fitted to the data points will give the actual failure distribution function for the sample data. However, there is no assurance that a plot of the data obtained from future observations will even resemble the general form of this function, unless extremely large sample sizes are used. If there is reasonable assurance that the data fit a specific type of statistical distribution, data of a relatively small sample size can be used to make reliability predictions with a high degree of statistical confidence. The technique generally used is to select a statistical distribution having a small number of parameters and then to estimate the values of the parameters in terms of the sample data. The life qualities used as a measure of reliability are then expressed in terms of these parameters.

GENERAL RELIABILITY CONCEPTS

Definitions

Before discussing formula derivations, it will be necessary to define some basic reliability concepts. The most common ones are as follows:

Let $N =$ Initial Sample Size

$R(t) =$ Reliability, also called probability_of_survival, is
defined as the probability that an item will perform its assigned task for a given time under the conditions specified. It is equal to the fraction of items surviving up to time t.

\[ f(t) = \text{Probability density function} \]
\[ F(t) = \text{Cumulative distribution function} \]

\[ F(t) = \text{Cumulative distribution function, which is the probability that a failure will occur prior to time } t. \text{ It is equal to the fraction of items failing before time } t. \text{ Graphically, it is the area under the curve of the probability density function truncated at time } t. \]

\[ h(t) = \text{Conditional probability of failure, or hazard rate, is defined as the limit of the ratio of the fraction of items failing per unit time during a given time interval to the fraction surviving up to the beginning of that interval.} \]

Mean life (mean time to failure, also called Mean Time Between Failures, MTBF, in the case of a large number of devices operating simultaneously) of a distribution of lifetimes is defined as the first moment about the vertical axis of the total cumulative distribution function.

Median life is defined as the time at which half of the population will have failed. It is equal to the value of \( t \) for which \( R(t) = \frac{1}{2} \).

**Mathematical Relationships**

Since the reliability function and the cumulative distribution function must add up to unity,

\[ R(t) = 1 - F(t) \]

The value of the cumulative distribution function is obtained by summing the probability density function from time zero to time \( t \). Therefore, for a continuous distribution

\[ F(t) = \int_{0}^{t} f(t) \, dt \]
It follows from the above that the probability density function is equal to the first derivative, with respect to \( t \), of the cumulative distribution function,

\[
f(t) = \frac{d}{dt} F(t)
\]

The definition of the hazard rate expressed in terms of mathematical symbols is as follows:

\[
h(t) = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t \cdot (1 - F(t))} = \frac{d}{dt} \frac{F(t)}{1 - F(t)} = \frac{f(t)}{R(t)}
\]

Where \( \Delta t \) is an arbitrarily small increment of time \( t \).

A mathematical expression for the reliability is derived from the equation for the hazard function. It follows from the definition of the hazard rate that

\[
h(t) = \frac{\frac{d}{dt} F(t)}{1 - F(t)} = \frac{\frac{d}{dt} [1 - R(t)]}{R(t)}
\]

This is equivalent to

\[
-h(t) \, dt = \frac{d[R(t)]}{R(t)}
\]

Integrating both sides of the above from 0 to \( t \),

\[
- \int_0^t h(t) \, dt = \int_0^t \frac{dR(t)}{R(t)} = \ln R(t)
\]

If we take the antilogarithm to the base \( e \) of both sides of the above equation, we have an expression for the reliability in terms of the hazard rate,

\[
R(t) = e^\int_0^t h(t) \, dt
\]

It was shown before that \( h(t) = f(t)/R(t) \). Rearranging this equation by solving for \( f(t) \), we obtain

\[
f(t) = h(t)R(t)
\]

If the statistical distribution of lifetimes under discussion is continuous, it follows from the definition that the mean life is equal to the integral from 0 to \( t \) of the probability density function multiplied by \( t \); therefore,

\[
\text{Mean Life} = \int_0^t \! t \, f(t) \, dt = \int_0^t \! t \, h(t) \, R(t) \, dt
\]

The above formula is based on the theoretical definition of the
population mean of a continuous distribution.

An estimate of the mean life (M) from the sample data is given by the formula $M = \sum t_i/N$, where $i$ is the time at which the $i$th failure occurred and $N$ is the sample size.

**Confidence Limits**

When a statistical distribution is fitted to a sample of items, the sample seldom represents the total population; however, the approximate values of the population distribution parameters are usually determined from the sample data by estimation procedures. Unless the sample size is equal to the total population size, there is some error involved in estimating the population parameters from the sample data. By statistical methods, it is possible to measure the magnitude of this error with a specified degree of assurance. This degree of assurance is called the confidence level. Confidence level is defined as the degree of desired trust or assurance in a given result. It measures the probability that a given assertion is true. It can be the probability that a parameter is not greater than a certain value. This value is called an upper confidence limit. It can also be the probability that a parameter is not less than a certain value, called the lower confidence limit. In either of these cases the value specified is a one-sided confidence limit. A third possibility is that the confidence level can be the probability that a parameter will fall between two specified values, the upper confidence limit and the lower confidence limit. In this case we have two-sided confidence limits. The range of values included between the confidence limits is the confidence interval. If $\theta$ is the parameter we are estimating, $\theta_L$ is a lower confidence limit on $\theta$ and $\theta_U$ is an upper confidence limit on $\theta$, and if the confidence level is chosen to be 0.9, then the probability is 90 percent that

$$\theta_L < \theta < \theta_U$$

Confidence limits for the exponential and Weibull distributions are discussed later in the report. For a discussion of confidence limits for the normal distribution, the reader is referred to references 2, 5, 6, 7, and 12.
THE EXPONENTIAL DISTRIBUTION

Properties of the Exponential Distribution

The most simple form of failure distribution is a one-parameter distribution of life times called the exponential distribution. This distribution is based on the assumption of a hazard function \( h(t) \) equal to a constant value for all values of time \( t \). The expression for the hazard function is represented by

\[
h(t) = \lambda, \quad \text{where } \lambda > 0
\]

The constant \( \lambda \) is the only parameter of the exponential distribution. Its value is estimated from the sample data, and knowledge of the value of \( \lambda \) together with the sample size will uniquely determine the form of this distribution and all life qualities associated with it. The reliability for the exponential distribution is given by the equation

\[
R(t) = e^{-\int_0^t h(t) dt} = e^{-\int_0^t \lambda dt} = e^{-\lambda t}
\]

Then the cumulative distribution function becomes

\[
F(t) = 1 - R(t) = 1 - e^{-\lambda t}
\]

The probability density function for this distribution is

\[
f(t) = h(t)R(t) = \lambda e^{-\lambda t}
\]

The value of the mean life for the exponential distribution is found as follows:

\[
\text{Mean life } \theta = \int_0^\infty t f(t) dt = \int_0^\infty t \lambda e^{-\lambda t} dt = \frac{1}{\lambda}
\]

It is seen that for the exponential distribution the mean life (mean time to failure) is equal to the reciprocal of the hazard rate.

It is important to emphasize that the exponential distribution is the only distribution for which this relationship is true. This reciprocal relationship between the mean life and the hazard rate enables us to express the reliability, the cumulative distribution function, and the probability density function in terms of the mean life. Thus,

\[
R(t) = e^{-\frac{t}{\theta}}
\]
\[ F(t) = 1 - e^{-\frac{t}{\theta}} \]

and

\[ f(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}} \]

Notice that when \( t = \theta \), the fraction failed, \( F(t) \), is equal to \( 1 - e^{-1} \) or 0.632.

The median life for the exponential distribution is obtained from the definition by solving \( R(t) = \frac{1}{2} \) for \( t \). Thus,

\[ e^{-\frac{t}{\theta}} = \frac{1}{2} \]

solving for \( t \),

\[ \text{Median life } t = \theta \ln 2 = 0.693 \theta \]

Therefore, the median life for the exponential distribution is approximately equal to the mean life multiplied by the fraction 0.693.

Parameter Estimation for the Exponential Distribution

The problem of parameter estimation for the exponential distribution consists of determining the hazard rate \( \lambda \). The true value of \( \lambda \) is seldom if ever known, and can at best be approximated by estimation from the sample data. The hazard rate is defined as the limiting value as \( \Delta t \) approaches zero of the function

\[ \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t \left[ 1 - F(t) \right]} \]

The numerator of the above expression is equal to the fraction of items failing during the time interval \( \Delta t \). The denominator is equal to the length of the time interval multiplied by the fraction of items surviving up to this interval.

If the value of \( \Delta t \) is chosen sufficiently small, the above expression can be used to approximate the hazard rate at any given time \( t \) for any distribution. To see how this is done, we first multiply both numerator and denominator by the sample size \( N \). Thus,

\[ h(t) \approx \frac{N [F(t + \Delta t) - F(t)]}{\Delta t N [1 - F(t)]} \]

where the symbol \( \approx \) denotes "is approximately equal to." The numerator \( N [F(t + \Delta t) - F(t)] \) is then equal to the number of items failing during the interval \( \Delta t \). The expression \( N [1 - F(t)] \) is equal to the number of items surviving up to this time interval,
and the denominator $N \frac{\frac{1}{2} - F(t)}{\frac{1}{2}}$ is equal to the total time of operation during the time interval $\Delta t$.

If $t$ is measured in hours, then it follows that at a given time $t$ and for a given increment $\Delta t$ of $t$ the hazard rate is

$$h(t) \sim \frac{\text{NUMBER OF FAILURES}}{\text{TOTAL NUMBER OF HOURS OF OPERATION}}$$

For the exponential distribution the above approximation to $h(t)$ is true, regardless of the length of the time interval chosen for $\Delta t$. However, for distributions other than the exponential, it is important to caution against the use of this formula unless $\Delta t$ is chosen sufficiently small, because in general $h(t)$ is not constant but depends on time.

Since for the exponential distribution the mean time to failure is equal to the reciprocal of the hazard rate, we have as an approximation to the mean time to failure ($\hat{\theta}$)

$$\hat{\theta} = \frac{\text{TOTAL NUMBER OF HOURS OF OPERATION}}{\text{TOTAL NUMBER OF FAILURES}}^{(1,3)}$$

Again it is emphasized that in general this expression is true only for the exponential distribution. For other distributions different formulas will have to be used unless all items have failed and complete information is given on the exact times to failure.

**Confidence Limits for the Exponential Distribution**

A two-sided confidence interval corresponding to the confidence level $\gamma$ for the mean life of the exponential distribution is given by the inequality

$$\frac{2N\hat{\theta}}{x^2 2N; \frac{1-\gamma}{2}} < \hat{\theta} < \frac{2N\hat{\theta}}{x^2 2N; \frac{1+\gamma}{2}}$$

Where $N$ is the sample size, $\hat{\theta}$ is the sample mean, $x^2$ is the value of the chi-square distribution corresponding to the quantities $\frac{1-\gamma}{2}$ and $\frac{1+\gamma}{2}$ for $2N$ degrees of freedom. The value $\frac{2N\hat{\theta}}{x^2 2N; \frac{1-\gamma}{2}}$ is the lower confidence limit and the value $\frac{2N\hat{\theta}}{x^2 2N; \frac{1+\gamma}{2}}$ is the upper confidence limit.

A two-sided confidence interval for the reliability of the exponential distribution is given by
The Weibull distribution of lifetimes is a three parameter distribution represented by the survivorship function

\[ R = e^{\frac{(t-T)^B}{\Theta-T}} \]

Where

- \( R = \) Reliability at time \( t \)
- \( B = \) Shape parameter, called Weibull slope
- \( T = \) Location parameter (minimum life)
- \( \Theta = \) Scale parameter or characteristic life (the time at which 63.2\% of the population has failed)

It is usually true that there is no time during the life of a component that is failure free. Therefore, the minimum life \( T \) is generally assumed to be zero. This reduces the Weibull distribution to a two-parameter distribution. For the two-parameter Weibull distribution, the hazard rate is given by the equation

\[ h(t) = \frac{B t^{B-1}}{\Theta^B} \]

Where \( B \) and \( \Theta \) are as defined above. When \( B < 1 \), \( h(t) \) is a decreasing function of \( t \). When \( B = 1 \), \( h(t) \) is constant and we have as a special case the exponential distribution, and when \( B > 1 \), \( h(t) \) is an increasing function of \( t \). The reliability for the two-parameter Weibull distribution is equal to

\[ R(t) = e^{-\int_0^t h(t) \, dt} = e^{-\int_0^t \frac{B t^{B-1}}{\Theta} \, dt} \]

Since

\[ \int_0^t \frac{B t^{B-1}}{\Theta} \, dt = \left[ \frac{t^B}{B} \right]_0^{\Theta} \]

We have

\[ R(t) = e^{-\frac{t^B}{\Theta}} \]
When the value of $B = 1$, the above becomes $R(t) = e^{-\frac{t}{\theta}}$ and it is noted that this is the formula for the exponential distribution reliability function. The cumulative distribution function for the Weibull distribution is:

$$F(t) = 1 - R(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^B}$$

Notice that when $t = 0$, the fraction failed, $F(t)$, is equal to $1 - e^{-1} = 0.632$.

The probability density function is then

$$f(t) = h(t)R(t) = \frac{B}{\theta} \left(\frac{t}{\theta}\right)^{B-1} e^{-\left(\frac{t}{\theta}\right)^B}$$

The mean life ($M$) is found from the definition as follows

$$M = \int_0^\infty tf(t)dt = \int_0^\infty t \frac{B}{\theta} \left(\frac{t}{\theta}\right)^{B-1} e^{-\left(\frac{t}{\theta}\right)^B} dt$$

When the above expression is integrated the formula reduces to

$$M = \theta \Gamma\left(\frac{1}{B} + 1\right)$$

Where

$$\Gamma(N) = \int_0^\infty x^{N-1} e^{-x}dx$$

Values of $\Gamma(N)$ for various values of $N$ can be found in any complete book of mathematical tables. When $B = 1$ we have

$$M = \theta \Gamma\left(\frac{1}{B} + 1\right) = \theta \Gamma(2) = \theta$$

When the Weibull slope is one and we have as a special case the exponential distribution, the mean life is identical to the characteristic life. For Weibull distributions having a Weibull slope other than one, the mean life is not equal to the characteristic life.

The median life for the Weibull distribution is found by equating the reliability function to $\frac{1}{2}$ and solving for $t$.

Thus,

$$R(t) = e^{-\left(\frac{t}{\theta}\right)^B} = \frac{1}{2} \quad \text{and} \quad t = \theta \ln 2^{\frac{1}{B}}$$

By equating the expressions for the mean life and the median life and solving for $B$, we can find for what value of $B$ the mean life and the median life of the Weibull distribution are equal. We have,

$$\theta \Gamma\left(\frac{1}{B} + 1\right) = \theta \ln 2^{\frac{1}{B}}$$

If the value $B = 3.5$ is substituted in the above equation, both sides are approximately equal. This tells us that the Weibull
slope is approximately 3.5, the mean is equal to the median and we have an almost symmetrical distribution which approximately coincides with the normal distribution. For values of $B$ other than 3.5 the Weibull distribution is skewed, and the greater $B$ differs from 3.5, the greater the skewness will be and the poorer the approximation of the Weibull distribution to the normal distribution.

Parameter Estimation for the Weibull Distribution

For the exponential distribution the value of the hazard rate was estimated directly from the sample data. For the Weibull distribution, this cannot usually be done because the hazard rate varies with time and there is seldom enough data available to justify plotting a graph of the hazard rate as a function of time.

Weibull distribution parameters which are estimated from the data are the slope $B$ and the characteristic life $\theta$. These parameters are usually determined graphically as follows. The cumulative probability of failure for each time to failure is estimated either by the mean rank or by the median rank. The mean rank of the $i$th failure in a sample size of $N$ failures is defined as the fraction $i/(N + 1)$. The median rank is a plotting position which equalizes the probability of obtaining positive and negative errors in the estimation of the cumulative probability of failure. Tables of median ranks for a sample size of 1-20 are given. When the values of the mean ranks or median ranks are obtained, the estimated cumulative probability of failure (mean rank or median rank) is plotted as a function of its corresponding failure age on Weibull probability paper, which transforms the Weibull function into a straight line. A straight line least square fitted to the sample points will provide an approximation to the true cumulative distribution function. The Weibull slope $B$ is estimated by measuring the slope of this line and the characteristic life $\theta$ is estimated by measuring the point on the failure age axis for which the cumulative probability of failure is 0.632.

If complete information is not given on exact times to failure of all items the characteristic life cannot be accurately determined graphically by plotting only the data on those items which failed. In the case of incomplete data, if the Weibull slope is
assumed known through prior testing or is determined graphically by plotting the failure data on Weibull probability paper, then an estimate of $\theta$ is given by 

$$\theta = \left[ \frac{\sum_{i=1}^{r} t_i^B + \frac{N-r}{r} \sum_{j=1}^{r} t_j^B}{r} \right]^{1/B}$$

Where $N$ is the number of items tested and $r$ is the number of failures observed. The $t_i$'s are the times accumulated to failure for the items which fail, and the $t_j$'s are the times accumulated for the items which are suspended before failure.

Once the values of $\theta$ and $B$ have been estimated from the data, it is possible to obtain an estimate of the mean life by substituting the sample values of $\theta$ and $B$ into the formula

$$M = \theta r \left( \frac{1}{B} + 1 \right)$$

If it happens that complete information is given on exact times to failure of all items, we can use the non-parametric estimate of the mean life obtained from the sample data. The formula for this estimate is $M = \sum t_i/N$. Where $t_i$ is the time at which the $ith$ failure occurred and $N$ is the sample size.

Confidence Limits for the Weibull Distribution

If the Weibull slope is assumed known, a two-sided confidence interval for the Weibull distribution is given by the inequality

$$\exp \left\{ -\frac{1}{B} \left[ \frac{x^2 2N; 1-\gamma}{2N} \right] \right\} < R < \exp \left\{ -\left[ \frac{1}{B} \left[ \frac{x^2 2N; 1+\gamma}{2N} \right] \right] \right\}$$

Where $N$ is the sample size, $\theta$ is the sample characteristic life, and $x$ is the value of the chi-square distribution corresponding to the quantities $(1-\gamma)/2$ and $(1+\gamma)/2$ for $2N$ degrees of freedom.

If the Weibull slope is not assumed known, two-sided confidence limits for the Weibull distribution are determined graphically by constructing two curves, one to the right and one to the left of the Weibull plot. These two curves will enclose the confidence interval for the reliability of the Weibull distribution. For example, if the confidence level is chosen to be 0.9, the points plotted for the confidence limits of the Weibull distribution will be values such that we are 90% confident that the true value of the reliabi-
liability is between the upper confidence limit and lower confidence limit.

The points for a two-sided confidence interval are arrived at by plotting against the times to failure the values obtained by solving the following equations for \( Y \).\(^{(12)}\)

\[
\int_0^Y \frac{N!}{(J-1)! (N-J)!} y^{J-1} (1-y)^{N-J} \, dy = \frac{1-\gamma}{2}
\]

and

\[
\int_0^Y \frac{N!}{(J-1)! (N-J)!} y^{J-1} (1-y)^{N-J} \, dy = \frac{1+\gamma}{2}
\]

Where \( Y \) is an upper or lower limit on the fraction failed up to item \( J \) in a collection of \( N \) items associated with a confidence level \( \gamma \). In the above equations if the confidence level is chosen to be 0.9, \((1-\gamma)/2 \) will be equal to 0.05 and \((1+\gamma)/2 \) will be equal to 0.95. The solutions are called the 5% rank and the 95% rank.

The first equation gives a lower confidence limit on the cumulative distribution function (upper confidence limit on reliability), and the second equation gives an upper confidence on the cumulative distribution function, or a lower confidence on the reliability.

The integral in the above equations is referred to as the incomplete beta function ratio, and is tabulated in Karl Pearson’s Tables of the Incomplete Beta Function, published by the Cambridge University Press. The solutions to the above equations for various values of \( \gamma \) are tabulated.\(^{(8)}\)

THE NORMAL DISTRIBUTION

The normal distribution of times to failure is a two-parameter continuous distribution described by the equation

\[
f(t) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\left[\frac{1}{2}\left(\frac{t-M}{\sigma}\right)^2\right]} \quad \text{where} \quad -\infty < t < \infty
\]

\[
\sigma > 0
\]

\[
-\infty < M < \infty
\]

The function \( f(t) \) is the probability density function of the normal distribution whose graph is a symmetrical bell-shaped curve. The extremities of this curve asymptotically approach the time axis and the mean, median, and mode of this distribution are all equal.
The time to failure is the variable \( t \), \( \mu \) is the mean life, and \( \sigma \) is the standard deviation of the times to failure. As was stated earlier, the form of this distribution approximately coincides with a Weibull distribution having a Weibull slope of 3.5.

Since the normal distribution is defined over the range for which \(-\infty < t < \infty\) it does not represent a proper probability distribution over the negative time axis. Therefore, if it is to represent the times to failure of a sample of items, it is necessary to truncate this distribution at the origin. This can be done without significant loss of accuracy if the mean is large relative to the standard deviation.

The cumulative distribution function for the normal distribution is (6)

\[
F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2} \left( \frac{t-\mu}{\sigma} \right)^2} dt
\]

and the reliability function is

\[
R(t) = 1-F(t) = \frac{1}{\sqrt{2\pi}} \int_{t}^{\infty} e^{-\frac{1}{2} \left( \frac{t-\mu}{\sigma} \right)^2} dt
\]

The hazard function is obtained from the relationship

\[ h(t) = \frac{f(t)}{R(t)} \]

Performing this division gives the hazard function as

\[
h(t) = \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2} \left( \frac{t-\mu}{\sigma} \right)^2}}{\int_{t}^{\infty} e^{-\frac{1}{2} \left( \frac{t-\mu}{\sigma} \right)^2} dt}
\]

The population mean (\( \mu \)) of the normal distribution is estimated by the sample value \( \bar{t} \), where (5)

\[
\bar{t} = \frac{\sum t_i}{N}
\]

The estimate of the population standard deviation (\( \sigma \)) is denoted by \( S \), where

\[
S = \sqrt{\frac{\sum t_i^2 - (\sum t_i)^2}{N(N-1)}}
\]
The procedures for estimating the parameters from incomplete data and for determining the confidence limits for the reliability of the normal distribution are more involved than for the other distributions discussed here. (2,5,6,7, and 12)

APPLICABILITY OF DISTRIBUTIONS

For a large number of devices of the same type for which several modes of failure contribute to the observed hazard rate, if the actual distribution of the hazard rate is plotted as a function of time, it will often resemble the general form of the following curve,

This curve will also apply in the case of a large group of different types of components possessing a variety of failure mechanisms, and which are drawn from several populations encompassing a diversity of failure distributions. Many types of components have a high initial hazard rate which is due to defects in such areas as design features, manufacturing processes and handling procedures. The manufacturer usually attempts to eliminate this portion of the curve by screening and/or acceptance tests. Therefore, for items which have been delivered to the customer, a high initial hazard rate may not be experienced. The initial hazard rate decreases sharply to some value at which the useful life portion begins. During this portion of the curve the hazard rate rises gradually for a considerable length of time until the wear-
out portion is reached. For simplification, the useful life portion is sometimes approximated by a constant hazard rate. However, for mechanical components, wearout failures may begin to appear early and there may be no portion of the hazard rate curve which is constant. At some point in the life of components failures begin to occur rapidly due to excessive wear and the hazard rate rises sharply. This portion of the curve is called the wearout portion, although early wearout failures will cause a gradual increase in the hazard rate before this portion is reached. The normal distribution is often regarded as being a good fit to this portion.

Since in the case of the Weibull distribution it is possible to have a decreasing, constant or increasing hazard rate, this distribution can be fitted to all three portions of the typical hazard rate curve. If the Weibull slope is less than one, we have a good fit to the debugging portion; if the Weibull slope is one, the exponential distribution results, which (for applications in which wearout is not a predominant failure mode) can be fitted to the useful life portion of the curve; and if the Weibull slope is greater than one, the wearout portion is described and a close approximation to the normal distribution results. Accordingly, if the Weibull distribution is to be fitted to all portions of the typical hazard rate curve for a given part or system, one single value of the Weibull slope $B$ will not always be sufficient to describe the distribution of lifetimes. In this case, the mixed Weibull model described by Kao\(^{10}\) can be used; however, the additional complexity introduced by the inclusion of more than one value of the Weibull slope is usually not justified. Since the predominant failure mode for mechanical components is wearout, it is typical to express more of an interest in that portion of the hazard rate curve for which wearout failures exist. Therefore, in this instance, a Weibull slope $B > 1$ usually provides a close fit to the data.

There are two main reasons advanced for preferring the Weibull distribution to the others discussed here.

1. It gives at least as good or better fit to the data than the normal or exponential distribution.
2. The methods of estimating its parameters and determining confidence limits are relatively simple. Figure 1 shows diagrams of the main characteristics of the normal, exponential, and Weibull distributions on a linear time scale.
Figure 1. Comparison of Distribution Characteristics
SECTION II - RELIABILITY REDUNDANCY CONSIDERATIONS

Redundancy Using Singly Cross-Strapped Elements

It can be shown (theoretically) that a unit could be broken down into smaller parts termed elements. If these elements are each made redundant and cross-strapped, the reliability is enhanced.

The question considered here is: Given elements of a particular failure rate ($\lambda$), how many can be used to obtain a specified failure rate for a specified time; or given so many (N) elements of a given failure rate, what is the obtainable probability of success (P) for specified times?

Here, all elements are considered to be in single standby redundancy with cross-strapping (i.e., one extra redundant, element per element).

Figures 5 through 10 show plots of P vs. N and $\lambda$ for 5,7 and 10 years. Values of $\lambda_s$ (the switching circuitry) of 50, 100, 200 and 500 x $10^{-9}$ were assumed. For the 5 and 10 year cases, the regions of major interest are expanded in Figures 7 and 10.

Voting Redundancy for Long Life

Voting redundancy uses an odd number of elements, all on and functioning and examines their outputs, selecting the majority decision (voting) as the true output. The simplest voting is single 3-fold voting shown in Figure 2.

![Diagram of Voting System](image)

**Figure 2. Single Three-Fold Voting**

The voter performs the logic function (+ denotes or, . denotes and) $D = AB + BC + AC$. Examination will show that, for any single failure in A, B, or C; D will provide the 2 out of 3 answer. Similar systems can be developed for 3 out of 5, etc., but are usually not necessary.
Multiple voting implies a chain of such elements, N in length. Voting occurs between each set of elements. The intermediate voters may be either single (Figure 3) or duplicated (Figure 4).

Figure 3. Multiple Voting with Single Voters

Figure 4. Multiple Voting with Duplicated Voters

With single voters, all voters must be internally redundant or a large loss in reliability results. With duplicated voters, only the last (final) voter need be made internally redundant. Since the part count is probably as high or higher for internally redundant voters as for three non-redundant voters, the configuration of Figure 4 is superior and is usually chosen.

Consider a "box" made up of N elements. To make this redundant using majority voting requires 3N elements and 3(N-1)+1=3N-2 voters. Each voter must perform the voting function for each line between elements. The problem is to determine the appropriate N (amount of subdivision of the "box" into elements) for particular requirements.

The applicable formula is (12)

\[ P = 1 - \lambda t - 3(N-1) \left[ \left( \frac{A}{N} + \lambda v \right)^2 + 2(N-1) \left( \frac{A}{N} + \lambda v \right)^2 - 3 \left( \frac{A}{N} \right)^2 + 2 \left( \frac{A}{N} \right)^3 \right] \]

Where:

- \( P \) = Probability of success of the redundant configuration.
- \( t \) = Time in hours.
- \( \lambda = \) Total failure rate of one non-redundant box in failures/hour.
\[ N = \text{Number of elements, assuming all elements are identical.} \]

\[ \lambda_v = \text{Failure rate of each intermediate voter.} \]

\[ \lambda_{fv} = \text{Failure rate of the final voter.} \]

Figure 5 is a plot of \( P \) vs \( N \) for 5, 7 and 10 years and values of \( \lambda \) ranging from 1,000 to \( 20,000 \times 10^{-9} \) failures per hour. For this plot, \( \lambda_v = 20 \times 10^{-9} \) failures per hour and \( \lambda_{fv} \) is assumed equal to zero.

As an example, assume the box must obtain a success probability of 0.92 for 7 years. The box has a non-redundant failure rate of 5,000 bits. Examining Figure 5, it can be seen that \( N = 4 \) is adequate.

A comparison to the same example as in the reference is also useful. For that case (functional cross-strapping) \( N \) was between 10 and 12 depending on assumptions made about the switching circuit. For this case, \( N \) would be in excess of 26. That case required only slightly over 2 times the weight and slightly over 1 times the power while the voting redundancy requires slightly over 3 times the weight and 3 times the power.

Which method to apply depends on system requirements. If the system must be kept operating at all times, only voting redundancy will work. If this is not a requirement, usually functional redundancy with cross-strapping will result in less weight (about 2/3), less power (about 1/3), less cost (about 2/3) and is easier to test.

For most systems, the application of voting redundancy for the "always required" elements (such as command circuits) and functional redundancy for all other elements, is the best solution.

Certain formulas have been particularly useful in working with applications of redundancy. Since many applications are now required to use redundancy to achieve specific system or unit reliability qualities, presented in Appendix A are some of these formulas (many of them approximations) of possible general use.

A few examples may be useful in understanding the use of the figures:
Example 1: For 5 years, and a requirement of \( P = 0.9 \) find the number of elements of \( \lambda = 2,000 \times 10^{-9} \) which may be used.

Using Figure 6, find \( N = 26 \) for \( \lambda_s = 50 \times 10^{-9} \) to \( N = 22 \) for \( \lambda_s = 500 \times 10^{-9} \).

Example 2: For 7 years, we have 16 elements of \( \lambda = 2,000 \times 10^{-9} \) and \( \lambda_s = 50 \times 10^{-9} \). What is \( P \)?

Using Figure 8, find \( P = 0.882 \).

Example 3: For 10 years, and 10 elements of \( \lambda = 1,500 \times 10^{-9} \); what is the effect of \( \lambda_s \) on \( P \)?

Using Figure 9, find \( P = 0.914 \) for \( \lambda_s = 50 \times 10^{-9} \) and \( P = 0.8965 \) for \( \lambda_s = 500 \times 10^{-9} \).
Figure 6. Probability of Success for N-Cross-Strapped Elements

$T = 5$ years

- $\lambda_S = 50 \times 10^{-9}$
- $\lambda_S = 100 \times 10^{-9}$
- $\lambda_S = 200 \times 10^{-9}$
- $\lambda_S = 500 \times 10^{-9}$
Figure 7.
Expansion of Figure 6.
Figure 8. Probability of Success for N-Cross-Strapped Elements

T = 7 years (λ₈ = 50 x 10⁻⁶)
Figure 9. Probability of Success for N-Cross-Strapped Elements

$T = 10$ years

- $\lambda_S = 50 \times 10^{-9}$
- $\lambda_S = 100 \times 10^{-9}$
- $\lambda_S = 200 \times 10^{-9}$
- $\lambda_S = 500 \times 10^{-9}$
SECTION III - A BAYESIAN/CLASSICAL APPROACH TO RELIABILITY DEMONSTRATION

The Bayesian Method in Reliability Demonstration

The selection of conventional reliability demonstration test plans, such as those found in MIL-STD-781B, is usually based on the values assigned to such factors as minimum acceptable (Mean Time Between Failure) MTBF ($\theta_1$), specified MTBF ($\theta_0$), consumer's risk ($\beta$), and the producer's risk ($\alpha$). Strictly speaking, the classical test plans do not provide for any consideration of past equipment operating experience. Each test is conducted with regard to any prior data or information which may be applicable to the equipment. For this reason, conventional plans usually require a considerable amount of testing before an accept/reject decision is reached.

In contrast, the Bayesian approach to reliability demonstration is dependent on prior information relative to the item under test. In the Bayesian method, past experience is combined with current test results and the accept/reject decision is based on this mathematical combination. The term "past experience" in the Bayesian expression can take on various forms, ranging from an estimate of equipment reliability based on engineering judgment to a completely identified MTBF probability density function developed from operational failure data on existing similar or identical equipments. The complexity of the Bayesian expression can similarly vary from a simple additive process to a more sophisticated approach requiring the combination of two probability density functions.

The approach described and applied in this report requires only simple addition because it uses as prior information the reliability prediction and its measure of uncertainty, the coefficient of variation.

The Reliability Prediction Used as Prior Information

Using the reliability prediction and its associated measure of uncertainty, the coefficient of variation as the "prior" in a Bayesian expression makes sense because:

28
a. Reliability predictions are a requirement in nearly every military electronic equipment development program;
b. Reliability prediction techniques are based on the pooled and organized experience of countless individuals and organizations. Thus, when one used these techniques, he benefits from this accumulated experience and does not have to guess, use engineering judgment, or rely on his own limited experience;
c. Reliability predictions have been shown to provide reasonable estimates of equipment operational performance;\(^{(16)}\)
d. The consumer and producer would be more likely to agree on the use of \(\lambda_p\), the predicted equipment failure rate, as prior evidence than most other types of prior information.

Since reliability predictions are not 100% accurate, it is necessary to include, in the prior, a measure of their inaccuracy. Such a measure was developed in the (Rome Air Development Center) RADC/Hughes study\(^{(17)}\) and is the coefficient of variation \(e_s\). Their value of \(e_s\) could not be used in this study, however, because it was based on equipment operating in an airborne environment. Since the present RADC internal study considers ground equipment only, it was necessary to compute a new \(e_s\), based on the ground environment.

The data used to calculate the coefficient of variation, \(e_s\), for this report were obtained on over one hundred ground electronic equipments and systems. The types of equipment included in this sample range from common ground communication receivers and transmitters to more complex ground-based radar and data processing systems. Data criteria were established to insure that each data point met the following qualifications:

a. The equipment operated in a fixed, ground environment;
b. The equipment did not contain an overabundance of mechanical or electro-mechanical devices and was primarily electronic in nature;
c. A predicted reliability was available and was computed using the stress analysis technique (the actual operating parameter of each resistor, capacitor, etc. is compared
to its rated parameter) with part failure rates obtained from the RADC Reliability Notebook (18) or MIL-HDBK-217 (19).

d. An observed field reliability was available and was based on valid and representative field operational data collected on mature equipments.

As shown in Appendix B, the ratio of observed-to-predicted MTBF was determined for each equipment, and after calculating the mean and standard deviation of the sample, the value of es was found to be equal to 1.38.

The Bayesian/Classical Reliability Demonstration Method

The B/C approach is a "semi-" or "quasi-" Bayesian method because it utilizes prior prediction information in a classical test plan. Specifically, the reliability prediction, \( \lambda_p \), and its coefficient of variation, \( e_s \), are transformed into pseudo test hours and pseudo test failures which are then combined with actual test hours and failures observed during the classical demonstration test. In other words, the prediction, \( \lambda_p \), and \( e_s \), are represented by a pseudo test situation in which \( x \) failures occur in \( t \) hours of test.

The pseudo test hours and failures represented by the prediction are calculated as follows: (17)

\[
\lambda_s = \frac{r + \frac{1}{(e_s)^2}}{t + \frac{1}{(e_s)^2 \lambda_p}}
\]

where:

- \( \lambda_s \) = Bayesian posterior failure rate of the equipment
- \( \lambda_p \) = prior predicted failure rate of the equipment
- \( e_s \) = prediction coefficient of variation = 1.38
- \( r \) = number of actual failures during test
- \( t \) = number of actual test hours

Analysis of the above equation reveals that, in the limit, if one is totally confident that the predicted failure rate is the true failure rate, i.e., \( e_s = 0 \), then the equation gives \( \lambda_s = \lambda_p \), irrespective of test data. Conversely, if \( e_s \) is infinite, i.e.,
totally uncertain, then \( \lambda_s = r/t \), based solely on the test results.

The following example shows how the B/C method is applied and compares its results with those of a purely classical demonstration method.

A display console is to undergo reliability demonstration testing per MIL-STD-781B, Test Plan IV. That is, the test plan gives the accept/reject criteria for a 20% risk. The following information is available:

\[
\lambda_c = \text{contract specified failure rate} = 3.33/1000 \text{ hours}
\]
\[
\lambda_p = \text{predicted failure rate} = 2.85/1000 \text{ hours}
\]

Also assume that the display console will experience one failure while in test, at \( t = 300 \) hours.

If the console is tested using the conventional method, its performance profile will be as shown by the solid line in Figure 11. With only one failure occurring, at \( t = 300 \) hours, the classical test would end at \( t = 630 \) hours with an accept decision. Using $30 as an average cost per hour for testing, the cost of the conventional test would be \( 630 \times 30 = 18,900 \). (Figure 11 shows only that portion of MIL-STD-781B test Plan IV which is applicable to this particular example.)

Using \( \lambda_c \) and \( \lambda_p \), as defined above, and also \( e_s = 1.38 \), the Bayesian/Classical method of reliability demonstration is now applied. First, the number of pseudo test hours and failures represented by the prediction and \( e_s \) are calculated using the equation on page 30:

\[
\lambda_s = \frac{r + \frac{1}{(e_s)^2}}{t + \frac{1}{(e_s)^2 \lambda_p}} = \frac{0 + \frac{1}{2}}{0 + \frac{1}{2 \times 2}} = \frac{0.53}{187} \sim 0.53 \text{ failures in 187 hours}
\]

Thus, the reliability prediction and the coefficient of variation are equivalent to a pseudo test situation in which 0.53 failures occur in 187 test hours. In other words, although no actual testing has been performed, we have taken advantage of the information contained in the prediction and will use this information in combination with the actual test hours \( t \) and failures \( r \) accumulated during the test. For example, after one hundred hours
of actual testing, we have:

\[
\lambda_s = \frac{0 + 0.53}{100 + 187} \sim 0.53 \text{ failures in 287 test hours}
\]

And after four hundred hours of actual testing, the Bayesian posterior failure rate, \( \lambda_s \), is:

\[
\lambda_s = \frac{1 + 0.53}{400 + 187} \sim 1.53 \text{ failures in 587 test hours}
\]

Referring to the B/C test performance profile in Figure 11 (the dotted line), application of the B/C test in this situation would result in an accept decision at \( t = 740 \) hours. However, the first 187 hours were "free", since these were not actual test hours but were hours represented by the prediction. Hence, only 553 actual test hours would be required using the B/C test, compared to the 630 hours required by the purely classical method. Thus, using the Bayesian/Classical demonstration test would result in an estimated savings of $2,310.

Since the selection of \( t = 300 \) hours for the first and only failure was not purely arbitrary (in the actual test only one failure was recorded and that was at \( t = 300 \) hours), the following table was computed showing the B/C test time and dollar savings with other possible failure combinations:

<table>
<thead>
<tr>
<th>Assumed Failure Combinations</th>
<th>Classical Test</th>
<th>B/C Test</th>
<th>B/C Savings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Failures @ hrs</td>
<td>Hours to Decision</td>
<td>Hours to Decision</td>
<td>Test Hrs</td>
</tr>
<tr>
<td>0</td>
<td>400 hrs</td>
<td>343 hrs</td>
<td>57</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>630</td>
<td>533</td>
</tr>
<tr>
<td>1</td>
<td>400</td>
<td>630</td>
<td>343</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>840</td>
<td>773</td>
</tr>
<tr>
<td>1</td>
<td>400</td>
<td>50</td>
<td>983</td>
</tr>
</tbody>
</table>
Case Histories

Herein are three case histories, showing the results of applying the B/C test and comparing these results with those of the original classical tests. Also included is a discussion of the use of the B/C method in a special situation in which accept/reject criteria had to be established and the testing time per equipment was only eight hours.

Case History 1

A. Given:

\[
\lambda_c = \text{contract specified failure rate} = 5.5/1000 \text{ hrs} \\
\lambda_p = \text{predicted failure rate} = 3.2/1000 \text{ hrs} \\
e_s = 1.38
\]

B. Original demonstration test results; Passed Test Plan IV of MIL-STD-781E at \( t = 252 \) hours with zero failures. Test Plan IV is shown in Figure 12.

C. Applying the Bayesian/Classical approach: Using the information given in A the number of pseudo test hours and failures represented by the prediction and \( e_s \) are
calculated as follows:

\[ \lambda_s = \frac{0.53}{164} \sim 0.53 \text{ failures in 164 test hours} \]

Thus, the reliability prediction and its associated coefficient of variation are equivalent to a pseudo test situation in which 0.53 failures occurred in 164 test hours. Since the original
demonstration test did not log any failures, the decision to use
the Bayesian Classical approach must be based on the 0.53 pseudo
failures only.

Entering Test Plan IV (Figure 12) at 0.53 failures, an accept
decision would have been reached at \( t = 306 \) hours. Of these, how­
ever, 164 hours were represented by the prediction, leaving only
142 actual test hours required by the B/C method to yield an accept
decision.

Since the B/C method needed 142 test hours to reach a decision
and the original demonstration test required 252 hours with no fail­
ure, a savings of 101 test hours are realized with the B/C test.
Using an average cost per test hour of $30, the dollar savings at­
tributed to the B/C demonstration method is 101 \( \times \) $30 = $3030.

Case History 2 (Radar Electronics)
A. Given
\[
\lambda_c = \text{contract specified failure rate} = 8.3/1000 \text{ hrs} \\
\lambda_p = \text{predicted failure rate} = 1.9/1000 \text{ hrs} \\
e_s = 1.38
\]
B. Original demonstration test results: Accepted at \( t = 760 \) hours with 6 failures. The failures occurred at
10, 30, 330, 420, 430 and 660 hours. The accept/reject
criteria (per Test Plan IV) used and the results of
the test are shown in Figure 13.
C. Applying the Bayesian/Classical approach: From the
equation on page 30 and the information given in A,
the number of pseudo test hours and failures represen­
ted by the prediction and its coefficient of variation,
\( e_s \), are calculated as follows:
\[
\lambda_s = \frac{0.53}{279} \sim 0.53 \text{ failures in 279 test hours}
\]

Thus, the reliability prediction and \( e_s \) are equivalent to a
279 hour life test with 0.53 failures. In applying the Bayesian/
Classical method consider the same number and sequence of failures
as in the original test, but adding 0.53 pseudo test failures and
279 test hours to the actual test failures (\( r \)) and hours (\( t \)) in the
above equation. The following table shows the total number of test
hours and failures used in the application of the B/C method.

<table>
<thead>
<tr>
<th>Total Test hours (Pseudo) + Actual</th>
<th>Total Test Failures (Pseudo) + Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>(279) + 0 = 279</td>
<td>(0.53) + 0 = 0.53</td>
</tr>
<tr>
<td>(279) + 10 = 289</td>
<td>(0.53) + 1 = 1.53</td>
</tr>
<tr>
<td>(279) + 30 = 309</td>
<td>(0.53) + 2 = 2.53</td>
</tr>
<tr>
<td></td>
<td>Accept decision at t = 475 hours</td>
</tr>
</tbody>
</table>

Figure 13 shows the results of the original reliability demonstration test compared with the B/C test. Notice that the B/C test results in an accept decision at \( t = 475 \) hours. Subtracting the 279 pseudo test hours reduces to 196 the actual hours required by this method to reach a decision. Since the original test required 760 hours of test, the use of prior prediction and prediction variation data results in a savings of 564 hours, or, a dollar savings equal to 564 \( \times \$30 = \$16,920 \).

Case History 3 (Data Processing and Display Equipment)

A. Given

\[
\lambda_c = \text{contract specified failure rate} = 1.83/1000 \text{ hrs}
\]

\[
\lambda_p = \text{predicted failure rate} = 1.73/1000 \text{ hrs}
\]

\[
e = 1.38
\]

B. Original demonstration test results: Accepted at \( t = 1200 \) hours with zero failures. Test Plan IV and the accept/reject criteria are shown in Figure 14.

C. Applying the Bayesian/Classical approach: Using the information given in A, the number of pseudo test failures and test hours represented by the reliability prediction are calculated using the equation on page 30.

\[
\lambda_s = \frac{0.53}{304} \sim 0.53 \text{ failures in 304 test hours}
\]

The reliability prediction and the coefficient of variation are thus equivalent to a pseudo test situation with 0.53 failures in 304 hours of test. Since the original demonstration test did not result in any failures, our decision using the Bayesian/Clas-
sical approach must be based on the 0.53 pseudo failures only.

Figure 14 shows that this particular test plan had a minimum operating time equal to \( t_0 = 2(545) = 1090 \) hours, where \( t_0 = \frac{1}{\lambda_c} \). (Although the test should have been terminated at this time, an additional 110 test hours were logged before official notice to terminate testing was received.) Using the Bayesian/Classical approach, we enter Figure 14 at 0.53 failures and find that an accept decision would have been reached at approximately \( t = 1640 \) hours. However, subtracting 304 pseudo hours leaves a total of 1336 actual test hours required by the B/C method. Since the original test required only 1200 hours, the use of the B/C method in this situation, with this test plan, would require 136 additional test hours and would, therefore, result in a net loss of 136 \( \times \$30 = \$4080 \).

The additional testing time required by the B/C method in the preceding Case History 3 was not entirely unexpected since the predicted failure rate was only slightly less than the contract specified failure rate (1.73 vs 1.83/1000 hours). This suggests that...
the reliability design (amount of derating, redundancy, screening, etc.) may have been just barely adequate to meet the equipment reliability requirements. In such cases, the use of predicted pseudo test hours and failures will reduce actual test time little if any. This case history clearly shows that the B/C method should be applied only in situations having favorable predictions. In this case history a predicted failure rate of 1.20/1000 hours of less would have resulted in a reduction of test hours using the B/C approach.

A Special Situation

The Bayesian/Classical demonstration method applies well in procurements not requiring a formal reliability demonstration test. A recent RADC equipment development program serves as an example.

In this particular situation, RADC had management responsibility over equipment being purchased by a foreign nation whose specifications did not include reliability demonstration requirements but did include short term qualification tests. RADC was reluctant to deliver the equipment without some verification of their reliability. It was decided, therefore, to use the Bayesian/Classical method during the short (eight hour) qualification test to determine whether the 100 hour MTBF reliability requirement would be satisfied.

Although a detailed stress analysis reliability prediction (defined earlier) was available it was not used in the B/C test. Instead, operational data obtained during Category I (all producer's qualification testing) and II (consumer's test to verify acceptability of producer's equipment) tests conducted on the same equipment in a previous program were used because they were believed to be more representative of the equipment's reliability. Thus, the following information was available:

\[ \lambda_c = \text{contract specified failure rate} = 10.00/1000\text{ hours} \]
\[ \lambda_{c, I, II} = \text{Previous Cat I and Cat II data, 22 failures in 2676 hrs} \]

Accept/reject criteria in the form of a 60% upper confidence limit (UCL) on the failure rate were selected. These were developed in the RADC/Hughes study. Specifically, an accept decision is
reached whenever the 60% upper confidence limit of the demonstrated failure rate is less than the contract specified failure rate (10.00/1000 hours) and is rejected when the 60% UCL is greater than \( \lambda_c \). The table below shows the 60% upper confidence limits for different numbers of failures. \(^{(17)}\) Judging criteria at 90%, 95%, etc., could also have been used; however, in this particular instance, 60% limits were considered adequate.

**ONE SIDED UPPER 60% CONFIDENCE LIMITS BASED ON THE NUMBER OF FAILURES**

<table>
<thead>
<tr>
<th>Number of Failures</th>
<th>60% Confidence Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.916</td>
</tr>
<tr>
<td>1</td>
<td>2.022</td>
</tr>
<tr>
<td>2</td>
<td>3.105</td>
</tr>
<tr>
<td>3</td>
<td>4.175</td>
</tr>
<tr>
<td>4</td>
<td>5.237</td>
</tr>
<tr>
<td>5</td>
<td>6.292</td>
</tr>
<tr>
<td>6</td>
<td>7.343</td>
</tr>
<tr>
<td>7</td>
<td>8.390</td>
</tr>
<tr>
<td>8</td>
<td>9.434</td>
</tr>
<tr>
<td>9</td>
<td>10.475</td>
</tr>
<tr>
<td>10</td>
<td>11.515</td>
</tr>
<tr>
<td>11</td>
<td>12.553</td>
</tr>
<tr>
<td>12</td>
<td>13.592</td>
</tr>
<tr>
<td>13</td>
<td>14.631</td>
</tr>
<tr>
<td>14</td>
<td>15.668</td>
</tr>
<tr>
<td>15</td>
<td>16.707</td>
</tr>
<tr>
<td>16</td>
<td>17.742</td>
</tr>
<tr>
<td>17</td>
<td>18.772</td>
</tr>
<tr>
<td>18</td>
<td>19.802</td>
</tr>
<tr>
<td>19</td>
<td>20.831</td>
</tr>
<tr>
<td>20</td>
<td>21.859</td>
</tr>
<tr>
<td>21</td>
<td>22.887</td>
</tr>
<tr>
<td>22</td>
<td>23.914</td>
</tr>
<tr>
<td>23</td>
<td>24.940</td>
</tr>
<tr>
<td>24</td>
<td>25.966</td>
</tr>
<tr>
<td>25</td>
<td>26.991</td>
</tr>
<tr>
<td>26</td>
<td>27.996</td>
</tr>
</tbody>
</table>

The qualification test was to be conducted on five equipments for eight hours each, a total of forty (40) equipment-operating hours. Since 2676 hours had already been accumulated and forty additional hours would be added during the qualification test, the decision to accept/reject was based on 2716 hours of test, assuming 0,1,2,3, etc., failures, in addition to the 22 failures already
There is a table showing the total number of failures and test hours, the 60% upper confidence limits and decision following each failure:

<table>
<thead>
<tr>
<th>Failures During Qual. Test</th>
<th>Failures</th>
<th>Total Test Time (hrs)</th>
<th>60% UCL Is &gt; than λ̂?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>from table</td>
</tr>
<tr>
<td>0</td>
<td>22</td>
<td>2716</td>
<td>8.79/K hrs</td>
</tr>
<tr>
<td>1</td>
<td>23</td>
<td>2716</td>
<td>9.17</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>2716</td>
<td>9.55</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>2716</td>
<td>9.93</td>
</tr>
<tr>
<td>4</td>
<td>26</td>
<td>2716</td>
<td>10.30</td>
</tr>
</tbody>
</table>

The table shows that the 60% UCL of the demonstrated failure rate is less than the contract-specified failure rate whenever the number of failures is three or less. Thus, if three or less failures are observed during the qualification test, the equipment should be accepted, whereas any more than three failures are cause for rejection. Of course, if other than 60% UCL criteria are used, then the accept/reject decisions will also change.

In this situation, the prior information was much "stronger" than the qualification test results since the number of failures and test hours in the prior far outweighed those of the qualification test. However, this example shows that even a short-term qualification test can be made profitable for reliability demonstration purposes. Also, it shows that the B/C method can be applied when formal decision criteria have not been established.

Conclusions

The reliability demonstration approach described herein is both efficient and economical when applied to equipments with favorable reliability predictions. Reductions in test hours amounted to 40% in Case History 1 and 74% in Case History 2, with corresponding dollar savings of $3030 and $16,920, respectively. Case History 3 shows that the Bayesian/Classical method can also result in longer tests when the reliability prediction is only slightly less than the specified failure rate. The B/C method is not recommended in that instance.
This method does increase the consumer's risk because, given a favorable prediction and barring any large number of failures, an accept decision will be based on fewer test hours than a purely classical test. However, this increased risk should be more than outweighed by the proven accuracy (the actual MTBF approaching the predicted MTBF) of the reliability prediction.
REFERENCES


RELIABILITY AND REDUNDANCY FORMULAS

1. If \( \lambda = \text{total failure rate (failures/hour)} \) of an element (a component, a collection of components, a circuit, a box, etc.), then the probability of success is:

\[
P = e^{-\lambda t} \quad \text{where} \quad t \text{ is in hours (8,760 hours = 1 year)}
\]

2. If \( \lambda t \leq 0.02 \), then

\[
P \approx 1 - \lambda t = 1 - Q
\]

where \( Q = \lambda t \) is the probability of failure.

3. A reliability improvement factor, \( F \), can be defined as:

\[
F = \frac{Q_{\text{non-red}}}{Q_{\text{red}}} = \frac{1 - P_{\text{non-red}}}{1 - P_{\text{red}}}
\]

4. If several elements are in series (each required), then the total reliability is:

\[
P_s = P_1 P_2 \cdots P_n \quad \text{or} \quad P_s = 1 - Q_s \quad \text{where}

\[
Q_s = Q_1 + Q_2 + \cdots + Q_n
\]

This uses the approximation that the product of numbers all near unity is equal to one minus the sum of their differences from unity. (Also, if \( P_s = P_a^n \), then \( Q_s = mQ \).)

5. For series or parallel redundancy, \( Q_s = Q^2 \) or \( Q_s = Q_a Q_b \) if \( Q_a \neq Q_b \). (Both elements always on). Also, \( P_s = P_a + P_b - P_a P_b \).

6. For quad redundancy, \( Q_s = 1.5Q^2 \)

7. If several elements are used, but only a few are needed, but all elements are always on: (all elements identical)

\[
\begin{align*}
a. & \quad 1 \text{ of } 2 \text{ not failed: } Q_s = Q^2 \\
b. & \quad 1 \text{ of } 3 \text{ not failed: } Q_s = Q^3 \\
c. & \quad 2 \text{ of } 3 \text{ not failed: } Q_s = 3Q^2 - 2Q^3 \\
d. & \quad 1 \text{ of } 4 \text{ not failed: } Q_s = Q^4 \\
e. & \quad 2 \text{ of } 4 \text{ not failed: } Q_s = 4Q^3 - 3Q^4 \\
f. & \quad 3 \text{ of } 4 \text{ not failed: } Q_s = 6Q^2 - 8Q^3 + 3Q^4 \\
g. & \quad 1 \text{ of } 5 \text{ not failed: } Q_s = Q^5 \\
h. & \quad 2 \text{ of } 5 \text{ not failed: } Q_s = 5Q^4 - 4Q^5 \\
i. & \quad 3 \text{ of } 5 \text{ not failed: } Q_s = 10Q^3 - 15Q^4 + 6Q^5 \\
j. & \quad 4 \text{ of } 5 \text{ not failed: } Q_s = 8Q^2 - 14Q^3 + 9Q^4 - 2Q^5 \\
k. & \quad 1 \text{ of } 6 \text{ not failed: } Q_s = Q^6 \\
l. & \quad 2 \text{ of } 6 \text{ not failed: } Q_s = 6Q^5 - 5Q^6 \\
m. & \quad 3 \text{ of } 6 \text{ not failed: } Q_s = 15Q^4 - 25Q^5 + 10Q^6
\end{align*}
\]
8. For voting redundancy, if the voter failure probability is $Q_v$, then
\[
Q_s = Q_v + 3Q_v^2 - 2Q_v^3
\]
for single 3-fold voting and
\[
Q_s = Q_{fv} + 3\left(\frac{Q_v}{N}\right)^2 + 3(N-1)(Q_v + \frac{Q_v}{N})^2 - 3\left(\frac{Q_v}{N}\right)^3 - 3(N-1)(Q_v + \frac{Q_v}{N})^3
\]
for multiple 3-fold voting, where $N =$ number of voted elements and
$Q_{fv}$ = failure probability of final voter
$Q =$ total failure probability of all elements

9. For standby redundancy, all elements on:
\[
Q_s = (Q_a + Q_{da} - Q_a Q_{da})(Q_b + Q_{db} - Q_b Q_{db}) \text{ or}
Q_s = Q^2 + Q(Q_{da} + Q_{db}) + Q_{da} Q_{db}
\]
where $Q_a = Q_b = Q$

Here, $Q_{da}$ is the decision element unreliability failing toward A, $Q_{db}$ toward B.

10. For standby redundancy, redundant element off until needed; and assuming the off element has a failure rate $r$ times lower when off:
\[
P_s = e^{-\lambda t} + \frac{\lambda e^{-\lambda t}}{\lambda_s + r\lambda} 1 - e^{-(\lambda_s + r\lambda)t},
\]
which may be written as:
\[
Q_s = Q - \frac{\lambda}{\lambda_s + r\lambda} (1 - Q)(Q_s + r\Omega - r\Omega_Q)
\]
Here, $\lambda_s, Q_s$ are the failure rate and probability of the switching element.

If $r = 0$, $Q_s = Q - \frac{\lambda}{\lambda_s} (1 - Q)Q_s$
DERIVATION OF $e_5$, THE COEFFICIENT OF VARIATION

The following computer program was used in the calculation of the coefficient of variation:

```
READY
$LIST
010 DIM A(500), B(500)
020 PRINT "WHAT IS THE NUMBER OF DATA ENTRIES";
030 INPUT N
040 FOR I=1 TO 5
050 PRINT
060 NEXT I
070 PRINT "A", "LOG(A)", "LOG(A)^2"
080 FOR I=1 TO N
090 READ A
100 LET A(I)=LOG(A)
110 LET B(I)=A(I)^2
120 PRINT A(I), B(I)
130 LET C=C+A(I)
140 LET D=D+B(I)
150 NEXT I
160 FOR I=1 TO 5
170 PRINT
180 NEXT I
190 PRINT "SUM OF LOG(A)"; C
200 PRINT "SUM OF LOG(A)^2"; D
210 DATA .607, .232, .170, .584, .411, .907, 1.20, 2.50, 2.33, .905
220 DATA 1.09, 1.03, 1.45, 5.26, 2.13, 2.53, 3.16, .973, .329, 1.22
230 DATA 2.00, 1.21, 1.26, 1.45, .878, 1.04, 1.47, 1.91, .817, .393
240 DATA .557, .585, .573, 2.10, 6.69, 1.75, .925, .793, 1.71, 2.45
250 DATA 1.13, .865, 3.41, .684, 1.36, .388, .139, .526, .535, 1.32
260 DATA 7.00, .830, .300, 270, 412, 1.82, .380, .465, .556, 0.90
270 DATA .101, 1.74, .667, 1.40, .555, .800, .706, .139, .124, 2.70
280 DATA 2.50, .443, .272, .750, .267, .389, .544, .827, .180, .720
290 DATA .554, 2.82, 1.08, 4.21, .298, .487, .92, .784, .242, .769
300 DATA .750, .933, .400, .400, .314, .558, .724, .434, .318, .771
310 DATA .660, .610, .556, .262, .727, .343, .762, .750, .575, 1.25
320 END
```

In the following computer printout, three columns of data are listed: the first column, $A$, is the ratio of predicted to observed MTBF; the second column is the log $A$; and the third column is log $(A)^2$.

<table>
<thead>
<tr>
<th>$A$</th>
<th>LOG $(A)$</th>
<th>LOG $(A)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.607</td>
<td>-0.499226</td>
<td>0.249227</td>
</tr>
<tr>
<td>0.232</td>
<td>-1.461020</td>
<td>2.134570</td>
</tr>
<tr>
<td>0.170</td>
<td>-1.771960</td>
<td>3.139830</td>
</tr>
<tr>
<td>0.584</td>
<td>-0.537854</td>
<td>0.289287</td>
</tr>
<tr>
<td>3.410</td>
<td>1.226710</td>
<td>1.504820</td>
</tr>
<tr>
<td>A</td>
<td>LOG (A)</td>
<td>LOG (A)12</td>
</tr>
<tr>
<td>-------</td>
<td>-----------</td>
<td>-----------</td>
</tr>
<tr>
<td>0.907</td>
<td>-9.76128E-02</td>
<td>9.52827E-03</td>
</tr>
<tr>
<td>1.200</td>
<td>0.182322</td>
<td>3.32412E-02</td>
</tr>
<tr>
<td>2.500</td>
<td>0.916291</td>
<td>0.839589</td>
</tr>
<tr>
<td>2.330</td>
<td>0.845868</td>
<td>0.715493</td>
</tr>
<tr>
<td>0.905</td>
<td>-9.98203E-02</td>
<td>9.96410E-03</td>
</tr>
<tr>
<td>1.090</td>
<td>8.61777E-02</td>
<td>7.42660E-03</td>
</tr>
<tr>
<td>1.030</td>
<td>2.95588E-02</td>
<td>8.73723E-04</td>
</tr>
<tr>
<td>1.450</td>
<td>0.371564</td>
<td>0.138059</td>
</tr>
<tr>
<td>5.260</td>
<td>1.660130</td>
<td>2.756030</td>
</tr>
<tr>
<td>2.180</td>
<td>0.779325</td>
<td>0.607347</td>
</tr>
<tr>
<td>2.580</td>
<td>0.947789</td>
<td>0.898305</td>
</tr>
<tr>
<td>3.160</td>
<td>1.150570</td>
<td>1.323820</td>
</tr>
<tr>
<td>0.973</td>
<td>-2.73712E-02</td>
<td>7.49182E-04</td>
</tr>
<tr>
<td>3.290</td>
<td>1.190890</td>
<td>1.418210</td>
</tr>
<tr>
<td>1.220</td>
<td>0.198851</td>
<td>3.95417E-02</td>
</tr>
<tr>
<td>2.000</td>
<td>0.693147</td>
<td>0.480453</td>
</tr>
<tr>
<td>1.120</td>
<td>0.113329</td>
<td>1.28434E-02</td>
</tr>
<tr>
<td>1.260</td>
<td>0.231112</td>
<td>5.34126E-02</td>
</tr>
<tr>
<td>1.450</td>
<td>0.371564</td>
<td>0.138059</td>
</tr>
<tr>
<td>0.878</td>
<td>-0.130109</td>
<td>1.69283E-02</td>
</tr>
<tr>
<td>1.040</td>
<td>3.92207E-02</td>
<td>1.53826E-03</td>
</tr>
<tr>
<td>0.679</td>
<td>-0.387134</td>
<td>0.149873</td>
</tr>
<tr>
<td>0.491</td>
<td>-0.711311</td>
<td>0.505964</td>
</tr>
<tr>
<td>0.817</td>
<td>-0.202116</td>
<td>4.08510E-02</td>
</tr>
<tr>
<td>0.893</td>
<td>-0.113169</td>
<td>1.28072E-02</td>
</tr>
<tr>
<td>0.557</td>
<td>-0.585190</td>
<td>0.342447</td>
</tr>
<tr>
<td>0.505</td>
<td>-0.682197</td>
<td>0.466758</td>
</tr>
<tr>
<td>0.373</td>
<td>-0.986177</td>
<td>0.972545</td>
</tr>
<tr>
<td>2.100</td>
<td>0.741937</td>
<td>0.550471</td>
</tr>
<tr>
<td>0.649</td>
<td>-0.432323</td>
<td>0.186903</td>
</tr>
<tr>
<td>1.760</td>
<td>0.565314</td>
<td>0.319580</td>
</tr>
<tr>
<td>0.925</td>
<td>-7.79615E-02</td>
<td>6.07800E-03</td>
</tr>
<tr>
<td>0.793</td>
<td>-0.231932</td>
<td>5.37925E-02</td>
</tr>
<tr>
<td>1.710</td>
<td>0.536493</td>
<td>0.287825</td>
</tr>
<tr>
<td>2.450</td>
<td>0.896088</td>
<td>0.802974</td>
</tr>
<tr>
<td>1.180</td>
<td>0.165514</td>
<td>2.73950E-02</td>
</tr>
<tr>
<td>0.865</td>
<td>-0.145026</td>
<td>2.10325E-02</td>
</tr>
<tr>
<td>3.120</td>
<td>1.137830</td>
<td>1.294660</td>
</tr>
<tr>
<td>0.684</td>
<td>-0.379797</td>
<td>0.144246</td>
</tr>
<tr>
<td>1.360</td>
<td>0.307485</td>
<td>9.45468E-02</td>
</tr>
<tr>
<td>0.388</td>
<td>-0.946750</td>
<td>0.896335</td>
</tr>
<tr>
<td>0.139</td>
<td>-1.973280</td>
<td>3.893840</td>
</tr>
<tr>
<td>0.526</td>
<td>-0.642454</td>
<td>0.412747</td>
</tr>
<tr>
<td>0.535</td>
<td>-0.625489</td>
<td>0.391236</td>
</tr>
<tr>
<td>1.320</td>
<td>0.277632</td>
<td>7.70794E-02</td>
</tr>
<tr>
<td>7.000</td>
<td>1.945910</td>
<td>3.786570</td>
</tr>
<tr>
<td>0.830</td>
<td>-0.186330</td>
<td>3.47187E-02</td>
</tr>
<tr>
<td>0.500</td>
<td>-0.693147</td>
<td>0.480453</td>
</tr>
<tr>
<td>0.270</td>
<td>-1.309330</td>
<td>1.714350</td>
</tr>
<tr>
<td>0.412</td>
<td>-0.886732</td>
<td>0.786294</td>
</tr>
<tr>
<td>0.182</td>
<td>-1.703750</td>
<td>2.902760</td>
</tr>
<tr>
<td>0.380</td>
<td>-0.967584</td>
<td>0.936219</td>
</tr>
<tr>
<td>A</td>
<td>LOG (A)</td>
<td>LOG (A)×2</td>
</tr>
<tr>
<td>-----</td>
<td>-----------</td>
<td>-----------</td>
</tr>
<tr>
<td>0.465</td>
<td>-0.763718</td>
<td>0.358324</td>
</tr>
<tr>
<td>0.556</td>
<td>-0.586987</td>
<td>0.344554</td>
</tr>
<tr>
<td>0.090</td>
<td>-2.407950</td>
<td>5.798200</td>
</tr>
<tr>
<td>0.101</td>
<td>-2.292630</td>
<td>5.256170</td>
</tr>
<tr>
<td>0.174</td>
<td>-1.748700</td>
<td>3.057950</td>
</tr>
<tr>
<td>0.667</td>
<td>-0.404965</td>
<td>0.163997</td>
</tr>
<tr>
<td>1.400</td>
<td>0.336472</td>
<td>0.113214</td>
</tr>
<tr>
<td>0.555</td>
<td>-0.588787</td>
<td>0.346670</td>
</tr>
<tr>
<td>0.800</td>
<td>-0.223144</td>
<td>4.97930E-02</td>
</tr>
<tr>
<td>0.706</td>
<td>-0.348140</td>
<td>0.121201</td>
</tr>
<tr>
<td>0.139</td>
<td>-1.973280</td>
<td>3.893840</td>
</tr>
<tr>
<td>0.124</td>
<td>-2.087470</td>
<td>4.357550</td>
</tr>
<tr>
<td>0.270</td>
<td>-1.309330</td>
<td>1.714350</td>
</tr>
<tr>
<td>0.250</td>
<td>-1.386290</td>
<td>1.921810</td>
</tr>
<tr>
<td>0.443</td>
<td>-0.814186</td>
<td>0.662898</td>
</tr>
<tr>
<td>0.272</td>
<td>-1.301950</td>
<td>1.695080</td>
</tr>
<tr>
<td>0.750</td>
<td>-0.287682</td>
<td>8.27610E-02</td>
</tr>
<tr>
<td>0.267</td>
<td>-1.320510</td>
<td>1.743740</td>
</tr>
<tr>
<td>0.429</td>
<td>-0.846298</td>
<td>0.716221</td>
</tr>
<tr>
<td>0.544</td>
<td>-0.608806</td>
<td>0.370645</td>
</tr>
<tr>
<td>0.827</td>
<td>-0.189951</td>
<td>3.60832E-02</td>
</tr>
<tr>
<td>1.800</td>
<td>0.587787</td>
<td>0.345493</td>
</tr>
<tr>
<td>0.720</td>
<td>-0.328504</td>
<td>0.107915</td>
</tr>
<tr>
<td>0.554</td>
<td>-0.590591</td>
<td>0.348797</td>
</tr>
<tr>
<td>2.820</td>
<td>1.036740</td>
<td>1.074820</td>
</tr>
<tr>
<td>1.080</td>
<td>7.69510E-02</td>
<td>5.92300E-03</td>
</tr>
<tr>
<td>4.210</td>
<td>1.437460</td>
<td>2.066300</td>
</tr>
<tr>
<td>0.298</td>
<td>-1.210660</td>
<td>1.465700</td>
</tr>
<tr>
<td>0.487</td>
<td>-0.719491</td>
<td>0.517668</td>
</tr>
<tr>
<td>0.902</td>
<td>-0.103141</td>
<td>1.06380E-02</td>
</tr>
<tr>
<td>0.784</td>
<td>-0.243346</td>
<td>5.92174E-02</td>
</tr>
<tr>
<td>0.242</td>
<td>-1.418820</td>
<td>2.013040</td>
</tr>
<tr>
<td>0.769</td>
<td>-0.262664</td>
<td>6.89925E-02</td>
</tr>
<tr>
<td>0.750</td>
<td>-0.287682</td>
<td>8.27610E-02</td>
</tr>
<tr>
<td>0.933</td>
<td>-6.93501E-02</td>
<td>4.80943E-03</td>
</tr>
<tr>
<td>0.400</td>
<td>-0.916291</td>
<td>0.839589</td>
</tr>
<tr>
<td>4.000</td>
<td>1.386290</td>
<td>1.921810</td>
</tr>
<tr>
<td>0.314</td>
<td>-1.158360</td>
<td>1.341800</td>
</tr>
<tr>
<td>0.558</td>
<td>-0.583396</td>
<td>0.340351</td>
</tr>
<tr>
<td>0.724</td>
<td>-0.322964</td>
<td>0.104306</td>
</tr>
<tr>
<td>0.434</td>
<td>-0.834711</td>
<td>0.696742</td>
</tr>
<tr>
<td>0.318</td>
<td>-1.145780</td>
<td>1.312640</td>
</tr>
<tr>
<td>0.771</td>
<td>-0.260067</td>
<td>6.76348E-02</td>
</tr>
<tr>
<td>0.660</td>
<td>-0.415515</td>
<td>0.172653</td>
</tr>
<tr>
<td>0.610</td>
<td>-0.494296</td>
<td>0.244329</td>
</tr>
<tr>
<td>0.556</td>
<td>-0.586987</td>
<td>0.344554</td>
</tr>
<tr>
<td>0.362</td>
<td>-1.016110</td>
<td>1.032480</td>
</tr>
<tr>
<td>0.727</td>
<td>-0.318829</td>
<td>0.101652</td>
</tr>
<tr>
<td>0.343</td>
<td>-1.070020</td>
<td>1.144950</td>
</tr>
<tr>
<td>0.762</td>
<td>-0.271809</td>
<td>7.38800E-02</td>
</tr>
<tr>
<td>0.750</td>
<td>-0.287582</td>
<td>8.27610E-02</td>
</tr>
<tr>
<td>0.575</td>
<td>-0.553385</td>
<td>0.306235</td>
</tr>
</tbody>
</table>
\[ \begin{array}{ccc}
\text{A} & \text{LOG (A)} & \text{LOG (A)^2} \\
1.250 & 0.223144 & 4.97930E-02 \\
\end{array} \]

SUM OF LOG(A) = -33.7648
SUM OF LOG (A)^2 = 90.7958

The calculations leading to the value of \( \epsilon_s = 1.38 \) are shown below:

a. \( \sum \log_e A = -33.7648 \)
b. \( \sum (\log_e A)^2 = 90.7958 \)
c. \( \log_e \bar{A} = -0.3069 \)
d. \( \text{antilog} (-0.3069) = \bar{A} = \text{geometric mean} = 0.73 \)
e. \( \sigma^2 \log_e A = 0.738 \)
f. \( \sigma \log_e A = 0.859 \)
g. \( \text{antilog} (-0.3069 \pm 1\sigma) = 1.74 \text{ to } 0.73 \text{ with a mean of } 0.73 = 1.74 - 0.73 = 1.01 \)
h. \( \epsilon_s = \text{coefficient of variation} = \frac{\sigma}{\bar{A}} = 1.38 \)