CALIFORNIA STATE UNIVERSITY, NORTHRI GE

NONLINEAR ANALYSIS BY FINITE ELEMENT METHODS

A project submitted in partial satisfaction of the requirements for the degree of Master of Science in Engineering

by

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iv</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>I. NONLINEAR ANALYSIS TECHNIQUES</td>
<td>10</td>
</tr>
<tr>
<td>II. EXAMPLE: INCREMENTAL METHOD</td>
<td>14</td>
</tr>
<tr>
<td>III. EXAMPLE: ITERATIVE METHOD</td>
<td>24</td>
</tr>
<tr>
<td>IV. COMPUTER PROGRAM</td>
<td>34</td>
</tr>
<tr>
<td>A. FLOW CHART FOR THE INCREMENTAL METHOD</td>
<td>34</td>
</tr>
<tr>
<td>B. FLOW CHART FOR THE ITERATION METHOD</td>
<td>36</td>
</tr>
<tr>
<td>C. MATRIX INVERSION BY DECOMPOSITION</td>
<td>38</td>
</tr>
<tr>
<td>D. INPUT INSTRUCTIONS FOR THE INCREMENTAL METHOD</td>
<td>44</td>
</tr>
<tr>
<td>E. INPUT INSTRUCTIONS FOR THE ITERATIVE METHOD</td>
<td>47</td>
</tr>
<tr>
<td>F. COMPUTER LISTING FOR THE INCREMENTAL METHOD</td>
<td>50</td>
</tr>
<tr>
<td>G. COMPUTER LISTING FOR THE ITERATIVE METHOD</td>
<td>57</td>
</tr>
<tr>
<td>V. BIBLIOGRAPHY</td>
<td>66</td>
</tr>
<tr>
<td>APPENDIX A - DERIVATION OF THE STIFFNESS MATRIX FOR A THREE DIMENSIONAL TRUSS</td>
<td>67</td>
</tr>
<tr>
<td>APPENDIX B - EXAMPLES USING THE INCREMENTAL AND THE ITERATIVE COMPUTER PROGRAMS</td>
<td>73</td>
</tr>
</tbody>
</table>
ABSTRACT

NONLINEAR ANALYSIS BY FINITE ELEMENT METHODS

by

Russell Howard England

Master of Science in Engineering

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This project deals with the solution of nonlinear structural problems by the use of the finite element method. The method is described for the general structural problem. Two computer programs are developed using nonlinear analysis. These computer programs solve a three dimensional nonlinear truss by the use of the incremental and the iterative methods. The theory behind these programs along with a user manual is presented herein. These results compare favorably with those obtained by exact methods.
INTRODUCTION

The finite element method has evolved since the development of the high speed digital computer. The finite element method requires the solution of a set of many simultaneous equations. With the use of the computer the solution to a set of simultaneous equations can be found rapidly with modern numerical methods.

In the past, structural analysis has proceeded as follows:

1. A free body cut of a structural member is taken and the six stress components are applied to their respective locations.
2. The six stress components are related through equilibrium.
3. The stresses are related to strains by the use of Hooke's law.
4. Strains are related to displacements.
5. From relating equilibrium a mathematical expression, often a differential equation is found and analysis is reduced to finding a solution to the differential equation that satisfies the boundary conditions.

Often it is not possible to determine a closed form solution to the mathematical model, so numerical methods are used to provide approximate solutions. The finite
element method divides the structure into a number of
discrete volumes by dividing it into an equivalent system
of smaller elements. Instead of trying to find a closed
form solution for the entire body, solutions are found for
each small element with various boundary conditions.

A model of the structure is divided into a finite
number of elements. These elements are connected to other
elements at nodes. A displacement function is assumed to
approximate the exact displacement for each element. The
amplitudes of the displacement function are determined
from the displacements of the nodes. The strains are
determined from the displacement function; then from
the potential energy of the element the stiffness matrix
is determined. The potential energy of an elastic
structure is the sum of the internal energy (strain
energy) plus the potential energy of the external loads.
If a structure is in a state of equilibrium then this
energy is a minimum.

The element stiffness matrix for a linear material
is derived as follows: The selection of a displacement
is chosen, generally a polynomial. A polynomial is chosen
for a displacement function because polynomials are easy
to differentiate and integrate. The order of the
polynomial is equal to the number of degrees of freedom
needed to describe the displacements of all the nodes.
A displacement function must be continuous within each element and displacements must be compatible between elements. Compatible means each element must deform without gaps or overlaps between elements. A displacement function must include rigid body displacements and also include constant states of strain. A constant state of strain is necessary because if an element approaches infinitesimally small the strains in each element will approach a constant.

Examples:

one dimensional

\[ u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \]

two dimensional

\[
\begin{align*}
    u(x, y) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 y + \alpha_5 y^2 + \alpha_6 y \\
    v(x, y) &= \alpha_7 + \alpha_8 x + \alpha_9 xy + \alpha_{10} x^2 + \alpha_{11} y^2 + \alpha_{12} y
\end{align*}
\]

three dimensional

\[
\begin{align*}
    u(x, y, z) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z \\
    v(x, y, z) &= \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 z \\
    w(x, y, z) &= \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} z
\end{align*}
\]

In matrix format

\[
\begin{bmatrix}
    u \\
    v \\
    w
\end{bmatrix} = 
\begin{bmatrix}
    u(x, y, z) \\
    v(x, y, z) \\
    w(x, y, z)
\end{bmatrix} = 
\begin{bmatrix}
    \phi
\end{bmatrix} 
\begin{bmatrix}
    \alpha
\end{bmatrix}
\]

Where

\[
\begin{bmatrix}
    u
\end{bmatrix} = \text{displacement function}
\]
\[ \begin{bmatrix} \phi \\ \alpha \end{bmatrix} \quad = \text{polynomials of the displacements} \\
\{ \alpha \} \quad = \text{the coefficients of the polynomials} \]

Since the number of degrees of freedom is equal to the number of coefficients, the coefficients can be determined from matrix algebra. The displacement function is evaluated at the nodal points.

\[
\begin{bmatrix} q \end{bmatrix} = \begin{bmatrix} U (\text{node 1}) \\ U (\text{node 2}) \\ U (\text{node 3}) \\ U (\text{node 4}) \\ \vdots \\ U (\text{node n}) \end{bmatrix} = \begin{bmatrix} \phi (\text{node 1}) \\ \phi (\text{node 2}) \\ \phi (\text{node 3}) \\ \phi (\text{node 4}) \\ \vdots \\ \phi (\text{node n}) \end{bmatrix} \{ \alpha \} = [A] \{ \alpha \} 
\]

From matrix algebra

\[
\{ \alpha \} = [A] \{ q \}
\]

Where

\[ [A] = \text{displacement transformation matrix} \]

Substituting for \( \alpha \) gives the displacement function as a function of the nodal displacements.

\[
\{ U \} = [\phi] [A]^{-1} \{ q \}
\]
The strains are functions of the displacement function and can be computed as direct function of the nodal displacement.

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} \\
\varepsilon_z &= \frac{\partial w}{\partial z}
\end{align*}
\]

\[
\begin{align*}
\sigma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\sigma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\
\sigma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}
\end{align*}
\]

\[
\{\varepsilon\} = \{f(\phi)\} \begin{bmatrix} A \end{bmatrix} \{\delta\} = \{B\} \{q\}
\]

Where

\[
\{\varepsilon\} = \text{vector of strains}
\]

From the generalized Hooke's law equation for elastic isotropic materials.

\[
\begin{align*}
\varepsilon_x &= \frac{1}{2} \left[ \sigma_x - v (\sigma_y + \sigma_z) \right] \\
\varepsilon_y &= \frac{1}{2} \left[ \sigma_y - v (\sigma_x + \sigma_z) \right] \\
\varepsilon_z &= \frac{1}{2} \left[ \sigma_z - v (\sigma_x + \sigma_y) \right]
\end{align*}
\]

\[
\begin{align*}
\sigma_{xy} &= \tau_{xy}/G \\
\sigma_{yz} &= \tau_{yz}/G \\
\sigma_{zx} &= \tau_{zx}/G
\end{align*}
\]

Where

\[
E = \text{Young's modulus} \quad G = \text{Shear modulus} \quad \nu = \text{Poisson's ratio}
\]
The stresses are functions of strains

\[ \{\sigma\} = [C] \{\varepsilon\} \]

Where

\[ \{\sigma\} = \text{vector of stress} \]

\[ [C] = \text{matrix of material constants} \]

\[ \{\sigma\} = [C][B] \{q\} \]

The stresses and strains are now a function of the nodal displacements \(\{q\}\).

\[ \{\varepsilon\} = [B] \{q\} \]

\[ \{\sigma\} = [C][B] \{q\} \]

In order to determine a method for calculating the element stiffness matrix it is necessary to determine the internal strain energy. Consider a cube with sides
dx, dy, and dz inside an elastic body subject to an external system of forces. The normal forces are

\[ \sigma_x dydz, \quad \gamma_y dxz, \quad \gamma_z dx dy \]

The shear forces are

\[ \tau_{xy} dydz, \quad \tau_{yz} dzdx, \quad \tau_{zx} dx dy \]
\[ \tau_{xz} dydz, \quad \tau_{yx} dzdx, \quad \tau_{zy} dx dy \]

Under the action of these forces the faces of the cube will displace in the normal direction by amounts \( \epsilon_x dx, \quad \epsilon_y dy, \quad \epsilon_z dz \) and will distort by amounts \( \gamma_{xy}, \quad \gamma_{yz}, \quad \gamma_{zx} \).

The total work done by these forces is the mechanical energy stored in the differential element volume dx dy dz. This mechanical energy is called strain energy. The strain energy does not depend on the manner in which the forces are applied but only on their final values. The relation between the normal force \( \sigma_x dydz \) and the displacement \( \epsilon_x dx \) is linear.
The corresponding work is equal to the area under the triangle.

\[ dU = \frac{1}{2} \sigma_x \epsilon_x \, dx \, dy \, dz = \frac{1}{2} \sigma_x \epsilon_x \, dV \]

For the other normal forces and shear forces the same procedure yields the following results.

\[ \frac{1}{2} \sigma_y \epsilon_y \, dV, \quad \frac{1}{2} \tau_{xy} \gamma_{xy} \, dV, \quad \frac{1}{2} \tau_{yz} \gamma_{yz} \, dV \]

\[ \frac{1}{2} \sigma_z \epsilon_z \, dV \quad \frac{1}{2} \tau_{xz} \gamma_{xz} \, dV \]

The total strain energy stored in a elastic body of volume \( V \) is

\[ U = \frac{1}{2} \, v (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz}) \, dV \]

\[ U = \frac{1}{2} \, v \{\epsilon\}^T \{\sigma\} \, dV \]

The internal strain energy is equal to the work done by the external loads \( \{Q\} \). The external work is equal to one half the sum of the products of the intensities of the external forces and the components of the displacements of their points of application in the directions of the external forces.

\[ U_E = \frac{1}{2} \, \sum_{i=1}^{N} Q_i \, q_i \]

\[ U_E = \frac{1}{2} \, \{q\}^T \{Q\} \]
From matrix algebra
\[
\{ \varepsilon \} = [B] \{ \delta \} \\
\{ \varepsilon \}^T = \{ \delta \} B
\]
\[
U_s = \frac{1}{2} \int_v \{ \varepsilon \}^T \{ \sigma \} \, dv
\]
\[
\begin{align*}
U_s &= \frac{1}{2} \{ \delta \}^T \left( \int_v [B]^T [C] [B] \, dv \right) \{ \delta \}
\end{align*}
\]

Clapeyron's theory states that the internal strain energy is equal to the external energy.
\[
U_e = U_s
\]
\[
\frac{1}{2} \{ \delta \}^T \{ \delta \} = \frac{1}{2} \{ \delta \}^T \left( \int_v [B]^T [C] [B] \, dv \right) \{ \delta \}
\]
\[
\{ Q \} = \left( \int_v [B]^T [C] [B] \, dv \right) \{ \delta \}
\]

By definition of the stiffness matrix the external loads are linear combination of the external displacements.
\[
\{ Q \} = [K] \{ \delta \}
\]

Therefore the stiffness matrix has been derived.
\[
[K] = \int_v [B]^T [C] [B] \, dv
\]
Section I
NONLINEAR ANALYSIS TECHNIQUES

Often in structural analysis it is convenient and accurate to use linear analysis. In reality all materials are nonlinear, some more than others and a linear analysis will often lead to results that are not accurate. In this case it is necessary to develop a nonlinear analysis technique. There are two types of problems in dealing with the finite element method. One is the nonlinear stress-strain relationship of a material and the other is a change in the geometry that results after loading of the structure. Nonlinear problems due to changes in geometry will be neglected, and nonlinear analysis will be based on the nonlinear relationship of the stress-strain of the materials. Geometry is assumed to remain constant.

As in elastic analysis, nonlinear analysis uses the basic conditions of equilibrium and compatibility. The major difference in the two analyses is the stress-strain relationship. In the inelastic problem the strain is not proportional to the stress since each strain component contains a linear elastic component and a plastic component not linearly related. The solution of nonlinear
problems by the finite element method can be solved by various methods. Two methods shall be described. One is the incremental or stepwise procedure, and the other is a step iterative or deformation theory. The incremental theory relates a increment of strain to a increment of stress for a given stress state. Hence the true load path is followed. The step iterative theory establishes a relationship between the stresses and the total strains. The path by which the stress distribution is reached does not influence the strains. This is only true for proportional loading. The two theories give equivalent answers for the case of proportional loadings.

The incremental stiffness method is used to solve nonlinear problems by dividing the load into several increments. The final stress and strain states are based on the accumulation of the results of the individual loading increments where the stiffness of the structure is based on the current stress and strain states.

Basically, the incremental procedure solves the nonlinear problem by a series of linear problems, or the analysis is based on a piece-wise linear stress-strain curve. If the loads are incremented smaller and smaller, the results become more and more accurate. If the structure is a truss or beam, in the limit the results are exact.
With the incremental method, the loads are first divided into increments usually but not necessarily equal.

\[ \{ \Delta Q \}_j \]

The total load applied to the structure in any given increment \( i \) is given by

\[ \{ Q \}_i = \sum_{j=1}^{i} \{ \Delta Q \}_j \]

The incremental displacement due to an incremental load can be determined from the stiffness matrix which is a function of the total displacements.

\[ \{ \Delta q \}_j = [K(\{ q \}_i)]^{-1} \Delta Q \]

\[ \{ q \}_i = \sum_{j=1}^{i} \{ \Delta q \}_j \]

The step iterative procedure differs from the incremental procedure in that the structure is completely loaded during each iteration. In the first iteration, Young's modulus is taken as the tangent modulus for a strain of zero. The stiffness matrix is evaluated and the displacement of the nodes are found. From these displacements the stresses are found from the stress-strain relationship for each element. Since the stiffness matrix was approximated, the internal loads produced in the structure will not equal the external loads applied to the structure. Therefore equilibrium will not be satisfied. After each iteration the load that is not in
balance is used for the loads to compute an additional
displacement. This process is continued until equilibrium
is approximated to a desirable solution.

First the displacements are calculated based on
applying all the load. The stiffness matrix is evaluated
for Young's modulus equal to the tangent modulus for
zero strain.

\[ \{ \mathbf{q}_i \} = [\mathbf{K}] \{ \mathbf{q} \} \]

This displacement gives loads that do not satisfy
equilibrium.

\[ \{ \Delta \mathbf{q}_i \} = \mathbf{f} (\{ \mathbf{q}_i \}) \]

These loads that do not satisfy equilibrium are used to
calculate a incremental displacement.

\[ \{ \Delta \mathbf{q}_i \} = [\mathbf{K}]^{-1} \{ \Delta \mathbf{q} \} \]
\[ \{ \mathbf{q}_i \} = \{ \mathbf{q}_i \} + \sum_{j=i}^{i} \{ \Delta \mathbf{q}_j \} \]

The ith displacement is used to compute the ith+1 loads
and the process is continued until equilibrium is
satisfied to a desired amount. The stiffness matrix
is based on the secant modulus at the end of the previous
iterative step.
Section II
EXAMPLE: INCREMENTAL METHOD

In this section a nonlinear three bar statically indeterminate truss is solved using the incremental method.

<table>
<thead>
<tr>
<th>Member #</th>
<th>Area</th>
<th>Length</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>200</td>
<td>30(^\circ)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>100</td>
<td>90(^\circ)</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>200</td>
<td>150(^\circ)</td>
</tr>
</tbody>
</table>

The material properties are the same for all three element members.
The stress-strain curve for all members

\[ \sigma = 10 \times 10^6 \epsilon - 6.25 \times 10^8 \epsilon^2 \]

The load of 50,000 will be applied in five increments. E for each cycle will be taken as the tangent modulus for each member's strain state. The tangent modulus can be defined as follows

\[ E_t = \frac{d\sigma}{d\epsilon} \]

For this problem

\[ \sigma = 10 \times 10^6 \epsilon - 6.25 \times 10^8 \epsilon^2 \]

\[ E_t = \frac{d\sigma}{d\epsilon} = 10 \times 10^6 - 1.25 \times 10^9 \epsilon \]

For the first increment the tangent modulus is evaluated at zero strain. Since this problem has been divided into five load increments, the first load applied will
be 10,000. The stiffness matrix is

\[
[K_1] = \frac{E_1 A_1}{L_1} \begin{bmatrix} \cos^2 \varepsilon, & -\sin \varepsilon \cos \varepsilon \\ -\sin \varepsilon \cos \varepsilon, & \sin^2 \varepsilon \end{bmatrix}
\]

\[
K_1 = \frac{E_1}{200} \begin{bmatrix} .75, & -.433 \\ -.433, & .25 \end{bmatrix}
K_2 = \frac{E_2}{100} \begin{bmatrix} .0, & .0 \\ .0, & 1. \end{bmatrix}
\]

\[
K_3 = \frac{E_3}{200} \begin{bmatrix} .75, & .433 \\ .433, & .25 \end{bmatrix}
\]

For the first increment $E_1$ is taken as the tangent modulus at zero strain.

\[
E_1 = 10,000,000
\]

The total stiffness matrix for the first increment

\[
K_t = K_1 + K_2 + K_3
\]

\[
K_t = 50,000 \begin{bmatrix} 1.5, & 0. \\ 0., & 2.5 \end{bmatrix}
\]

The deflections for the first increment

\[
\{\Delta q\}_1 = [K_t]^{-1}\{\Delta Q\}_1
\]
\[
\begin{bmatrix}
\Delta q_1 \\
\Delta q_2
\end{bmatrix}
= \frac{1}{50,000}
\begin{bmatrix}
0.666, & 0 \\
0, & 0.4
\end{bmatrix}
\begin{bmatrix}
0 \\
10,000
\end{bmatrix}
\]

\[
\begin{bmatrix}
q_1 \\
q_2
\end{bmatrix}
= \begin{bmatrix}
0. \\
0.08
\end{bmatrix}
\]

The strain for each member

\[
\Delta \varepsilon_i = \frac{1}{L} \begin{bmatrix}
\cos \theta, & \sin \theta
\end{bmatrix}
\begin{bmatrix}
\Delta q_1 \\
\Delta q_2
\end{bmatrix}
\]

\[
\Delta \varepsilon_{11} = \frac{1}{200} \begin{bmatrix}
0.866, & 0.5
\end{bmatrix}
\begin{bmatrix}
\Delta q_1 \\
\Delta q_2
\end{bmatrix}
\]

\[
\Delta \varepsilon_{21} = \frac{1}{100} \begin{bmatrix}
0, & 1.
\end{bmatrix}
\begin{bmatrix}
\Delta q_1 \\
\Delta q_2
\end{bmatrix}
\]

\[
\Delta \varepsilon_{31} = \frac{1}{200} \begin{bmatrix}
-0.866, & 0.5
\end{bmatrix}
\begin{bmatrix}
\Delta q_1 \\
\Delta q_2
\end{bmatrix}
\]

For the first increment

\[
\begin{align*}
\Delta \varepsilon_{11} &= 0.0002 \\
\Delta \varepsilon_{21} &= 0.0008 \\
\Delta \varepsilon_{31} &= 0.0002
\end{align*}
\]

The change in the stress can be calculated

\[
\Delta \sigma_i = E_i \Delta \varepsilon_i
\]
\[ \Delta \sigma_{1T} = 2,000 \]
\[ \Delta \sigma_{2T} = 8,000 \]
\[ \Delta \sigma_{3T} = 2,000 \]

After the first increment the total stresses, strains, and displacements are equal to the first increment.

\[ \sigma_{1T} = \Delta \sigma_{1} \]
\[ \sigma_{2T} = \Delta \sigma_{2} \]
\[ \sigma_{3T} = \Delta \sigma_{3} \]
\[ \varepsilon_{1T} = \Delta \varepsilon_{1} \]
\[ \varepsilon_{2T} = \Delta \varepsilon_{2} \]
\[ \varepsilon_{3T} = \Delta \varepsilon_{3} \]

\[ q_{1T} = \Delta q_{1} \]
\[ q_{2T} = \Delta q_{2} \]

For the second increment the tangent modulus for the strains after the first increment are calculated as

\[ E_{1} = 10 \times 10^{6} - 1.25 \times 10^{9} \]
\[ E_{1} = 9.75 \times 10^{6} \]
\[ E_{2} = 9.00 \times 10^{6} \]
\[ E_{3} = 9.75 \times 10^{6} \]

The stiffness matrix for the second increment

\[ K_{t} = K_{1} + K_{2} + K_{3} = 10^{5} \begin{bmatrix} .7312, & .0 \\ .0, & 1.144 \end{bmatrix} \]
\[
\begin{align*}
\{\Delta q\}_\Pi &= [k_t] \{\Delta Q\}_\Pi \\
\{\Delta q_1\}_\Pi &= 10^{-5} \begin{bmatrix} 1.137, & 0 \\ 0. & .8744 \end{bmatrix} \{0, 10,000\} \\
\{\Delta q_2\}_\Pi &= \begin{cases} 0. \\ .08744 \end{cases}
\end{align*}
\]

\[
\begin{align*}
\{\Delta \varepsilon_{\Sigma}\}_\Pi &= \begin{bmatrix} .000218600 \end{bmatrix} \\
\{\Delta \varepsilon_{\Sigma}\}_\Pi &= \begin{bmatrix} .000874320 \end{bmatrix} \\
\{\Delta \varepsilon_{\Sigma}\}_\Pi &= \begin{bmatrix} .000218600 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\Delta \sigma_{\Pi} &= E_1 \Delta \varepsilon_{\Sigma} \\
\{\Delta \sigma_1\}_\Pi &= \{2,131.155\} \\
\{\Delta \sigma_2\}_\Pi &= \{7,868.5\} \\
\{\Delta \sigma_3\}_\Pi &= \{2,131.155\}
\end{align*}
\]

The total displacements, stresses, and strains after the second increment

\[
\begin{align*}
\varepsilon_{\Sigma} &= \Delta \varepsilon_{\Sigma I} + \Delta \varepsilon_{\Sigma \Pi} \\
\sigma_{\Sigma} &= \Delta \sigma_{\Sigma I} + \Delta \sigma_{\Sigma \Pi} \\
\varphi_{\Sigma} &= \Delta \varphi_{\Sigma I} + \Delta \varphi_{\Sigma \Pi}
\end{align*}
\]

\[
\begin{align*}
\varepsilon_{\Sigma I} &= .00041858 \\
\varepsilon_{\Sigma I} &= .00167432 \\
\varepsilon_{\Sigma I} &= .00041858 \\
\varepsilon_{\Sigma \Pi} &= .00167432 \\
\varepsilon_{\Sigma \Pi} &= .00041858 \\
\sigma_{\Sigma I} &= 4,131.155 \\
\sigma_{\Sigma I} &= 15,868.5 \\
\sigma_{\Sigma \Pi} &= 4,131.155 \\
\varphi_{\Sigma I} &= 0 \\
\varphi_{\Sigma \Pi} &= .167422
\end{align*}
\]
For the third increment the tangent modulus is calculated

\[
E_{1\text{III}} = 9.4768 \times 10^6 \\
E_{2\text{III}} = 7.9071 \times 10^6 \\
E_{3\text{III}} = 9.4768 \times 10^6
\]

For the third increment the stiffness matrix is

\[
[K_{t}]_{\text{III}} = 10^5 \begin{bmatrix}
0.71076, & 0. \\
0.0, & 1.0276
\end{bmatrix}
\]

\[
\{\Delta q\}_{\text{III}} = [K_{t}]_{\text{III}}\{\Delta Q\}_{\text{III}}
\]

\[
\{\Delta q\}_{\text{III}} = 10^{-5} \begin{bmatrix}
1.4069, & 0. \\
0.0, & .973113
\end{bmatrix} \begin{bmatrix}
0 \\
10,000
\end{bmatrix}
\]

\[
\Delta \varepsilon_{1\text{III}} = .00024328 \\
\Delta \varepsilon_{2\text{III}} = .00097311 \\
\Delta \varepsilon_{3\text{III}} = .00024328 \\
\Delta \varepsilon_{4\text{III}} = 0. \\
\Delta \varepsilon_{5\text{III}} = .093133 \\
\Delta \varepsilon_{6\text{III}} = 2,305.516 \\
\Delta \varepsilon_{7\text{III}} = 7,694.5 \\
\Delta \varepsilon_{8\text{III}} = 2,305.516
\]

The total displacements, stresses, and strains after the third increment

\[
\varepsilon_{iT} = \Delta \varepsilon_{i\text{I}} + \Delta \varepsilon_{i\text{II}} + \Delta \varepsilon_{i\text{III}}
\]

\[
\sigma_{iT} = \Delta \sigma_{i\text{I}} + \Delta \sigma_{i\text{II}} + \Delta \sigma_{i\text{III}}
\]

\[
\gamma_{iT} = \Delta \gamma_{i\text{I}} + \Delta \gamma_{i\text{II}} + \Delta \gamma_{i\text{III}}
\]
\[\varepsilon_{1T} = 0.00066186 \quad \sigma_{1T} = 6436.671\]
\[\varepsilon_{2T} = 0.00264733 \quad \sigma_{2T} = 23563.\]
\[\varepsilon_{3T} = 0.00066186 \quad \sigma_{3T} = 6436.671\]

\[\delta_{1T} = 0.\]
\[\delta_{2T} = 0.260535\]

For the fourth increment the tangent modulus is calculated

\[E_1 = 9.1727 \times 10^6\]
\[E_2 = 6.6908 \times 10^6\]
\[E_3 = 9.1727 \times 10^6\]

For the fourth increment the stiffness matrix is

\[
\begin{bmatrix}
   K_t \\
\end{bmatrix} = 10^5 \begin{bmatrix}
   0.68795, & 0. \\
   0., & 0.89840
\end{bmatrix}
\]

\[
\{ \Delta q \}_{IV} = \begin{bmatrix} K_t \end{bmatrix} \{ \Delta Q \}_{IV}
\]

\[
\{ \Delta q \}_{IV} = 10^{-5} \begin{bmatrix} 1.4536, & 0. \\
   0., & 1.1131 \end{bmatrix} \begin{bmatrix} 0 \\
   10,000 \end{bmatrix}
\]

\[\Delta \varepsilon_{1IV} = 0.0027827 \quad \Delta \varepsilon_{1IV} = 2552.52\]
\[\Delta \varepsilon_{2IV} = 0.00111309 \quad \Delta \varepsilon_{2IV} = 7447.48\]
\[\Delta \varepsilon_{3IV} = 0.0027827 \quad \Delta \varepsilon_{3IV} = 2552.52\]

\[\Delta \varepsilon_{4IV} = 0.\]
\[\Delta \varepsilon_{5IV} = 0.111309\]
The total displacements, stresses, and strains after the fourth increment

\[ \varepsilon_{iT} = \Delta \varepsilon_{iT} + \Delta \varepsilon_{iII} + \Delta \varepsilon_{iIII} + \Delta \varepsilon_{iIV} \]
\[ \sigma_{iT} = \Delta \sigma_{iT} + \Delta \sigma_{iII} + \Delta \sigma_{iIII} + \Delta \sigma_{iIV} \]
\[ \varphi_{iT} = \Delta \varphi_{iT} + \Delta \varphi_{iII} + \Delta \varphi_{iIII} + \Delta \varphi_{iIV} \]

\[ \varepsilon_{1T} = .00094014 \quad \sigma_{1T} = 8,989.15 \]
\[ \varepsilon_{2T} = .00376042 \quad \sigma_{2T} = 31,010.48 \]
\[ \varepsilon_{3T} = .00094014 \quad \sigma_{3T} = 8,989.15 \]

\[ \varphi_{1T} = 0, \quad \varphi_{2T} = .371844 \]

For the fifth increment the tangent modulus is calculated

\[ E_{1Y} = 8.8248 \times 10^6 \]
\[ E_{2Y} = 5.2995 \times 10^6 \]
\[ E_{3Y} = 8.8248 \times 10^6 \]

For the fifth increment the stiffness matrix is

\[ [K_t]_Y = 10^5 \begin{bmatrix} .66186, & 0. \\ 0. & .75057 \end{bmatrix} \]
\[ \{\Delta q\}_Y = [K_t]^{-1}\{\Delta q\}_Y \]
\[ \{\Delta q\}_Y = 10^{-5} \begin{bmatrix} 1.51089, & 0. \\ 0. & 1.33232 \end{bmatrix} \{ 0 \} \]
\[ \Delta q_1 = 0. \]
\[ \Delta q_2 = 1.33232 \]

\[ \Delta \varepsilon_{11} = 0.00033308 \quad \Delta \varepsilon_{22} = 2.939.366 \]
\[ \Delta \varepsilon_{12} = 0.00133232 \quad \Delta \sigma_{12} = 7.060.634 \]
\[ \Delta \varepsilon_{22} = 0.00033308 \quad \Delta \sigma_{22} = 2.939.366 \]

The total displacements, stresses and strains after the fifth increment give the approximate final results for the nonlinear truss subject to the given loads.

\[ \varepsilon_{1T} = \Delta \varepsilon_{11} + \Delta \varepsilon_{12} + \Delta \varepsilon_{22} + \Delta \varepsilon_{13} + \Delta \varepsilon_{23} + \Delta \varepsilon_{14} + \Delta \varepsilon_{24} \]
\[ \sigma_{1T} = \Delta \sigma_{11} + \Delta \sigma_{12} + \Delta \sigma_{13} + \Delta \sigma_{14} + \Delta \sigma_{23} + \Delta \sigma_{24} \]
\[ \varepsilon_{2T} = \Delta \varepsilon_{21} + \Delta \varepsilon_{22} + \Delta \varepsilon_{23} + \Delta \varepsilon_{33} + \Delta \varepsilon_{24} + \Delta \varepsilon_{34} \]
\[ \sigma_{2T} = \Delta \sigma_{21} + \Delta \sigma_{22} + \Delta \sigma_{23} + \Delta \sigma_{33} + \Delta \sigma_{24} + \Delta \sigma_{34} \]
\[ \varepsilon_{3T} = \Delta \varepsilon_{31} + \Delta \varepsilon_{32} + \Delta \varepsilon_{33} + \Delta \varepsilon_{13} + \Delta \varepsilon_{23} + \Delta \varepsilon_{34} \]
\[ \sigma_{3T} = \Delta \sigma_{31} + \Delta \sigma_{32} + \Delta \sigma_{33} + \Delta \sigma_{13} + \Delta \sigma_{23} + \Delta \sigma_{34} \]

\[ \varepsilon_{1T} = 0.0012732 \quad \sigma_{1T} = 11,928.516 \]
\[ \varepsilon_{2T} = 0.0050924 \quad \sigma_{2T} = 38,071.114 \]
\[ \varepsilon_{3T} = 0.0012732 \quad \sigma_{3T} = 11,928.516 \]

\[ \varepsilon_{1T} = 0. \]
\[ \varepsilon_{2T} = 0.505076 \]

This problem was solved for five increments. The actual displacements, stresses, and strains will be slightly different than the results calculated for five increments. The more increments used the more accurate the results. If the number of increments goes to infinity the solution is exact.
Section III
EXAMPLE: ITERATIVE METHOD

In this section the same example solved in section III with the incremental method is solved using the iterative method.

\[ q_1 \]
\[ q_2 \]

50,000

<table>
<thead>
<tr>
<th>Member #</th>
<th>Area</th>
<th>Length</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>200</td>
<td>30°</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>100</td>
<td>90°</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>200</td>
<td>150°</td>
</tr>
</tbody>
</table>

The material properties are the same for all three element members.
The stress-strain curve for all members

\[ \sigma = 10 \times 10^6 \varepsilon - 6.25 \times 10^8 \varepsilon^2 \]

The load of 50,000 will be solved for five iterations. E for each cycle will be taken as the secant modulus for each member's stress state. The stiffness matrix is

\[
\begin{bmatrix}
K_1
\end{bmatrix} = E_1 A_1 / L_1
\begin{bmatrix}
cos^2 \varepsilon, & -\sin \varepsilon \cos \varepsilon \\
-\sin \varepsilon \cos \varepsilon, & \sin^2 \varepsilon
\end{bmatrix}
\]

\[
K_1 = E_1 / 200
\begin{bmatrix}
.75, & -0.433 \\
-0.433, & 0.25
\end{bmatrix}
\]

\[
K_2 = E_2 / 100
\begin{bmatrix}
0., & 0. \\
0., & 1.
\end{bmatrix}
\]

\[
K_3 = E_3 / 200
\begin{bmatrix}
.75, & .433 \\
.433, & .25
\end{bmatrix}
\]

For the first increment \( E_1 \) is taken as the secant or tangent modulus at zero strain.
\[ E_1 = 10,000,000 \]

The stiffness matrix for the first iteration

\[
\begin{bmatrix} K_t \end{bmatrix} = K_1 + K_2 + K_3
\]

\[
\begin{bmatrix} K_t \end{bmatrix} = 50,000 \begin{bmatrix} 1.5 & 0 \\ 0 & 2.5 \end{bmatrix}
\]

The deflection for the first iteration

\[
\{ q \}_1 = (K_t)^{-1} \{ Q_1 \}
\]

\[
\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \frac{1}{50,000} \begin{bmatrix} .666 & 0 \\ 0 & .4 \end{bmatrix}
\]

\[
\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ .4 \end{bmatrix}
\]

The strains for each member

\[
\varepsilon_i = \frac{1}{L_i} \begin{bmatrix} \cos \theta, \sin \theta \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]

\[
\varepsilon_1 = \frac{1}{200} \begin{bmatrix} .866, .5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]

\[
\varepsilon_2 = \frac{1}{100} \begin{bmatrix} 0,.1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]

\[
\varepsilon_3 = \frac{1}{200} \begin{bmatrix} -.866, .5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]
For the first iteration

\( \varepsilon_1 = .001 \)
\( \varepsilon_2 = .004 \)
\( \varepsilon_3 = .001 \)

From the stress–strain curve the stresses due to the strains for the first iteration

\( \sigma_1 = 9,375 \)
\( \sigma_2 = 30,000 \)
\( \sigma_3 = 9,375 \)

The load in each member for the first iteration is

\[ P_i = \sigma_i A_i \]
\( P_1 = 9,375 \)
\( P_2 = 30,000 \)
\( P_3 = 9,375 \)

The loads of all three members produce a load in the \( Q_2 \) direction of

\[ Q_2 = P_1 \sin 30 + P_2 + P_3 \sin 30 \]
\( Q_2 = 39,375 \)

If this load would have equal the applied load of 50,000 the solution would have been reached. Since the truss is not in equilibrium a second iteration is needed with a
load equal to \((50,000 - 39,375)\) or 10,625. The load applied for the second iteration is 10,625. For the second iteration the secant modulus is

\[
E_1 = 9.375 \times 10^6 \\
E_2 = 7.50 \times 10^6 \\
E_3 = 9.375 \times 10^6
\]

The stiffness matrix for the second iteration

\[
K_t = 10^5 \begin{bmatrix}
0.703125, & 0. \\
0., & 0.984375
\end{bmatrix}
\]

\[
\{\Delta q\}_I = [K_t] \{Q\}
\]

\[
\{\Delta q\}_II = 10^{-5} \begin{bmatrix}
1.4222, & 0. \\
0., & 1.01587
\end{bmatrix} \{0, 10,625\}
\]

\[
\{\Delta q_1\}_II = \begin{bmatrix}
0. \\
0.107937
\end{bmatrix}
\]

The total deflection after the second iteration

\[
q_{i_T} = q_{i_I} + q_{i_II}
\]

\[
q_{1_T} = 0.
\]

\[
q_{2_T} = .507937
\]

The total strains are calculated from the total displacements.
\[ \epsilon_{\text{IT}} = .0012698 \]
\[ \epsilon_{\text{LT}} = .00507937 \]
\[ \epsilon_{\text{XT}} = .0012698 \]

From the stress-strain curve the stress after the second iteration are

\[ \sigma_{\text{IT}} = 11,690.19 \]
\[ \sigma_{\text{LT}} = 34,668.73 \]
\[ \sigma_{\text{XT}} = 11,690.19 \]

These stresses produce a load in the Q_2 direction of 46,358.9. A third iteration is solved with a load of (50,000 - 46,358.9) or 3,641.1. For the third iteration the secant modulus is

\[ E_1 = 9.2063 \times 10^6 \]
\[ E_2 = 6.8254 \times 10^6 \]
\[ E_3 = 9.2063 \times 10^6 \]

The stiffness matrix for the third iteration

\[
\left[ K_t \right] = 10^5 \begin{bmatrix} .69047, & 0. \\ 0., & .91269 \end{bmatrix}
\]

\[
\{ \Delta q \}_{\text{III}} = \left[ K_t \right] \{ \Delta Q \}_{\text{III}}
\]

\[
\{ \Delta q \}_{\text{III}} = 10^{-5} \begin{bmatrix} 1.4482, & \end{bmatrix} \begin{bmatrix} 0 \\ 0., & 1.09565 \end{bmatrix} \begin{bmatrix} 3,641.1 \end{bmatrix}
\]
\[
\begin{bmatrix}
\Delta q_1 \\
\Delta q_2
\end{bmatrix}_{\text{III}} = \begin{bmatrix}
0. \\
.03989
\end{bmatrix}
\]

The total deflection after the third iteration

\[
q_{iT} = q_{i1} + \Delta q_{i1} + \Delta q_{iIII}
\]

\[
q_{1T} = 0.
\]

\[
q_{2T} = .547831
\]

The total strains are calculated for the total displacements

\[
\varepsilon_{1T} = .00136958
\]

\[
\varepsilon_{2T} = .00547831
\]

\[
\varepsilon_{3T} = .00136958
\]

From the stress-strain curve the stress after the third iteration are

\[
\sigma_{1T} = 12,523.46
\]

\[
\sigma_{2T} = 36,025.68
\]

\[
\sigma_{3T} = 12,523.46
\]

The stresses produce a load in the \(Q_2\) direction of 48,549.14. A fourth iteration is solved for a load of (50,000 - 48,549.14) or 1,450.86. For the fourth iteration the secant modulus is

\[
E_1 = 9.144 \times 10^6
\]

\[
E_2 = 6.576 \times 10^6
\]

\[
E_3 = 9.144 \times 10^6
\]
The stiffness matrix for the fourth iteration

\[
[K_t] = 10^5 \begin{bmatrix}
.6858, & 0. \\
0., & .8862
\end{bmatrix}
\]

\[
\{\Delta q\}_4 = [K_t]^{-1} \{\Delta q\}_4
\]

\[
\{\Delta q\}_4 = 10^{-5} \begin{bmatrix}
1.458, & 0 \\
0., & 1.1284
\end{bmatrix} \begin{bmatrix}
0. \\
1,450.85
\end{bmatrix}
\]

\[
\begin{cases}
\Delta q_1 \\
\Delta q_2
\end{cases} = \begin{bmatrix}
0. \\
.016372
\end{bmatrix}
\]

The total deflection after the fourth iteration

\[
q_{\text{IT}} = q_{\text{I}} + \Delta q_{\text{II}} + \Delta q_{\text{III}} + \Delta q_{\text{IV}}
\]

\[
q_{1\text{IT}} = 0.
\]

\[
q_{2\text{IT}} = .5642
\]

The total strains are calculated for the total displacements

\[
\varepsilon_{\text{IT}} = .001411
\]

\[
\varepsilon_{2\text{IT}} = .005642
\]

\[
\varepsilon_{3\text{IT}} = .001411
\]

From the stress-strain curve the stresses after the third iteration are

\[
\sigma_{\text{IT}} = 12,865.67
\]
These stresses produce a load in the $Q_2$ direction of 49,390 which is approximately equal to the applied load of 50,000. If more accuracy is desired, the iteration process can be continued for more cycles. For this problem, this result will suffice. The final results for the displacements, stresses, and strains are

$$\begin{align*}
\sigma_{\ell T} &= 36,524.97 \\
\sigma_{\delta T} &= 12,865.67
\end{align*}$$

These results are approximately equal to the results obtained for the incremental method. For the exact results, see the same example in the appendix. In general, the iterative method is faster than the incremental method.
The exact solution for strains, stress and deflections are

\[
\begin{align*}
\sigma_1 &= 13,000 & \varepsilon_1 &= 0.00143 \\
\sigma_2 &= 37,000 & \varepsilon_2 &= 0.00572 \\
\sigma_3 &= 13,000 & \varepsilon_3 &= 0.00143
\end{align*}
\]

\[
\begin{align*}
\delta_1 &= 0 \\
\delta_2 &= 0.572
\end{align*}
\]

The largest error occurs in the deflection at \( \delta_2 \). For the incremental method, the percent error in deflection is 11.9%. For the iterative method, the error is 1.4%. The results can be improved by taking more steps.
Section IV
COMPUTER PROGRAM

IV-A FLOW CHART FOR THE INCREMENTAL METHOD

1. Read data

2. Zero out:
   - Total deflections
   - Total stresses
   - Total strains

3. Calculate elements tangent modulus based on the total element strains

4. Assemble total system stiffness matrix

5. Assemble reduced stiffness matrix and reduced incremental loads

6. Calculate reduced incremental deflections
Calculate total system incremental deflections

Calculate element incremental stresses and strains

Calculate the sum of all the incremental stresses, strains, deflections and loads

Does the sum of all incremental loads equal the total applied load

Yes

Print results

End

No
IV-B FLOW CHART FOR THE ITERATION METHOD

1. Read data
2. Zero out Total deflections
3. Calculate element secant modulus based on the total element strains
4. Assemble total system stiffness matrix
5. Assemble reduced stiffness matrix and reduced resultant load
6. Calculate reduced iterative deflections based on the resultant loads
7. Calculate total system iterative deflection
Calculate total deflection based on the sum of all iterative deflections

Calculate element strains

Calculate element stresses based on element strains

Calculate resultant loads based on the applied loads minus the loads produced by element stresses

Are the resultant loads approximately equal to the applied loads

Yes

Print results

End

No
The finite element method requires the inversion of the stiffness matrix $[K]$. Much of the computer calculation time is involved in inverting this stiffness matrix. An efficient method for inverting the stiffness matrix is desirable. One fact about the stiffness matrix is that it is symmetrical. This can be proved by Maxwell's reciprocity theorem.

1. First a load $P_1$ is applied to a structure and gives a deflection at $P_1$ equal to $x_{11}$.

![Diagram showing deflection $x_{12}$](image)

The external energy is equal to

$$U_1 = \frac{1}{2} P_1 x_{11}$$

2. Now a load $P_2$ is applied to the structure and produces an additional deflection at $P_2$ equal to $x_{22}$ and an additional deflection at $P_1$ equal to $x_{12}$. 
The external energy for both loads applied

\[ U_{1+2} = \frac{1}{2} P_1 x_{11} + \frac{1}{2} P_2 x_{22} + P_1 x_{12} \]

3. Now a load \( P_2 \) is first applied and it produces a deflection at point 2 equal to \( x_{22} \).

The external energy is equal to

\[ U_2 = \frac{1}{2} P_2 x_{22} \]

4. Now a load \( P_1 \) is applied and produces a deflection at \( P_1 \) equal to \( x_{11} \) and an additional deflection at \( P_2 \) equal to \( x_{21} \).
The external energy for both loads applied

\[ U_{1+2} = \frac{1}{2} P_2 x_{22} + P_2 x_{21} + \frac{1}{2} P_1 x_{11} \]

Since from superposition the external energy is independent of the order in which the loads are applied.

\[ \frac{1}{2} P_1 x_{11} + \frac{1}{2} P_2 x_{22} + P_1 x_{12} = \]

\[ = \frac{1}{2} P_2 x_{22} + \frac{1}{2} P_1 x_{11} + P_2 x_{21} \]

\[ P_1 x_{12} = P_2 x_{21} \]

Since

\[ x_{12} = \frac{P_2}{k_{12}} \quad x_{21} = \frac{P_1}{k_{21}} \]

\[ P_1 P_2 / k_{12} = P_2 P_1 / k_{21} \]

Therefore

\[ k_{12} = k_{21} \]

Similarly it can be proved that

\[ k_{i,j} = k_{j,i} \]
The symmetrical stiffness matrix can be decomposed into the product of three matrices

\[
K = [U]^T[D][U].
\]

Where \([D]\) is a diagonal matrix and \([U]\) is an upper triangular matrix

\[
[D] = \begin{bmatrix}
D_{11} & 0 & \cdots & 0 \\
0 & D_{22} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & D_{nn}
\end{bmatrix}
\quad [U] = \begin{bmatrix}
U_{12} & \cdots & U_{1n} \\
0 & \cdots & \vdots \\
\vdots & \ddots & \ddots \\
0 & \cdots & 0
\end{bmatrix}
\quad [U]^T = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
U_{12} & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
U_{1n} & \cdots & 1 & 1
\end{bmatrix}
\]

From matrix algebra

\[
[K]' = [U]'[D]'[U]'^T
\]

The coefficients of \([U], [U]^T\) and \([D]\) can be determined from matrix multiplication

\[
K_{11} = D_{11} \\
K_{12} = D_{11}U_{12} \\
K_{13} = D_{11}U_{13} \\
K_{1n} = D_{11}U_{1n} \\
K_{22} = U_{12}^2 + D_{11} + D_{22} \\
K_{23} = U_{12}U_{13}D_{11} + D_{22}U_{23} \\
K_{24} = U_{12}U_{14}D_{11} + D_{22}U_{24} \\
K_{2n} = U_{12}U_{1n}D_{11} + D_{22}U_{2n}
\]
From a more exhaust matrix multiplication, the recursion formulas can be determined as

\[ D_{11} = A_{11} \]

\[ U_{11} = 1 \quad \text{for } i=1,n \]

\[ U_{1j} = K_{1j}/D_{11} \quad j \geq 2 \]

\[ D_{ii} = K_{ii} - \sum_{k=1}^{i-1} U_{ki}^2 D_{kk} \quad i \geq 2 \]

\[ U_{ij} = 1/D_{ii} \left( K_{ij} - \sum_{k=1}^{i-1} U_{ki} U_{kj} D_{kk} \right) \quad i \geq 2, j \geq i+1 \]

The inversion of \([D]\), \([U]\), and \([U]\) can also be found from matrix multiplication.

\[ D_{11} = 1/D_{11} \]

\[ D_{ii} = 0 \quad \text{for } i \neq i \]

\[ [U]^T [U] = [I] \]

\[
\begin{bmatrix}
U_{11} & U_{12} & \cdots & U_{1n} \\
U_{21} & U_{22} & \cdots & U_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & U_{nn}
\end{bmatrix}
\begin{bmatrix}
1 & U_{12} & \cdots & U_{1n} \\
0 & 1 & \cdots & U_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 1
\end{bmatrix} = [I]
\]
From matrix multiplication

\[ l = U_{11}^{-1} \]
\[ l = U_{22}^{-1} \]
\[ l = U_{ii}^{-1} \text{ for } i = 1, n \]
\[ 0 = U_{12} + U_{12}^{-1} \]
\[ 0 = U_{13} + U_{23} U_{12}^{-1} + U_{13}^{-1} \]
\[ 0 = U_{14} + U_{24} U_{12}^{-1} + U_{34} U_{13}^{-1} + U_{14}^{-1} \]

From an exhausted matrix multiplication, the recursion formulas can be determined as

\[ U_{11}^{-1} = l \text{ for } i = 1, n \]
\[ U_{1j}^{-1} = \sum_{k=1}^{j-1} U_{ik}^{-1} U_{kj} \text{ for } i < j \]
\[ [U]^{-1} = [U]^{-T} \]

Now the inverse of the stiffness matrix can be determined from matrix multiplication

\[ [K]^{-1} = [U]^{-1} [D]^{-1} [U]^{-T} \]
The capacity of the program depends on the size of the dimension statements. At present, the program is dimensioned to handle a maximum of 10 nodes, 30 elements and 10 different materials. The input instructions are listed as follows:

I Control cards

There are two control cards. The first card reads the total number of nodes, total number of elements, and the total number of materials. The second card reads the number of nodes with loads and the total number of increments. (Note: If NLI=1, it is for the elastic case.)

Read (60,501) NNT,NNE,NMT
501 Format (3I5)
Read (60,502) NL,NLI
502 Format (2I5)

II Material cards

The total number of material cards depend upon the total number of materials and the type of material. The first card reads the type of material, 0 for elastic plastic and 1 for a third order polynomial tangent modulus verses strain curve. If the material type if 0, then the next card reads the Young's modulus for the
elastic portion and the strain when the tangent modulus is zero. If the material type is 1, then the next 2 cards read 4 values of tangent modulus on the first card and four values of corresponding strains on the second card.

```
Do 10 I = 1, NMT
   Read (60, 503) ITYPE(I)
503 Format (I5)
   For ITYPE(I) = 0
      Read (60, 505) EL(I), SP(I)
505 Format (2F10.5)
      Go to 10
   For ITYPE(I) = 1
      Read (60, 504) EL(I), E2(I), E3(I), E4(I)
      Read (60, 504) S1(I), S2(I), S3(I), S4(I)
504 Format (4F10.5)
10 Continue
```

III Nodal cards

The total number of cards is equal to the total number of nodes. These cards read node number, degree of freedom in the I, J, K direction, (0 for free, 1 for fixed), and the X, Y, Z location for the particular node. The nodes must be read in node order.

```
Read (60, 506) NN, IDF(I), IDF(J), IDF(K), X, Y, Z
506 Format (4I5, 3F10.5)
```

IV Element cards

The total number of cards is equal to the total number of elements. These cards read the element number,
the two nodes located on the end of the element, the material number, and the area of the element.

\[
\text{Read (60,507) NE,NNI(NE),NNJ(NE),MN(NE),AREA(NE)}
\]
\[
507 \text{ Format (4I5,F10.5)}
\]

V Load cards

The total number of cards is equal to two times the total number of nodes with loads. The first card in a sequence reads the node being loaded and the second reads the load in the X, Y, Z direction.

\[
\text{Read (60,508) NN}
\]
\[
508 \text{ Format (I5)}
\]
\[
\text{Read (60,509) P(I),P(J),P(K)}
\]
\[
509 \text{ Format (3F10.5)}
\]
IV-E INPUT INSTRUCTIONS FOR THE ITERATIVE METHOD

The capacity of the program depends on the size of the dimension statements. At present, the program is dimensioned to handle a maximum of 10 nodes, 30 elements and 10 different materials. The input instructions are listed as follows:

I  Control cards

There are two control cards. The first card reads the total number of nodes, total number of elements, and the total number of materials. The second card reads the number of nodes with loads and the total number of iterations. (Note: If NLI=1, it is for the elastic case.)

Read (60,501) NNT,NNE,NMT  
501 Format (3I5)  
Read (60,502) NL,NLI  
502 Format (2I5)

II  Material cards

The total number of material cards depend upon the total number of materials and the type of material. The first card reads the type of material, 0 for elastic plastic and 1 for a third order polynomial stress verses strain curve. If the material type is 0, then the next card reads the Young's modulus for the elastic portion
and the strain when the tangent modulus is zero. If the material type is 1, then the next 2 cards read 4 values of stresses on the first card and four values of corresponding strains on the second card.

```plaintext
Do 10 I = 1, NMT
   Read (60, 503) ITYPE(I)
   503 Format (I5)
   For ITYPE(I) = 0
   Read (60, 505) EL(I), SB(I)
   505 Format (2F10.5)
   Go to 10
   For ITYPE(I) = 1
   Read (60, 504) SI1(I), SI2(I), SI3(I), SI4(I)
   Read (60, 504) S1(I), S2(I), S3(I), S4(I)
   504 Format (4F10.5)
10 Continue
```

III Nodal cards

The total number of cards are equal to the total number of nodes. These cards read node number, degree of freedom in the I, J, K direction, (0 for free, 1 for fixed), and the X, Y, Z location for the particular node. The nodes must be read in node order.

```plaintext
Read (60, 506) NN, IDF(I), IDF(J), IDF(K), X, Y, Z
506 Format (4I5, 3F10.5)
```

IV Element cards

The total number of cards are equal to the total number of elements. These cards read the element number,
the two nodes located on the end of the element, the material number, and the area of the element.

\[
\text{Read (60,507) NE,NNI(NE),NNJ(NE),MN(NE),AREA(NE)}
\]
\[
507 \text{ Format (4I5,F10.5)}
\]

V Load cards

The total number of cards are equal to two times the total number of nodes with loads. The first card in a sequence reads the node being loaded and the second reads the load in the X, Y, Z direction.

\[
\text{Read (60,508) NN}
\]
\[
508 \text{ Format (I5)}
\]
\[
\text{Read (60,509) P(I),P(J),P(K)}
\]
\[
509 \text{ Format (3F10.5)}
\]
IV-F COMPUTER LISTING FOR THE INCREMENTAL METHOD
ANSI FORTRAN(2,3) MASTER  INTEGER WORD SIZE = 1, * OPTION IS OFF, * OPTION IS OFF

LN 0001 COMMON A(10), B(10), C(10), D(10), STRAIN(I30), SP(I30), E13G, E1(10)
LN 0002 COMMON S1(I30), S2(I30), S3(I30), S4(I30), TYPE(10), MN(10), NET
LN 0003 DIMENSION E(10), E(10), E(10), E(10), IF(10), X(130), Y(130), Z(130)
LN 0004 DIMENSION U(30, 30), P(30, 30), STRESS(30), PR(30)
LN 0005 DIMENSION X(30, 30), Y(30, 30), Z(30), S(30, 30), D(30)
LN 0006 DIMENSION X(I30), Y(I30), Z(I30), S(I30), D(I30)
LN 0007 DIMENSION D(I30), STRAIN(I30), STRESS(I30), LENGTH(I30)
LN 0008 C READ NUMBER OF NODES, NUMBER OF ELEMENTS, NUMBER OF MATERIALS
LN 0009 READ (66,501) MN, NET, NM
LN 0010 501 FORMAT (3I5)
LN 0011 C READ NUMBER OF NODES WITH LOADS, AND NUMBER OF INCREMENTS TO DIVIDE LOADS.
LN 0012 C NOTE IF NLI IS EQUAL TO 1 IT IS FOR ELASTIC CASE
LN 0013 READ (66,502) NL, NLI
LN 0014 502 FORMAT (2I5)
LN 0015 GO TO 11
LN 0016 C READ I, YTYPE(I) 0 FOR ELASTIC PLASTIC, 1 FOR 3RD ORDER POLYNOMIAL
LN 0017 READ (66,503) I, YTYPE(I)
LN 0018 503 FORMAT (2I5)
LN 0019 IF (YTYPE(I) .EQ. 0) GO TO 20
LN 0020 C READ A VALUES FOR F AND CORRESPONDING STRAINS
LN 0021 C TWD CARDS, E ON FIRST CARD STRAINS ON SECOND
LN 0022 READ (66,504) E(1), E(2), E(3), E(4)
LN 0023 READ (66,504) S1(I), S2(I), S3(I), S4(I)
LN 0024 504 FORMAT (4F10.5)
LN 0025 READ (66,505) A(I)
LN 0026 READ (66,505) B(I), C(I), D(I), E(I), F(I)
LN 0027 505 FORMAT (5F10.5)
LN 0028 C(I) = C(I) + E(I) * A(I) - C(I) * A(I) * B(I)
LN 0029 D(I) = D(I) + E(I) * A(I) - D(I) * A(I) * B(I)
LN 0030 E(I) = E(I) + E(I) * A(I) - E(I) * A(I) * B(I)
LN 0031 GO TO 10
LN 0032 C READ FOR ELASTIC PLASTIC MATERIAL E ELASTIC, AND STRAIN WHEN E=0
LN 0033 20 READ (66,506) E(I), SP(I)
LN 0034 506 FORMAT (2F10.5)
LN 0035 10 CONTINUE
LN 0036 I = I + 1
LN 0037 GO TO 20
LN 0038 C READ NODE NUMBER DEGRE OF FREEDOM IN I,J,K
LN 0039 C X,Y,Z. NOTE MUST BE READ IN NODE ORDER 1,2,3,4 ETC.
LN 0040 READ (66,506) NX, IODF(I), IODF(I+1), IODF(I+2), XI(I), YJ(I), ZJ(I)
LN 0041 506 FORMAT (4I5,3F10.5)
LN 0042 I = I + 1
LN 0043 30 CONTINUE
LN 0044 GO TO 40
LN 0045 C READ ELEMENT NUMBER, ITH NODE NUMBER, JTH NODE NUMBER, MATERIAL NUMBER, AREA
LN 0046 READ (66,507) NE, NN1(I), NN2(I), MNI(I), AREAI
LN 0047 507 FORMAT (6I5,F15.5)
LN 0048 NE = NE + 1
LN 0049 I = I + 1
LN 0050 XI = XI(I)
LN 0051 YJ = YJ(I)
LN 0052 ZK = ZK(I)
LN 0053 LENGTH(I) = SQRT(XI**2 + YJ**2 + ZK**2)
LN 0054 BNE = I = INL
ANSI FORTRAN IZ.31/MASTER

INTEGER WORD SIZE = 1, * OPTION IS OFF, O OPTION IS

LN 0055  BINE,2) = YLY
LN 0056  BINE,3) = ZLZ
LN 0057  BINE,4) = XLY
LN 0058  BINE,5) = YLY
LN 0059  40 BINE,6) = ZLZ
LN 0060  NDF=NNT=3
LN 0061  C ZERO OUT DISPLACEMENTS, STRAINS, STRESSES, LOADS
LN 0062  DO 58 I=1,NDF
LN 0063  X(I) = 0.
LN 0064  50 P(I) = 0.
LN 0065  DO 60 I=1,NET
LN 0066  STRAIN(I) = 0.
LN 0067  60 STRESS(I) = 0.
LN 0068  I=COUNT=1
LN 0069  C READ NODE LOADS (NODE NUMBER, AND LOAD IN X,Y,Z)
LN 0070  DO 70 I=1,NL
LN 0071  READ (60,508) NN
LN 0072  500 FORMAT (IS)
LN 0073  N3=NNX3=3
LN 0074  70 READ (60,509) P(N3+1), P(N3+2), P(N3+3)
LN 0075  509 FORMAT (3F10.5)
LN 0076  C ASSEMBLING REDUCED P COLUMN MATRIX
LN 0077  NR=0,7
LN 0078  IF=1
LN 0079  DO 80 I=1,NDF
LN 0080  IF=10 IF(I).NE.0 GO TO 80
LN 0081  NR=NR+1
LN 0082  IF=IP+1
LN 0083  IP=IP+1
LN 0084  80 CONTINUE
LN 0085  60 WRITE (61,600) NNT, NET, NMT
LN 0086  600 FORMAT (* NUMBER OF ELEMENTS = *,I5,//)
LN 0087  1* NUMBER OF ELEMENTS = *,I5,//
LN 0088  2* NUMBER OF MATERIALS = *,I5,/////)
LN 0089  WRITE (61,601)
LN 0090  601 FORMAT (I)
LN 0091  1* NODE DEGREE OF FREEDOM LOCATION OF COORDINATES*/*
LN 0092  2* NUMBER X Y Z
LN 0093  DO 90 I=1,NET
LN 0094  N3=N3+1
LN 0095  70 WRITE (61,602) I, IDF(N3+1), IDF(N3+2), IDF(N3+3), X(I), Y(I), Z(I)
LN 0096  602 FORMAT (4I5,F10.5)
LN 0097  WRITE (61,603)
LN 0098  603 FORMAT (/////)
LN 0099  4* ELEMENT NODE POINT NUMBERS MATERIAL AREA*/*
LN 0100  2* NUMBER I J
LN 0101  DO 100 I=1,NMT
LN 0102  710 WRITE (61,604) I, NN(I), NNJ(I), MN(I), AREA(I)
LN 0103  604 FORMAT (I,2I10,F10.5)
LN 0104  WRITE (61,605)
LN 0105  605 FORMAT (I)////I
LN 0106  DO 720 I=1,NMT
LN 0107  IF (TYPE(I).NE.0) GO TO 730
LN 0108  WRITE (61,606) I
<table>
<thead>
<tr>
<th>Line</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>LN 0109</td>
<td>606 FORMAT (* MATERIAL NUMBER =<em>,I5,5X,<em>TYPE(3</em>RD ORDER POLYNOMIAL)</em>)</td>
</tr>
<tr>
<td>LN 0110</td>
<td>WRITE (6,607) E1(I),E2(I),E3(I),E4(I)</td>
</tr>
<tr>
<td>LN 0111</td>
<td>607 FORMAT (* E1=*,E10.3,5X,<em>E2=</em>,E10.3,5X,<em>E3=</em>,E10.3,5X,<em>E4=</em>,E10.3)</td>
</tr>
<tr>
<td>LN 0112</td>
<td>WRITE (6,608) S1(I),S2(I),S3(I),S4(I)</td>
</tr>
<tr>
<td>LN 0113</td>
<td>608 FORMAT (* S1=*,E10.3,5X,<em>S2=</em>,E10.3,5X,<em>S3=</em>,E10.3,5X,<em>S4=</em>,E10.3)</td>
</tr>
<tr>
<td>LN 0114</td>
<td>GO TO 720</td>
</tr>
<tr>
<td>LN 0115</td>
<td>720 WRITE 161,605 I,ELIII,SPIII</td>
</tr>
<tr>
<td>LN 0116</td>
<td>FORHAT 1 MATERIAL NUMBER =,I5,5X,TYPEIELASTIC-PLASTICJ,</td>
</tr>
<tr>
<td>LN 0117</td>
<td>I=x,E=*,E10.3,5X,<em>PLASTIC STRAIN =</em>,E10.3</td>
</tr>
<tr>
<td>LN 0118</td>
<td>GO-TO 730 _____</td>
</tr>
<tr>
<td>LN 0119</td>
<td>C CALCULATE YOUNG'S MODULUS FOR EACH ELEMENT</td>
</tr>
<tr>
<td>LN 0120</td>
<td>CALL YOUNG</td>
</tr>
<tr>
<td>LN 0121</td>
<td>C ZERO STIFFNESS MATRIX</td>
</tr>
<tr>
<td>LN 0122</td>
<td>DO 120 I=1,NDF</td>
</tr>
<tr>
<td>LN 0123</td>
<td>DO 100 J=1,NDF</td>
</tr>
<tr>
<td>LN 0124</td>
<td>100 S1=0.</td>
</tr>
<tr>
<td>LN 0125</td>
<td>C CALCULATE GLOBAL STIFFNESS, AND TOTAL SYSTEM STIFFNESS</td>
</tr>
<tr>
<td>LN 0126</td>
<td>DO 120 K=1,NET</td>
</tr>
<tr>
<td>LN 0127</td>
<td>NI=NNI(K)</td>
</tr>
<tr>
<td>LN 0128</td>
<td>NJ=NNJ(K)</td>
</tr>
<tr>
<td>LN 0129</td>
<td>DO 110 I=1,6</td>
</tr>
<tr>
<td>LN 0130</td>
<td>DO 110 J=1,6</td>
</tr>
<tr>
<td>LN 0131</td>
<td>110 G(K,1)=B(K,1)*B(I,1)*AREA(K)*E(K)/(LENGTH(K)**3)</td>
</tr>
<tr>
<td>LN 0132</td>
<td>DO 120 I=1,J</td>
</tr>
<tr>
<td>LN 0133</td>
<td>DO 120 J=1,I</td>
</tr>
<tr>
<td>LN 0134</td>
<td>M1=NT**3-3*I</td>
</tr>
<tr>
<td>LN 0135</td>
<td>N2=NT**3-3*I</td>
</tr>
<tr>
<td>LN 0136</td>
<td>N3=NT**3-3*I</td>
</tr>
<tr>
<td>LN 0137</td>
<td>N4=NT**3-3*I</td>
</tr>
<tr>
<td>LN 0138</td>
<td>S(N1,N2)=G(K,1)+S(N1,N2)</td>
</tr>
<tr>
<td>LN 0139</td>
<td>S(N3,N4)=G(K,1)+S(N3,N4)</td>
</tr>
<tr>
<td>LN 0140</td>
<td>S(N1,N4)=G(K,1)+S(N1,N4)</td>
</tr>
<tr>
<td>LN 0141</td>
<td>120 S(N3,N2)=G(K,1)+S(N3,N2)</td>
</tr>
<tr>
<td>LN 0142</td>
<td>C ASSEMBLING REDUCED STIFFNESS MATRIX</td>
</tr>
<tr>
<td>LN 0143</td>
<td>IA=0</td>
</tr>
<tr>
<td>LN 0144</td>
<td>ISR=1</td>
</tr>
<tr>
<td>LN 0145</td>
<td>DO 130 I=1,NDF</td>
</tr>
<tr>
<td>LN 0146</td>
<td>JSR=1</td>
</tr>
<tr>
<td>LN 0147</td>
<td>ISR=ISR+IA</td>
</tr>
<tr>
<td>LN 0148</td>
<td>IA=0</td>
</tr>
<tr>
<td>LN 0149</td>
<td>DO 130 J=1,NDF</td>
</tr>
<tr>
<td>LN 0150</td>
<td>IF (ISR(J).NE.0) GO TO 130</td>
</tr>
<tr>
<td>LN 0151</td>
<td>IF (ISR(I).NE.0) GO TO 130</td>
</tr>
<tr>
<td>LN 0152</td>
<td>IA=1</td>
</tr>
<tr>
<td>LN 0153</td>
<td>SR=ISR<em>JSR</em>JSR</td>
</tr>
<tr>
<td>LN 0154</td>
<td>JSR=ISR*ISR</td>
</tr>
<tr>
<td>LN 0155</td>
<td>130 CONTINUE</td>
</tr>
<tr>
<td>LN 0156</td>
<td>C SIZE OF REDUCED 'MATRIX SR(NR,NR)</td>
</tr>
<tr>
<td>LN 0157</td>
<td>C CALCULATING THE INVERSE OF REDUCED MATRIX</td>
</tr>
<tr>
<td>LN 0158</td>
<td>CALL INVERSE (NR,SR,SM)</td>
</tr>
<tr>
<td>LN 0159</td>
<td>C CALCULATE DIFFERENTIAL CHANGE IN LENGTH OF REDUCED DEFLECTIONS</td>
</tr>
<tr>
<td>LN 0160</td>
<td>DO 150 I=1,NR</td>
</tr>
<tr>
<td>LN 0161</td>
<td>DQ&amp;R(I)=0.</td>
</tr>
<tr>
<td>LN 0162</td>
<td>DO 150 K=1,NR</td>
</tr>
</tbody>
</table>
ANSI FORTRAN(2.3)/MASTER

INTEGER WORD SIZE = 1  * OPTION IS OFF  * 0 OPTION IS 0

LN 0163  150 DXR(I)=DXR(I)+SK(3,I)*PR(I)/NLI
LN 0164  C CALCULATION OF DIFFERENTIAL CHANGE IN LENGTH OF TOTAL SYSTEM
LN 0165  IDX=1
LN 0166  DO 160 IDX=IDX+1
LN 0167  IF (IDF(I).NE.0) GO TO 170
LN 0168  OX(I)=DXR(IDX)
LN 0169  GO TO 160
LN 0170  170 OX(I)=0.
LN 0171  160 CONTINUE
LN 0172  C CALCULATION OF ELEMENB NUMBER STRAINS, STRESSES
LN 0173  DO 168 1=1,NOF
LN 0174  OX(I)=O:N(K)=0.
LN 0175  NON(K)=10.*N(K)/3+1
LN 0176  N2=NN(K)+5-3+1
LN 0177  N3=NN(K)+5-3+1
LN 0178  DSTRAIN(K)=O:STRAIN(K)+8(K,I)*DX(N1)+8(K,I+1)*DX(N2)
LN 0179  DSTRAIN(K)=DSTRAIN(K)/(1+LENGTH(K)**2)
LN 0180  OSTRAIN(K)=E(K)*DSTRAIN(K)
LN 0181  OSTRAIN(K)=OSTRAIN(K)+OSTRAIN(K)
LN 0182  OSTRAIN(K)=OSTRAIN(K)+OSTRAIN(K)
LN 0183  C CALCULATION OF TOTAL DISPLACEMENTS
LN 0184  DO 185 1=1,NOF
LN 0185  GO TO 220
LN 0186  200 X(I)=X(I)+DX(I)
LN 0187  IF (ICOUNT.EQ.NLI) GO TO 220
LN 0188  ICOUNT=ICOUNT+1
LN 0189  GO TO 90
LN 0190  90 CONTINUE
LN 0191  611 FORMAT (//)
LN 0192  1* NODE DEFLECTION OF NODE*,/,
LN 0193  2* NUMBER X Y Z*
LN 0194  5* MAT
LN 0195  10 3=1-3
LN 0196  330 WRITE (61,611) 1,XT(N3+1),XT(N3+2),XT(N3+3)
LN 0197  611 FORMAT (15,4X,E10.3,5X,E10.3,5X,E10.3)
LN 0198  612 WRITE (61,612)
LN 0199  612 FORMAT (///,
LN 0200  1* ELEMENT NUMBER STRAIN STRESS*)
LN 0201  240 WRITE (61,613) 1,STRAIN(I),STRESS(I)
LN 0202  613 FORMAT (19,14X,E10.3,10X,E10.3)
LN 0203  END

USASI FORTRAN DIAGNOSTIC RESULTS FOR FINMAIN

NO ERRORS
SUBROUTINE INVERT (N,XK,XXI)
DIMENSION D(I30),UI(30,30),XK(30,30),XXI(30,30),UI(30,30)
DO 10 I=1,N
10 UI(I,I)=0.
DO 10 I=1,N
UI(I,J)=D(I)
10 CONTINUE
DO 20 I=1,N
XXI(I,J)=UI(I,I)*UI(I,J)+XXI(I,J)
20 CONTINUE
DO 30 K=1,N
IF (NK.LT.K) GO TO 30
UI(I,J)=UI(I,J)-UI(K,J)*UI(K,I)
30 CONTINUE
RETURN
END
SUBROUTINE YOUNG

COMMON A(10), B(10), C(10), D(10), STRAIN(30), SP(10), E(30), EL(10)

COMMON S1(10), S2(10), S3(10), S4(10), JTYPE(10), MN(10), NET

GO TO 10

I = I + 1,

M = MN(I)

IF (I(TYPE(M), EQ.0) GO TO 20

E(I) = A(I) + B(I) * (STRAIN(I) - S1(I)) +

C(I) * (STRAIN(I) - S1(I)) * (STRAIN(I) - S2(I)) +

E(I) = (STRAIN(I) - S3(I)) * (STRAIN(I) - S3(I))

GO TO 10

GO TO 20 IF (STRAIN(I), LT, SP(M)) GO TO 30

GO TO 10

E(I) = 0.

GO TO 10

GO TO 10

RETURN

END

USASI FORTRAN Diagnostic Results for YOUNG

NO ERRORS
IV-G COMPUTER LISTING FOR THE ITERATIVE METHOD
```
| LN 0001 | COMMON A(101),I(101),C(131),D(131),S(131),SP(131),E(131),EL(101) |
| LN 0002 | COMMON SI(131),S2(131),SI(131),J=1,101,ITYPE(131),MN(131),NET,STRF,SS(131) |
| LN 0003 | DIMENSION 31(131),32(131),31(131),25(131),IDF(131),Y(31),X(31) |
| LN 0004 | DIMENSION 21,30,14,31,15,14,58,10,IP(10),IPX(10) |
| LN 0005 | DIMENSION XI(101),XI2(101),XI3(101),XI4(101),XI5(101),XI6(101),XI7(101),XI8(101) |
| LN 0006 | DIMENSION OFX(30),LENGTH(31) |
| LN 0007 | C READ NUMBER OF NODES, NUMBER OF ELEMENTS, NUMBER OF MATERIALS |
| LN 0008 | READ (5,50) MN,NET,STRF |
| LN 0009 | C READ FORMAT (3,5) |
| LN 0010 | C READ FORMAT (10) |
| LN 0011 | C READ FORMAT(10) |
| LN 0012 | C READ FORMAT (50) |
| LN 0013 | C READ N(5,50) NL,HLI |
| LN 0014 | C READ FORMAT (10) |
| LN 0015 | COV 1=1,4,HL |
| LN 0016 | C READ FORMAT (3,5) FOR ELASTIC PLASTIC, 1 FOR INC ORDER POLYNOMIAL |
| LN 0017 | IF ITYPE(1)=0 GO TO 23 |
| LN 0018 | C READ 4 VALUES FOR SIGMA AND CORRESPONDING STRAINS, TWO CARDS. SIGMA * 000 |
| LN 0019 | C ON THE FIRST STRAIN IS ON THE SECOND |
| LN 0020 | READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0021 | IF ITYPE(1)=6 GO TO 23 |
| LN 0022 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0023 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0024 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0025 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0026 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0027 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0028 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0029 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0030 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0031 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0032 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0033 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0034 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0035 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0036 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0037 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0038 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0039 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0040 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0041 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0042 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0043 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0044 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0045 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0046 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0047 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0048 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0049 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0050 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0051 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0052 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0053 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
| LN 0054 | C READ (5,50) S11(1),S12(1),S13(1),S23(1) |
```
<table>
<thead>
<tr>
<th>Line</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>LN 0109</td>
<td>IF (ITYPE(I),=0.0) GO TO 730</td>
</tr>
<tr>
<td>LN 0111</td>
<td>WRITE (61,6,97) S1(1),S1(2),S1(3),S1(4)</td>
</tr>
<tr>
<td>LN 0113</td>
<td>WRITE (61,6,36) S1(i),S2(i),S3(i),S4(i)</td>
</tr>
<tr>
<td>LN 0115</td>
<td>WRITE (61,6,38) S1(i),S2(i),S3(i),S4(i)</td>
</tr>
<tr>
<td>LN 0116</td>
<td>WRITE (61,6,39) S1(i),S2(i),S3(i),S4(i)</td>
</tr>
<tr>
<td>LN 0117</td>
<td><em>E11,5.5x,STRAIN=</em>/E0.3</td>
</tr>
<tr>
<td>LN 0118</td>
<td>G0 TO 720</td>
</tr>
<tr>
<td>LN 0119</td>
<td>WRITE (61,6,15) I,E(IJ,1,1)</td>
</tr>
<tr>
<td>LN 0120</td>
<td>WRITE (61,6,17) I,E(IJ,1,2)</td>
</tr>
<tr>
<td>LN 0121</td>
<td>WRITE (61,6,18) I,E(IJ,1,3)</td>
</tr>
<tr>
<td>LN 0122</td>
<td>WRITE (61,6,19) I,E(IJ,1,4)</td>
</tr>
<tr>
<td>LN 0123</td>
<td>CALL YOUNG</td>
</tr>
<tr>
<td>LN 0124</td>
<td>C ZERO STIFFNESS MATRIX</td>
</tr>
<tr>
<td>LN 0125</td>
<td>DO 100 I=1,NDF</td>
</tr>
<tr>
<td>LN 0126</td>
<td>DO 100 J=1,NDF</td>
</tr>
<tr>
<td>LN 0127</td>
<td>100 CONTINUE</td>
</tr>
<tr>
<td>LN 0128</td>
<td>C CALCULATE GLOBAL STIFFNESS, AND TOTAL SYSTEM STIFFNESS</td>
</tr>
<tr>
<td>LN 0129</td>
<td>DO 100 K=1,NET</td>
</tr>
<tr>
<td>LN 0130</td>
<td>N=NN(J,K)</td>
</tr>
<tr>
<td>LN 0131</td>
<td>N=NN(J,K)</td>
</tr>
<tr>
<td>LN 0132</td>
<td>DO 110 J=1,NDF</td>
</tr>
<tr>
<td>LN 0133</td>
<td>DO 110 J=1,NDF</td>
</tr>
<tr>
<td>LN 0134</td>
<td>110 J=1,NDF</td>
</tr>
<tr>
<td>LN 0135</td>
<td>DO 120 J=1,NDF</td>
</tr>
<tr>
<td>LN 0136</td>
<td>DO 120 J=1,NDF</td>
</tr>
<tr>
<td>LN 0137</td>
<td>N=NI(J,K)</td>
</tr>
<tr>
<td>LN 0138</td>
<td>N=NI(J,K)</td>
</tr>
<tr>
<td>LN 0139</td>
<td>N=NI(J,K)</td>
</tr>
<tr>
<td>LN 0140</td>
<td>N=NI(J,K)</td>
</tr>
<tr>
<td>LN 0141</td>
<td>SI(NJ)=SK1(J*K1)</td>
</tr>
<tr>
<td>LN 0142</td>
<td>SI(NJ)=SK1(J*K1)</td>
</tr>
<tr>
<td>LN 0143</td>
<td>SI(NJ)=SK1(J*K1)</td>
</tr>
<tr>
<td>LN 0144</td>
<td>SI(NJ)=SK1(J*K1)</td>
</tr>
<tr>
<td>LN 0145</td>
<td>C ASSEMBLING REDUCED STIFFNESS MATRIX</td>
</tr>
<tr>
<td>LN 0146</td>
<td>T4=0</td>
</tr>
<tr>
<td>LN 0147</td>
<td>T5=1</td>
</tr>
<tr>
<td>LN 0148</td>
<td>DO 130 I=1,NDF</td>
</tr>
<tr>
<td>LN 0149</td>
<td>JS=I</td>
</tr>
<tr>
<td>LN 0150</td>
<td>TS=TS+14</td>
</tr>
<tr>
<td>LN 0151</td>
<td>I4=0</td>
</tr>
<tr>
<td>LN 0152</td>
<td>DO 130 J=1,NDF</td>
</tr>
<tr>
<td>LN 0153</td>
<td>IF (ITP(I,J),=0.0) GO TO 133</td>
</tr>
<tr>
<td>LN 0154</td>
<td>IF (ITP(I,J),=0.0) GO TO 133</td>
</tr>
<tr>
<td>LN 0155</td>
<td>I4=1</td>
</tr>
<tr>
<td>LN 0156</td>
<td>JS=JS+1</td>
</tr>
<tr>
<td>LN 0157</td>
<td>JS=JS+1</td>
</tr>
<tr>
<td>LN 0158</td>
<td>130 CONTINUE</td>
</tr>
<tr>
<td>LN 0159</td>
<td>C SIZE OF REDUCED MATRIX SUMTV=TV1</td>
</tr>
<tr>
<td>LN 0160</td>
<td>C CALCULATING THE INVERSE OF REDUCED MATRIX</td>
</tr>
<tr>
<td>LN 0161</td>
<td>CALL INVE (AR,NV,SV)</td>
</tr>
<tr>
<td>LN 0162</td>
<td>C CALCULATE DIFFERENTIAL CHANGE IN LENGTH OF REDUCED DEFORMATIONS</td>
</tr>
</tbody>
</table>
ANSI FORTRAN 2.31/MASTER  INTEGRAL WORD SIZE = 1, * OPTION IS JFF, O OPTION IS OP

LN 0161 00 159 I=1,NA
LN 0164 0XRI[I]=3.
LN 0165 00 159 K=1,UP
LN 0166 150 DXRI[I]=DXRI[I]+SRRI[I]*I*PRRI[I]
LN 0167 C CALCULATION OF DIFFERENTIAL CHANGE IN LENGTH OF TOTAL SYSTEM 000
LN 0169 10=1
LN 0169 00 160 I=1,NDF
LN 0170 IF (TOF(I),166,0) GO TO 170
LN 0171 DXRI[I]=DXRI[I]
LN 0172 10=10+1
LN 0174 GO TO 160
LN 0175 160 CONTINUE
LN 0176 C CALCULATE TOTAL SYSTEM DEFLECTIONS 000
LN 0177 00 163 0=1,NC
LN 0178 180 XT[I]=XT[I]+DX[I]
LN 0179 C CALCULATION OF ELEMENT STRAINS 000
LN 0180 00 167 0=1,NET
LN 0181 SPAINK=9.
LN 0182 77 200 I=1,3
LN 0183 NINNITK=I+3+1
LN 0184 02=NINJXK=I+3+1
LN 0186 190 STRAI[IK]=STRAIN[IK]/(STRENGTH[IK]**2)
LN 0187 CALL SIGMA
LN 0188 00 205 I=1,NDF
LN 0189 205 DPI[I]=O.
LN 0190 00 210 K=1,NET
LN 0191 70 220 I=1,3
LN 0192 NINNITK=I+3+1
LN 0193 02=NINJXK=I+3+1
LN 0194 DPRI[I]=DPRI[I]*STRK[IK]*STREK[IK]/STRENGTH[IK]
LN 0195 220 DPI[I]=DPRI[I]+STRESS[IK]*XT[IK]/STRENGTH[IK]
LN 0196 210 CONTINUE
LN 0197 IP=1
LN 0198 03 77 J=1,NDF
LN 0199 IF (CJ[I,J]) GO TO 230
LN 0200 DPI[I]=DPRI[I]
LN 0201 230 CONTINUE
LN 0202 IF (COUNT,EQ,NLI) GO TO 240
LN 0203 230 CONTINUE
LN 0204 COUNT=COUNT+1
LN 0205 GO TO 91
LN 0206 90 WRIFF (N+1,NL)
LN 0207 #10 FORMIT (/////,)
LN 0208 1* NODE
LN 0209 2 NUMERICAL Y Z
LN 0210 070 N=1,501,MY
LN 0211 N=I+3
LN 0212 200 WRIFF (61,6111,1,XT[N+3],XT[N+3],XT[N+3],XT[N+3])
LN 0213 611 FORMAT (7.3,7.0,7.0,7.0,7.0)
LN 0214 WRIFF (61,612)
LN 0215 612 FORMIT (/////,)
LN 0216 #10 FORMIT (/////,)
ANSI FORTRAN(2.3)/MASTER

INTEGER WORD SIZE = 1

* OPTION IS OFF, O OPTION IS OFF

LN 0217  DO 260 I=1,N
LN 0218  260 WRITE (61,613) I,STRAIN(I),STRESS(I)
LN 0219  613 FORMAT (I3,14X,E13.1,10X,E13.3)
LN 0220  END

USASI FORTRAN DIAGNOSTIC RESULTS FOR FTN.MAIN

NO ERRORS
INTEGER WORD SIZE = 1, * OPTION IS OFF, 3 OPTION IS OFF

LN 0001 SUBROUTINE INCUKT (N, X, XI)
LN 0002 DIMENSION UIT(30,30), UI(30,30), X(30)
LN 0004 DO 10 I=1,N
LN 0005 XI(I)=.1
LN 0006 DO 10 J=1,N
LN 0007 UI(I,J)=0.
LN 0004 10 UI(I,J)=I
LN 0010 XI(I)=XI(I)+XI(I)
LN 0011 UI(I,I)=1.
LN 0012 XI(I)=.1
LN 0013 K=1
LN 0014 IF (NX.LT.K) GO TO 50
LN 0015 UI(I,I)=UI(I,I)-UI(I,K)*UI(K,I)*DIK
LN 0016 K=K+1
LN 0017 GO TO 40
LN 0018 JN=I+1
LN 0019 DO 50 J=JN,N
LN 0020 K=1
LN 0021 IF (NX.LT.K) GO TO 50
LN 0022 UI(I,I)=UI(I,I)-UI(I,J)*UI(K,J)*DIK
LN 0023 K=K+1
LN 0024 GO TO 50
LN 0025 DO 50 UI(I,J)=UI(I,J)+UI(I,J)*DI(J)
LN 0026 J=J+1
LN 0027 UI(I,I)=UI(I,I)+UI(I,J)*DI(J)
LN 0028 JN=I+1
LN 0029 DO 50 J=JN,N
LN 0030 NX=K-1
LN 0031 K=1
LN 0032 IF (NX.LT.K) GO TO 70
LN 0033 UI(I,I)=UI(I,I)+UI(I,K)*UI(K,J)
LN 0034 K=K+1
LN 0035 GO TO 50
LN 0036 CONTINUE
LN 0037 DO 10 I=1,N
LN 0038 DO 10 J=1,N
LN 0039 UI(I,J)=-UI(I,J)+DI(J)
LN 0040 DO 10 I=1,N
LN 0041 DO 10 J=1,N
LN 0042 XI(I,J)=.1
LN 0043 DO 10 I=1,N
LN 0044 DO 10 J=1,N
LN 0045 XI(I,J)=XI(I,J)+UI(I,J)*UI(J,I)
LN 0046 IF (X<0.1) IF (X+0.1) 34,1,34
LN 0047 CONTINUE
LN 0048 IF (X<0.1) IF (X+0.1) 34,1,34
LN 0049 34 IF (X<0.1) IF (X+0.1) 34,1,34
LN 0050 END
ANSI FORTRAN(2,3)/MASTER

<table>
<thead>
<tr>
<th>INTEGER WORD SIZE=1, 0 OPTION IS OFF, 0 OPTION IS ON</th>
</tr>
</thead>
</table>

| LN 0001 | SUBROUTINE YOUNG                                   |
| LN 0002 | COMMON A(10),B(10),C(10),D(10),E(10),F(10),G(10) |
| LN 0003 | COMMON S1(30),S2(30),S3(30),S4(30),S5(30),S6(30),S7(30),S8(30),S9(30),S10(30) |
| LN 0004 | DO 10 I=1,NET                                      |
| LN 0005 | H=44(1)                                           |
| LN 0006 | IF (TYPE(4)=3,0) GO TO 20                        |
| LN 0007 | C(1)=I(1)+C(1)*S1(4)+S2(1)+S3(1)*S4(1)*S5(1) |
| LN 0008 | END                                              |

| LN 0009 | GO TO 10                                          |
| LN 0010 | 20 C(1)=C(1)                                    |
| LN 0011 | 10 CONTINUE                                      |
| LN 0012 | RETURN                                           |
| LN 0013 | END                                              |

USASI FORTRAN DIAGNOSTIC RESULTS FOR YOUNG

OBJ: LOG

NO ERRORS
USASI FORTRAN DIAGNOSTIC RESULTS FOR SIGMA

NO ERRORS
Section V

BIBLIOGRAPHY


Appendix A
DERIVATION OF THE STIFFNESS MATRIX FOR A
THREE DIMENSIONAL TRUSS

In this section the stiffness matrix is derived for a three dimensional truss using the method described in the introduction. The stiffness matrix for other types of structural elements can be derived using this method.

The displacement function is

\[ U(s) = \alpha_1 + \alpha_2 s \]

\( U(s) \) is the position of any point on the truss as a function of s. \( q_1 \) and \( q_2 \) are the position of the ends of the truss after the loads have been applied.

\[ \{ \mathbf{U} \} = [\phi] \{ \alpha \} \]
Where \( \{ U \} \) is the displacement function, \( \phi \) is the polynomials of \( s \) and \( \{ \alpha \} \) is the coefficients of the polynomials.

\[
\{ q \} = [\phi \text{ At Nodes}] \{ \alpha \} = [A] \{ \alpha \}
\]

Where \( \{ q \} \) is the displacements of the nodes, \( [A] \) is the value of the dependent variables at their particular node.

\[
\begin{align*}
\{ q_1 \} &= \begin{bmatrix} 1, & 0 \end{bmatrix} \{ \alpha \} \\
\{ q_2 \} &= \begin{bmatrix} 1, & L \end{bmatrix} \{ \alpha \}
\end{align*}
\]

From matrix algebra \( \{ \alpha \} \) can be determined.

\[
\{ \alpha \} = [A] \{ q \}
\]

Where \( A \) is the displacement transformation matrix

\[
\begin{align*}
\{ \mathcal{L}_1 \} &= \begin{bmatrix} 1, & 0 \end{bmatrix} \{ q_1 \} \\
\{ \mathcal{L}_2 \} &= \begin{bmatrix} -1/L, & 1/L \end{bmatrix} \{ q_2 \}
\end{align*}
\]

\[
\{ U \} = [\phi] [A] \{ q \}
\]

\[
\{ U(s) \} = [1, \ s] \begin{bmatrix} 1, & 0 \end{bmatrix} \{ q_1 \}
\]

\[
\{ U(s) \} = [1, \ s] \begin{bmatrix} -1/L, & 1/L \end{bmatrix} \{ q_2 \}
\]
\[
\begin{align*}
\{ U(s) \} &= \begin{bmatrix} 1 - s/L, & s/L \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\
\{ \varepsilon_s \} &= \begin{bmatrix} -1/L, & 1/L \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\
\sigma_s &= E \varepsilon_s = E \begin{bmatrix} -1/L, & 1/L \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\end{align*}
\]

The strain can now be determined from the displacement function.

The stress can be determined from the strain. For a truss

\[
\sigma_s = E \varepsilon_s = E \begin{bmatrix} -1/L, & 1/L \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]

The internal strain energy

\[
U_I = \frac{1}{2} \int \int \int_V \{ \varepsilon \}^T \{ \sigma \} \, dV
\]

\[
U_I = \frac{1}{2} \int_0^L \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \begin{bmatrix} -1/L, & 1/L \end{bmatrix} EA \begin{bmatrix} -1/L, & 1/L \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \, ds
\]

Where \( A \) is the cross sectional area

\[
U_I = \frac{1}{2} \int_0^L \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \begin{bmatrix} 1/L^2 & 1 \end{bmatrix} \begin{bmatrix} 1, & -1 \\ -1, & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} EA \, ds
\]

\[
U_I = \frac{1}{2} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \begin{bmatrix} 1/L & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]

\[
U_I = \frac{1}{2} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} EA/L \begin{bmatrix} 1, & -1 \\ -1, & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]
The external work, the work done by the external forces \([\mathbf{q}]\):
\[
U_E = \frac{1}{2} \mathbf{q}^T \mathbf{Q} \mathbf{q}
\]
\[
U_E = \frac{1}{2} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{bmatrix}
\]

The external work must equal the internal work

\[
U_E = U_I
\]

\[
\frac{1}{2} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}^T \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}
\]

\[
\begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}
\]

Therefore the stiffness matrix is

\[
[K] = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

Transformation from the \(s\) coordinate system into the \(x,y,z\) coordinates

\[
Q_{1x} = 1 \ Q_1 
\]
\[
Q_{2x} = 1 \ Q_2 
\]
\[
Q_{1y} = m \ Q_1 
\]
\[
Q_{2y} = m \ Q_2 
\]
\[
Q_{1z} = n \ Q_1 
\]
\[
Q_{2z} = n \ Q_2 
\]
Where $l$, $m$, $n$ are the direction cosines between the $x$, $y$, $z$ coordinates and the $s$ axis respectively.

\[
\begin{align*}
\{Q_{1x}\} &= \begin{bmatrix} l, 0 \\ m, 0 \\ n, 0 \end{bmatrix}, \\
\{Q_{1y}\} &= \begin{bmatrix} 0, 1 \\ 0, m \\ 0, n \end{bmatrix}, \\
\{Q_{1z}\} &= \frac{E/A}{L} \begin{bmatrix} 1, -1 \\ -1, 1 \end{bmatrix} \{q_1\}, \\
\{Q_{2x}\} &= \begin{bmatrix} 1, m, n, o, o, o \end{bmatrix}, \\
\{Q_{2y}\} &= \begin{bmatrix} 0, 0, 0, 1, m, n \end{bmatrix}, \\
\{Q_{2z}\} &= \begin{bmatrix} 1, m, n, o, o, o \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
q_1 &= 1 q_{1x} + m q_{1y} + n q_{1z} \\
q_2 &= 1 q_{2x} + m q_{2y} + n q_{2z}
\end{align*}
\]
Therefore the stiffness matrix is

\[
[K] = \frac{EA}{L} \begin{bmatrix}
1, 0 & m; 0 & n, 0 & 0, 1 & 0, m & 0, n \\
-1, 0 & l, -1, 1, m, n, 0, 0, 0 \\
-1, 1, 0, 0, 1, m, n \\
-1, m, -1n, 1m, 1n \\
-1n, -mn, 12, 1m, mn \\
-12, 1m, -1n, 1m, mn \\
\end{bmatrix}
\]
Appendix B

EXAMPLES FOR THE INCREMENTAL AND ITERATIVE METHOD

Two separate examples are shown for both the incremental and iterative method. The input and output for the computer programs are shown in this section. The first example is a two dimensional 3 bar truss where all elements are composed of one material. The second example is a three dimensional truss with 5 bars. There are two types of materials in the second example. For the first material, stress is a function of strain and a third order polynomial fit is used in the program. The second material is elastic-plastic. The number of total increments for both examples was taken as 50. The number of total iterations for both examples was 10. It was found that much fewer cycles were necessary to achieve the same accuracy with the iterative method.
Example 1
<table>
<thead>
<tr>
<th>Node number</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
<th>IDF(I)</th>
<th>IDF(J)</th>
<th>IDF(K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.</td>
<td>0.</td>
<td>0.</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>173.2</td>
<td>0.</td>
<td>0.</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>346.3</td>
<td>0.</td>
<td>0.</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>173.2</td>
<td>100.</td>
<td>0.</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Element number</th>
<th>Material number</th>
<th>Node(I)</th>
<th>Node(J)</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>1.</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>1.</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>1.</td>
</tr>
</tbody>
</table>

The only node that has an applied load is node 4. The load applied is 50,000. The material properties are as follows

\[
\sigma = 10 \times 10^6 \varepsilon - 6.25 \times 10^8 \varepsilon^2
\]

\[
E_t = \frac{d\sigma}{d\varepsilon} = 10 \times 10^6 - 1.25 \times 10^9 \varepsilon
\]

For the iterative method, any four different values of stress and corresponding values of strain are input for the material properties. For the incremental method, any four different values of tangent modulus and corresponding values of strain are input for the material properties.
COMPUTER OUTPUT FOR THE INCREMENTAL METHOD
EXAMPLE 1
NUMBER OF ELEMENTS = 4

NUMBER OF ELEMENTS = 3

NUMBER OF MATERIALS = 1

<table>
<thead>
<tr>
<th>NODE</th>
<th>DEGREE OF FREEDOM</th>
<th>LOCATION OF COORDINATES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ELEMENT</th>
<th>NODE POINT NUMBERS</th>
<th>MATERIAL NUMBER</th>
<th>AREA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1, 1</td>
<td>1</td>
<td>1.00000</td>
</tr>
<tr>
<td>2</td>
<td>2, 3</td>
<td>1</td>
<td>1.00000</td>
</tr>
<tr>
<td>3</td>
<td>2, 3</td>
<td>1</td>
<td>1.00000</td>
</tr>
</tbody>
</table>

MATERIAL NUMBER = 1

TYPE (3RD ORDER POLYNOMIAL)

<table>
<thead>
<tr>
<th>E1</th>
<th>E2</th>
<th>E3</th>
<th>E4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.100E+08</td>
<td>0.075E+07</td>
<td>0.750E+07</td>
<td>0.625E+07</td>
</tr>
</tbody>
</table>

S1 = 0.000E+00 S2 = 0.100E-02 S3 = 0.200E-02 S4 = 0.300E-02

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<th>DEFLECTION OF NODE</th>
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<tr>
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</tr>
<tr>
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<td>0.000E+00</td>
</tr>
<tr>
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<td>0.000E+00</td>
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<tr>
<td>4</td>
<td>0.908E-11</td>
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<th>STRESS</th>
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<tbody>
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<td>0.130E+05</td>
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<tr>
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<tr>
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<td>---</td>
<td></td>
</tr>
<tr>
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<th>NODE POINT NUMBERS</th>
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<th>AREA</th>
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<tbody>
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<td>J</td>
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<table>
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<tr>
<td>TYPE (3RD ORDER POLYNOMIAL)</td>
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</tr>
<tr>
<td>STRESS1 = 0.000E+00</td>
<td>STRESS2 = 0.937E+04</td>
</tr>
<tr>
<td>STRAIN1 = 0.000E+00</td>
<td>STRAIN2 = 0.100E-02</td>
</tr>
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<table>
<thead>
<tr>
<th>NODE NUMBER</th>
<th>DEFLECTION OF NODE</th>
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</thead>
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<tr>
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<tr>
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<td>0.300E+00</td>
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<td>0.340E-11</td>
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<table>
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<tr>
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<th>STRESS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.144E-02</td>
<td>0.131E+05</td>
</tr>
<tr>
<td>2</td>
<td>0.577E-02</td>
<td>0.369E+05</td>
</tr>
<tr>
<td>3</td>
<td>0.144E-02</td>
<td>0.131E+05</td>
</tr>
</tbody>
</table>
Example 2

○ Node number
○ Element number
△ Material number
<table>
<thead>
<tr>
<th>Node number</th>
<th>x</th>
<th>y</th>
<th>z</th>
<th>IDF(I)</th>
<th>IDF(J)</th>
<th>IDF(K)</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>0.</td>
<td>0.</td>
<td>0.</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>100.</td>
<td>100.</td>
<td>0.</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0.</td>
<td>100.</td>
<td>0.</td>
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<td>1</td>
<td>1</td>
</tr>
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<td>5</td>
<td>50.</td>
<td>50.</td>
<td>0.</td>
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<td>1</td>
</tr>
<tr>
<td>6</td>
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<td>50.</td>
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</table>

<table>
<thead>
<tr>
<th>Element number</th>
<th>Material number</th>
<th>Node(I)</th>
<th>Node(J)</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>1.</td>
</tr>
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<td>5</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>1.</td>
</tr>
</tbody>
</table>

The only node that a load is applied is node 6. The load applied is 100,000. There are two types of materials. The first has the following properties:

\[
\sigma = 8.33 \times 10^6 \varepsilon - 1.563 \times 10^8 \varepsilon^2 - 1.302 \times 10^{10} \varepsilon^3
\]

\[
\varepsilon_t = \frac{d\sigma}{d\varepsilon} = 8.33 \times 10^6 - 3.126 \times 10^8 \varepsilon - 3.906 \times 10^{10} \varepsilon^2
\]

For the iterative method, any four different values of stress and corresponding values of strain are input for the material properties. For the incremental method,
COMPUTER OUTPUT FOR THE INCREMENTAL METHOD

EXAMPLE 2
any four different values of tangent modulus and corresponding values of strain are input for the material properties. The second material is elastic-plastic.

\[ \sigma \]

\[ 50,000 \]

\[ \epsilon \]

\[ \cdot005 \]

The input for elastic-plastic materials are the same for both the incremental and iterative methods.

\[ E = 10,000,000 \]

\[ \epsilon_p = .005 \]
**NUMBER OF ELEMENTS = 6**

**NUMBER OF ELEMENTS = 5**

**NUMBER OF MATERIALS = 2**

<table>
<thead>
<tr>
<th>NODE</th>
<th>DEGREE OF FREEDOM</th>
<th>LOCATION OF COORDINATES</th>
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<tr>
<td>1</td>
<td>1 1 1 1</td>
<td>0.00000 0.00000 0.00000</td>
</tr>
<tr>
<td>2</td>
<td>1 1 1 1</td>
<td>0.00000 0.00000 0.00000</td>
</tr>
<tr>
<td>3</td>
<td>1 1 1 1</td>
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<td>0.00000 100.00000 0.00000</td>
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<td>5</td>
<td>1 1 1 1</td>
<td>50.00000 50.00000 0.00000</td>
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<td>6</td>
<td>0 0 0 0</td>
<td>50.00000 50.00000 200.00000</td>
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<table>
<thead>
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<th>ELEMENT</th>
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<th>MATERIAL</th>
<th>AREA</th>
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<td>2 2 6</td>
<td>1</td>
<td>1.0000</td>
</tr>
<tr>
<td>3</td>
<td>3 3 6</td>
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**MATERIAL NUMBER = 1**

<table>
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<tr>
<th>TYPE (3RD ORDER POLYNOMIAL)</th>
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<tbody>
<tr>
<td>$E_1$ = 0.838E+07 $E_2$ = 0.796E+07 $E_3$ = 0.755E+07 $E_4$ = 0.710E+07 $S_1$ = 0.000E+00 $S_2$ = 0.100E+00 $S_3$ = 0.200E+00 $S_4$ = 0.300E+00</td>
</tr>
</tbody>
</table>

**MATERIAL NUMBER = 2**

<table>
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<tr>
<th>TYPE (ELASTIC-PLASTIC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$ = 0.100E+08 $E_2$ = 0.100E+08 $E_3$ = 0.100E+08 $E_4$ = 0.100E+08 $S_1$ = 0.100E+08 $S_2$ = 0.100E+08 $S_3$ = 0.100E+08 $S_4$ = 0.100E+08</td>
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</tbody>
</table>

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<th>DEFORMATION OF NODE</th>
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<tr>
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</tr>
<tr>
<td>4</td>
<td>0.000E+00 0.000E+00 0.000E+00</td>
</tr>
<tr>
<td>5</td>
<td>0.000E+00 0.000E+00 0.000E+00</td>
</tr>
<tr>
<td>6</td>
<td>0.000E+00 0.100E+00 0.100E+00</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>ELEMENT NUMBER</th>
<th>STRAIN</th>
<th>STRESS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0.100E+05</td>
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<tr>
<td>2</td>
<td>0.100E-02</td>
<td>0.100E+05</td>
</tr>
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<td>0.100E-02</td>
<td>0.100E+05</td>
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<td>0.100E-02</td>
<td>0.100E+05</td>
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<tr>
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<td>0.100E-02</td>
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COMPUTER OUTPUT FOR THE ITERATIVE METHOD
EXAMPLE 2
<table>
<thead>
<tr>
<th>NODE</th>
<th>DEGREE OF FREEDOM</th>
<th>LOCATION OF COORDINATES</th>
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<tbody>
<tr>
<td>NUMBER</td>
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<th>AREA</th>
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<td>6</td>
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<tbody>
<tr>
<td>STRESS1 = 0.000E+00</td>
<td>STRESS2 = 0.000E+00</td>
<td>STRESS3 = 0.500E+05</td>
</tr>
<tr>
<td>STRAIN1 = 0.000E+00</td>
<td>STRAIN2 = 0.400E-02</td>
<td>STRAIN3 = 0.400E-02</td>
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<tr>
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<table>
<thead>
<tr>
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<th>STRAIN</th>
<th>STRESS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0.330E+03</td>
</tr>
<tr>
<td>2</td>
<td>0.560E-02</td>
<td>0.330E+03</td>
</tr>
<tr>
<td>3</td>
<td>0.560E-02</td>
<td>0.330E+03</td>
</tr>
<tr>
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<tr>
<td>5</td>
<td>0.560E-02</td>
<td>0.330E+03</td>
</tr>
</tbody>
</table>
**PROGRAM**

**INCREMENAL METHOD (EXAMPLE 1)**

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<tr>
<th>STATEMENT NUMBER</th>
<th>FORTRAN STATEMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>CONTROL CARDS</strong></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
</tr>
</tbody>
</table>

|                  | **MATERIAL CARDS** |
| 1                | 100000000.0       |
| 2                | 8750000.0        |
| 3                | 7500000.0        |
| 4                | 6250000.0        |

|                  | **NODAL CARDS** |
| 1                | 0.0              |
| 2                | 0.0              |
| 3                | 346.4            |
| 4                | 173.2            |

|                  | **ELEMENT CARDS** |
| 1                | 1                |
| 2                | 4                |
| 3                | 4                |

|                  | **LOAD CARDS** |
| 4                | 50000.0         |

*Example: A standard card form, IBM model 000157, is available for punching statements from this form.*
<table>
<thead>
<tr>
<th>STATEMENT NUMBER</th>
<th>CONTENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0. 9400. 30000. 34700. 0.000 0.001 0.004 0.00508</td>
</tr>
<tr>
<td>2</td>
<td>1 1 1 1 173.2 0</td>
</tr>
<tr>
<td>3</td>
<td>1 1 1 346.4 0</td>
</tr>
<tr>
<td>4</td>
<td>0 0 0 1 173.2 100.0</td>
</tr>
</tbody>
</table>

**CONTROL CARDS**

**MATERIAL CARDS**

**NODAL CARDS**

**ELEMENT CARDS**

**LOAD CARDS**

*A standard card form, IBM 188157, is available for punching statements from this form*
### FORTRAN Coding Form

**Program: Incremental Method (Example 2)**

<table>
<thead>
<tr>
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<th>PUNCHING</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>INSTRUCTIONS</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>STATEMENT</th>
<th>FORTRAN STATEMENT</th>
</tr>
</thead>
</table>

1. **CONTROL CARDS**

<table>
<thead>
<tr>
<th>1</th>
<th>5</th>
<th>2</th>
</tr>
</thead>
</table>

2. **MATERIAL CARDS**

<table>
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</thead>
</table>

3. **NODAL CARDS**

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<th>0.</th>
<th>0.</th>
</tr>
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<table>
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<th>0.</th>
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<table>
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<th>100.</th>
<th>0.</th>
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</table>

4. **ELEMENT CARDS**

<table>
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<tr>
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<th>5</th>
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</tr>
</thead>
</table>

5. **LOAD CARDS**

| 6 | 1 |

---

*A standard card form, IBM electro 088157, is available for punching statements from this form.*
<table>
<thead>
<tr>
<th>COLUMN STATEMENT NUMBER</th>
<th>COLUMN</th>
<th>FORTRAN STATEMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0. 100000. 0.</td>
</tr>
<tr>
<td>STATEMENT NUMBER</td>
<td>FORTRAN STATEMENT</td>
<td></td>
</tr>
<tr>
<td>------------------</td>
<td>-------------------</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>CONTROL CARDS</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>MATERIAL CARDS</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>NODAL CARDS</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>ELEMENT CARDS</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>LOAD CARDS</td>
<td></td>
</tr>
</tbody>
</table>

- **CONTROL CARDS:**

```
6 5 2
```

- **MATERIAL CARDS:**

```
1 10
0.30000, 50000, 55000.
.000 .004 .008 .012
0
100000000, .005
```

- **NODAL CARDS:**

```
1 1110 0 0 0
2 1111 0 0 0
3 1111 100 0 0 0
4 1110 0 100 0 0 0
5 1111 50 50 0 0 0
6 1111 50 50 0 0 0
```

- **ELEMENT CARDS:**

```
1 1 6 1
2 2 6 1
3 3 6 1
4 4 6 1
5 5 6 2
```

- **LOAD CARDS:**

```
6
```

*A standard card form, IBM 888157, is available for punching statements from this form.*
<table>
<thead>
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<th>STATEMENT NUMBER</th>
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