ANALYSIS AND DESIGN OF A DIGITAL ACTUATOR

A Project submitted in partial satisfaction of the requirements for the degree of Master of Science in Engineering

by

Zacharias Vorgias

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\( K_T \) = Constant associated with stiffness of transmission
\( K_{V_i}, K_{V_2}, \ldots \) = Orifice constants in branches of digital flow control actuator
\( K_{V_S} \) = Velocity constant of unity feedback system
\( L \) = Leakage coefficient (inertial) for hydraulic transmission
\( N \) = Word size of controller
\( P_1, P_2 \) = Pressures on driven and return sides of transmission respectively
\( Q_{\text{TOTAL}} \) = Total flow in digital flow control actuator
\( Q_1, Q_2, \ldots \) = Flow in branches of digital flow control actuator
\( T \) = Time delay between updating digital commands
\( T_L \) = Load torque
\( T_m \) = Motor torque
\( V \) = Volume of fluid under pressure
\( X_i \) = Input to incremental actuator at \((i)\)th sampling instant
\( X_{i-1} \) = Input to incremental actuator at \((i-1)\)th sampling instant
\( X_P \) = Position of hydraulic ram
\( \dot{X}_P \) = Velocity of hydraulic ram
\( Y_i \) = Output of incremental actuator between \((i)\) and \((i+1)\)
\( Y_{i-1} \) = Output of incremental actuator between \((i-1)\) and \((i)\) sampling instants
\( Y_1, Y_2, Y_3 \) = Binary weighted analog quantities to be summed in the binary input digital actuator model

\( Y_0 \) = Output of binary input digital actuator model

\( Y_0' \) = Sum of the binary weighted quantities in the binary input digital actuator model

\( c_d(t) \) = Desired system output

\( e_i \) = Controlled input (system error before processing)

\( e_o \) = Controller output (processed error)

\( f(n) \) = Controller variable

\( h \) = Quantization interval

\( j \) = Imaginary number

\( K_r \) = Ripple factor

\( P_1, P_2, P_3, \ldots \) = Poles of \( G(Z) \)

\( v_1, v_2, v_3, \ldots \) = Inputs of hold circuit (binary inputs)

\( v_1', v_2', v_3', \ldots \) = Outputs of hold circuit

\( \Theta \) = Laplacian operator

\( \Theta_i \) = Simple poles of \( G(s) \)

\( \Theta_j \) = Repeated poles of \( G(s) \)

\( \Theta_K \) = Poles of \( G(s) \)

\( U_1 \) = Controller output (time domain)

\( X_1, X_2, X_3, \ldots \) = System state variables

\( Z \) = \( Z \)-transform operator

\( z_1, z_2, z_3, \ldots \) = Roots of \( G(z) \)

\( \beta \) = Bulk modulus of hydraulic fluid

\( \epsilon(t) \) = Steady state error due to sampling ripples
\[ \varepsilon^2(n\delta) \quad = \quad \text{Means squared output error due to quantization} \]

\[ \Theta_o(2) \quad = \quad \text{Z-transform of system output} \]

\[ \Theta_i(2) \quad = \quad \text{Z-transform of system input} \]

\[ \Theta_o \quad = \quad \text{Output of control system (load position)} \]

\[ \Theta_i \quad = \quad \text{Input of control system (commands)} \]

\[ \Theta_m \quad = \quad \text{Hydraulic motor shaft position} \]

\[ \Theta_p \quad = \quad \text{Angular velocity of pump shaft rotation} \]

\[ z \quad = \quad \text{Damping ratio of hydraulic transmission} \]

\[ \tau \quad = \quad \text{Valve actuator transport lags} \]

\[ \omega \quad = \quad \text{Radian frequency of test input} \]

\[ \omega_n \quad = \quad \text{Natural frequency of hydraulic transmission} \]

\[ \omega_s \quad = \quad \text{Radian frequency of sampling} \]

\[ \sigma^2_q \quad = \quad \text{Second moment of quantization error distribution} \]
ABSTRACT

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The typical control system configuration consists of a controller, a controlled process (the plant) and a set of one or more feedback elements. The controller in practice, is customarily an electrical device while the plant is often made up of mechanical components and processes. The conversion from electrical to mechanical energy in such a system is made in an electro-mechanical transducer (the actuator).

Present technology finds the controller taking the form of a digital computer with digital-to-analog conversion between the controller and the actuator. However, recent advances in the design of actuators have made possible the use of actuators which utilize digital inputs, thus eliminating the necessity for D/A conversion.
This project discusses the use of digital actuators in the design of closed loop control systems. It is shown how linear models can be derived for such actuators, based on the necessary functions which must be performed and the physical principles which govern the operation of the device.

A hybrid control system design is carried out in this project to illustrate the use of this type of actuator. The design is based on a sampled-data model of the system, and utilizes the graphical root-locus technique in the design of the discrete controller.

Aspects of hybrid control system design and analysis, such as quantization error, intersampling ripple and transient responses are discussed. Finally, conclusions and recommendations on the use of digital actuator in hybrid control systems have been included in the paper.
Digital actuation is a recent development, having evolved from a desire on the part of engineers to utilize the inherent accuracy and insensitivity to drift of digital techniques on the mechanical responder portion of a control system. While digital design is relatively simple for producing electrical outputs in response to electrical inputs, it is much less straightforward when electrical to mechanical energy conversion is involved. The design of a digital actuator is essentially the invention of a technique to combine digital methods with standard forms of high response electro-mechanical energy conversion while retaining the advantages of both.

This project does not propose to dwell on actuator design problems, except to discuss them in general terms and to describe some actuators that are commercially available. The use of such devices to solve a typical control problem will be the major subject. The special problems presented by digital actuation and the analysis methods required to verify the design will receive the most attention. While digital actuators are frequently designed for specific uses in open-loop applications because of their accuracy, in actual practice they often become a part of an overall feedback control system. This case will be considered here because of its practical significance, (e.g. flight control systems, radar directors).
The design of the digital controller (digital electronics) to be used in the system will be strongly influenced by the choice of the actuator. The actuator will determine the plant variable to be controlled, the form of the digital input, and to some extent, the necessary time constants in the controller. A portion of this paper will be devoted to establish design criteria for a system and carry through a design for a typical set of criteria. A system simulation will also be part of the investigation.

The term "hybrid" when used with reference to systems of this type denotes the digital - analog nature of the systems. All digitally controlled processes are "hybrid". However, in this paper all discussion will be limited to those systems in which the digital - to analog interface is in the mechanical, rather than the electrical, portion of the system.
DIGITAL ACTUATORS

2.1 General Types:

Accepting a broad definition of digital actuators would require the inclusion in this discussion of all electro-mechanical devices having discrete input-output characteristics. This would include all on-off devices, such as solenoids and pulse-operated devices, such as stepping motors, as well as the more recently developed incremental and binary input actuators. To reduce the scope of this study, the first type of device will be totally ignored and heavy emphasis will be placed on the binary input actuator.

2.2 Incremental Actuators:

An incremental actuator is often driven by a circuit which provides inputs to the actuator that are proportional to the difference between the most recent command and the command which preceded it. This type of actuator has memory and is capable of being driven by electrical pulses instead of requiring an external storage element.

To mathematically describe the action of our incremental actuator, difference equations are used. The block diagram of Figure 1 is referred to for the nomenclature to be used.
\[ Y_l = C_a (X_l - X_{l-1}) + Y_{l-1} \]  

(1)

Also

\[ Y_{i-1} = C_a (X_{i-1} - X_{i-2}) + Y_{i-2} \text{ e.t.c} \]  

(2)

\[ Y_i = C_a \sum_{k=0}^{i} (X_k - X_{k-1}) \]  

(3)

of course this last equation simplifies to:

\[ Y_i = C_a X_i \]  

(4)

This illustrates that an incremental actuator, when used in conjunction with a digital circuit designed to compute the first difference of the digital input, can be made to respond as a complete valve device. This permits the use of such an actuator in a system where otherwise its response (due to memory) would be unacceptable.

The incremental actuator, while it is a discrete device, is not usually designed to accept inputs in digital form without data conversion. Pulse inputs or pulse-width-modulated inputs are required for most common devices of this class. This means that a special type of a D/A interface must be incorporated in the electronic driven circuits. An example of this is shown in Figure 2.

Because the incremental actuator has the property of requiring derivative inputs, it is a natural device to use in conjunction with the Digital Differential Analyzer type of computer. DDA is a truly digital machine, containing memory, gating and switching.
Figure 1. BLOCK OF DIAGRAM OF INCREMENTAL ACTUATOR

Figure 2. A DATA CONVERSION SYSTEM FOR USE WITH AN INCREMENTAL ACTUATOR
circuits. It processes all data in discretized, binary form. It is designed specifically for the selection of ordinary linear or nonlinear differential equations or sets of such equations. It is compact in size and very light. The basic component is a digital integrator. The mathematical operations are performed sequentially as a series type. In the DDA, derivatives of the variables are available due to the manner in which the equations are mechanized in the computer. This does away with the requirements for a separate differencing circuit to be used with the actuator.

2.3 Binary Input Actuator:

The binary input actuator has the advantage of requiring no digital-to-analog conversion of the input. This is obtained at the expense of a more complicated design since the operations of binary weighting, summing and energy conversion must be combined in a single device. An external storage function is usually necessary for smooth operation of the actuator. The binary input actuator is a freely digital actuator, since it may be utilized on a system whose electrical control section contains only logical functions.

While it is difficult to discuss binary input actuators in general terms, there are certain characteristics of such devices which may be illustrated by referring to a general model. Figure 3.
The variable $Y_0'$ is seen to be a summation of the parallel inputs after each has been operated on by a zero-order hold circuit and a transfer function $G_a$. Some actuators have an additional transfer function associated with the summation, $G_o$. This is, of course, determined by the principles of operation of the actuator. The equations describing the model are given below, where $G_{ho}$ is the transfer function of a Zero-order hold.

\begin{align}
Y_n &= G_{ho} G_a v_n \quad \left[ v'_n = v_n G_{ho} \right] \quad (5) \\
Y_0' &= G_{ho} \left[ G_a v_1 + G_a v_2 + \cdots + G_a v_n \right] \quad (6) \\
Y_0 &= G_{ho} G_o \left[ G_a v_1 + G_a v_2 + \cdots + G_a v_n \right] \quad (7)
\end{align}

The $Y$ variables are continuous quantities and not discrete as is the case with the "$r" variables. The digital inputs can only assume the values of "1" or "0". In general, the transfer functions $G_a$ may be written as follows:

\begin{equation}
G_a = K_n H_n(s) \quad (8)
\end{equation}

where "$s" is the Laplacian operator. Also

\begin{equation}
\frac{K_n}{K_{n+1}} = 1/2 \quad \text{(see equation (16)}
\end{equation}

for binary weighting of the inputs and, in general

\begin{equation}
H_1(s) \neq H_2(s) \neq H_3(s) \ldots.
\end{equation}

This description of the binary input actuator leads to the following conclusions as to its response. The steady state error of such a device is very low provided the binary weighting of the inputs is accurate. The transient response, however, can be very
rough and complicated because of differences which can exist in the response of the various sections of the actuator; this is particularly true of some designs.
Figure 3. A GENERAL MODEL FOR A BINARY INPUT ACTUATOR

Figure 4. THE VOLUME ADDER
3.1 Introduction:

The following actuators are described for the purpose of illustrating some of the concepts previously discussed. These are examples of practical devices (commercially available) which have been designed to respond to digital or discrete input signals. All are hydraulic actuators, although they differ considerably in regard to their principles of operations.

3.2 The Volume Adder:

Figure 4 is an illustration of an incremental type of actuator. It has been called a volume adder by the company which developed it.

To describe its operation, suppose the direction valve is in a position to allow high pressure fluid to be ported to the actuating valve. If the actuating valve is driven so as to port fluid to the lower side of the piston in the calibrated volume, a known quantity of fluid flows into the actuating cylinder, thus displacing the output ram. When the excitation is removed from the actuating valve solenoid, high pressure is applied to the upper side of the piston on the calibrated volume, producing an additional increment of displacement of the output ram. The direction of displacement of the output ram can be reversed by reversing the position of the direction valve.
Figure 5. INPUT AND OUTPUT OF VOLUME ADDER

Figure 6. DIGITAL POSITION CONTROL DEVICE
A current input of the type shown on Figure 5 will produce an output displacement as shown. This is, of course, an idealized response, the actual response can be derived from a more detailed model of the device.

3.3 Binary Input Actuators:

In Figure 6 is shown a digital position control device, which has the capability of being positioned in any one of seven discrete positions by venting or pressurizing the lettered ports. This is a binary input actuator, since the transfer valve solenoids can be driven directly by a digital binary signal source, (such as a register). Commercially, these actuators are usually designed to contain a larger number of binary sections for high resolution. Hydraulic or pneumatic pressure can be used to provide the driving force.

A characteristic of this particular actuator is that each binary section can have a different response time, depending upon the transfer valve design. In the model of Figure 3, the Gm terms must be included for proper modeling of this actuator.

Another example is the flow summing actuator shown in Figure 7. This is a binary device requiring an input similar to the previous example. Provision for a sign output is made to allow reversal of the actuator. Referring again to the model of Figure 3, this
actuator requires the inclusion of the Go transfer function of the position of the output ram is considered the output variable to be controlled. It can be seen that Go will contain an integration, or pulse lag term. This actuator is very similar to the first one discussed with the difference being in the type of input signals required.
Figure 7. FLOW SUMMING ACTUATOR
THE CONTROLLED SYSTEM

4.1 Description:

A power drive system for a heavy inertial load will be the subject of this study. This type of system often finds applications in the design of large tracking devices, such as antennas and telescopes, as well as missile launchers and anti-aircraft gun turrets.

The basic drive unit for the system is a variable-speed hydraulic transmission, utilizing a variable-stroke piston pump hydraulically coupled to a fixed displacement piston motor. To match the torque-speed characteristics of the transmission to the requirements of the heavy inertial load, a gear ratio of 70:1 is necessary between the motor output shaft and the load.

4.2 System Equations:

Using the symbols listed below the basic equation of this system may be derived from Fluid Mechanics. The transmission is a fluid flow circuit and if the external leakage is assumed to be replenished in some manner, a flow equation can be written as follows:
\[ D_P \dot{\theta}_P = D_m \dot{\theta}_m + L(P_1 - P_2) + \frac{1}{2}(K_e + \frac{V}{\beta}) \frac{d}{dt} (P_1 - P_2) \] (9)

Also

\[ D_P \dot{\theta}_P = K_P \chi_P \]

- \( D_P \) - pump displacement, \( \text{in}^3/\text{rad} \)
- \( \dot{\theta}_P \) - pump speed, \( \text{rad/sec} \)
- \( D_m \) - motor displacement, \( \text{in}^3/\text{rad} \)
- \( \dot{\theta}_m \) - motor speed, \( \text{rad/sec} \)
- \( L \) - leakage coefficient (internal), \( \text{in}^5/\text{lb} \cdot \text{sec} \)
- \( P_1, P_2 \) - pressures on driven and return sides of transmission, respectively, \( \text{lb/in}^2 \)
- \( K_e \) - coefficient of elasticity of hydraulic passage, \( \text{in}^5/\text{lb} \)
- \( V \) - volume of fluid under pressure, \( \text{in}^3 \)
- \( \beta \) - bulk modules of hydraulic fluid, \( \text{lb/in}^2 \)
- \( K_P \) - a constant related to pump design, \( \text{in}^2/\text{sec} \)
- \( \chi_P \) - pump stroke, \( \text{in} \)

The torque output of the motor can be equated to the load torque as follows:

\[ T_m = (P_1 - P_2)D_m = T_L = J_L \ddot{\theta}_m \] (10)

Here \( J_L \) is set equal to the reflected load inertia.
4.3 **Transfer Functions:**

The above equations may be used to solve for a transfer function relating the output shaft angle to the distance moved by the pump stroke servo.

\[
K_P X_P = D_m \ddot{\theta}_m + L(J_L/D_m)\dot{\theta}_m + 1/2(K_e + V/\beta)(J_L/D_m)\ddot{\theta}_m
\]  

(11)

Laplace transforming the above expression and rearranging terms gives the desired transfer function.

\[
\frac{\dot{\theta}_m}{X_P} = \frac{K_P/K_{pm}}{s^2 + L(J_L/D_m)K_{pm}s + D_m/K_{pm}}
\]  

(12)

\[
K_{pm} = 1/2 \left( K_e + V/\beta \right) \left( J_L/D_m \right)
\]  

(13)

With a gear ratio of 70:1 our equation becomes

\[
\frac{\dot{\theta}_m}{X_P} = \frac{K_P/70K_{pm}}{s^2 + \left( L(J_L/D_m)K_{pm} \right)s + D_m/K_{pm}}
\]  

(14)

The latter is a transfer function relating the load motion to the pump stroke servo displacement.

4.4 **Non-Linearities:**

Some other important properties of this particular plant need to be included at this point. It is a power limited drive and is therefore incapable of driving a load at velocities above
a maximum level and cannot accelerate the load beyond the acceleration permitted by the torque capability of the transmission. These are important limitations on the ability of the drive system to respond to transient inputs. It is impossible to take these saturation levels into account during a linear analysis and design; however, a more thorough treatment of this problem can be made using strictly time-domain equations and this will be shown later.

4.5 The System Specification:

The system which is to be designed must be capable of accurately positioning a reflected load inertia of 1.205 slug-feet squared with the following error specifications:

Allowable error for a sine wave input.

- 30 degree amplitude, 9 second period - 12 minutes
- 15 degree amplitude, 4.5 second period - 42 minutes

Allowable error for a constant velocity input.

- 30 degrees per second - 7 minutes
- 75 degrees per second - 30 minutes

Response to a step input-

System must be capable of moving load 60 degrees in 2 seconds and 90 degrees in 2.5 seconds.

Interpretation of these specifications in terms of control system characteristics is necessary to establish a meaningful...
system specification. It is assumed that the final system design will utilize unity feedback of the load position.

Frequency Response ($\omega$):
- $0.695 \text{ rad/sec} \pm 0.06 \text{ db}$
- $1.39 \text{ rad/sec} \pm 0.40 \text{ db}$

This specification can be considered a restriction on the shape of the closed-loop frequency response curve.

Velocity Constant ($K_{Vs}$): 30 deg/sec $K_{Vs} = 267$
- 75 deg/sec $K_{Vs} = 150$

It is difficult to interpret the step response specification as a rise time which could be used to develop a transient response specification. The saturation characteristics of the plant will not permit linear operation of the system when subjected to large step inputs. Thus, this specification will be more useful in devising a non-linear control strategy.
Figure 8. Diagram showing the component parts of the controlled system.

Figure 9. Functional block diagram of the system.
THE MATHEMATICAL MODEL OF THE SYSTEM

5.1 Preliminary System Design:

The preliminary system design is shown in Figure 9. The primary power drive element, the hydraulic transmission, is controlled by adjusting the length of the pump stroke using a linear digital actuator. The actuator is controlled by the output from a digital compensator, which includes a zero-order hold for storing the processed error. Feedback is derived from a digital shaft position encoder mechanically coupled to the load.

5.2 The Digital Actuator:

The actuator chosen will be the parallel binary actuator as illustrated in Figure 7. The actuator will be modeled as a linear device with the model based on the general outline as shown in Figure 3. The quantities which are summed in this actuator are the flow rates in the branches of the actuator supply and the output is the position of the hydraulic ram. Thus referring to the symbology on the general model

\[ Y_1 = Q_1, \ Y_2 = Q_2, \ldots, Y_n = Q_n \]

\[ Q_1, Q_2, \ldots, Q_n \] are flow rates in the respective branches of the hydraulic supply.

This represents a summation of flow rates.
\[ Y_0' = Q_{\text{TOTAL}} \]
\[ Y_0 = x_p \]

where \( x_p \) is the position of the ram.

\[ Y_1 = G_{a_1}v_1', \ldots, Y_n = G_{a_n}v_n' \]

It may be recalled at this point that the variables are binary variables, that is they can only assume values of "1" or "0", \( G_{a_1}, \ldots, G_{a_n} \) are transfer functions relating the flow rates through the branches to the electrical inputs to the solenoid valves contained within the respective branches. This term may be accurately represented by a transport lag; using the Laplacian expression for a time lag:

\[ G_{a_i} = K_{v_i}e^{-st} \]  \hspace{1cm} (15)

is the time delay inherent in the operation of a solenoid valve. Also

\[ G_{a_1}(s) = G_{a_2}(s) = \ldots = G_{a_n}(s) \]

since all the solenoids will have very similar characteristics.
$K_{V_1}$ is a constant associated with the size of a calibrated orifice placed in the smallest branch of the actuator.

$$K_{V_2} = 2K_{V_1}, \quad K_{V_3} = 2K_{V_2}, \cdots, \quad K_{V_n} = 2K_{V(n-1)} \quad (16)$$

These relations must be accurately held in a binary input actuator such as this.

Rewriting all of the above into a single expression, a simplified equation for $Q_{\text{total}}$ is derived.

$$Q_{\text{TOTAL}} = \left[ K_{V_1}r_1' + 2K_{V_1}r_2' + \cdots + 2^{(n-1)}K_{V_1}r_n' \right] e^{-sc}$$

The expression in the brackets is seen by inspection to be equal to the word value of the digital binary input word multiplied by the scale factor constant associated with the least significant bit of the actuator.

Equations of continuity for the hydraulic flow may be used in deriving the transfer function $G_o$. In this case (as previously) it will be assumed that the hydraulic fluid compressibility and leakage will have a negligible effect on the transfer function.

Let $A_p$ be the area of the actuator piston. Then,
\[ A_p X_p = Q_{\text{TOTAL}} \]  \hspace{1cm} (18)

Laplace transforming,

\[ \frac{X_p}{Q_{\text{TOTAL}}} = \frac{1}{A_p s} \]  \hspace{1cm} (19)

Combining the above transfer functions together with the Laplacian expression for a zero order hold results in the actuator, mathematical model to be used in this analysis.

Let \( G_1(s) = \) the actuator transfer function = \( X_p G_{\text{ho}} \)

\[ G_1(s) = \frac{K_v (1-e^{-sT}) e^{-sT}}{A_p s^2} \]  \hspace{1cm} (20)

\( T \) is the time delay between digital commands.

\[ \frac{K_v}{A_p} = C_v \]  \hspace{1cm} (21)

is the scale factor of the least significant bit of the actuator and may be given the dimensions in/sec/input unit since it represents the ratio between the steady state output velocity (of the hydraulic ram) and a single unit of input.

5.3 The Feedback Element:

The use of a shaft angle encoder to produce the digital feedback signal presents little or no difficulty in linear modeling. This is due to the high quality such instruments have attained.\(^9\)
Shaft angle encoders are designed to measure the rotation of the shaft to which they are coupled with precision and are capable of high readout rates. The data rate associated with a shaft angle encoder is a function of the capabilities of the digital logic elements used in the readout circuits. Consequently, the shaft encoder represents no dynamic lags or special date transfers which must be incorporated into the system model.

The feedback digital word value at the sampling instant \((nT)\) equals \(\Theta_0(nT)\), where \(\Theta_0\) is the load position as previously defined.

The system model is depicted in Figure 10, as a unity feedback, error-sampled control system. The sampling is introduced into the system due to the use of a digital controller and a digital actuator. The presence of sampling in the system model will require the use of sampled-data analysis techniques in the remainder of this paper.
Figure 10. LINEAR MODEL OF THE SYSTEM
A DISCUSSION OF WORD SIZE AND SAMPLE RATE

6.1 Special Problems Raised by the Use of Digital Control:

Introduction of a digital actuator and digital controller into the system will raise the questions:

a) what should be the number of digits used to express the digitized variables in the digital portion of the system?
b) At what rate should commands to the actuator be updated so that the system specifications may be met?

6.2 Quantization Error:

The first question listed above is approached by considering the quantization error, which is the error in the system output due to a quantization of a variable at some point in the digital part of the system. A quantization occurs at any point in the system where a roundoff operation must be performed to accurately represent the value of a digitized variable in a word of the chosen length.

In this system quantization occurs at three distinct points: at the input, in the feedback element and in the digital actuator. Actually, the quantization associated with the digital actuator is a truncation accomplished within the digital controller so that the word size of the output of the controller is matched to the requirements of the digital actuator. Figure II is an illustration of the transfer characteristic of a quantizer.
Figure 11: TRANSFER CHARACTERISTIC OF A QUANTIZER

Figure 12. (a) REPRESENTATION OF A QUANTIZER  
(b) PROBABILITY DENSITY FUNCTION FOR QUANTIZATION ERROR MEETING WIDROW'S CRITERIA
Digital actuators, as commercially manufactured, are not extremely high resolution devices and their cost increases sharply as their resolution is raised. This is also true of shaft angle encoders and digital computer arithmetic units, although to a lesser extent.

Widrow 10, 11 has shown that the quantization process can be considered, in a statistical sense, as analogous to periodic sampling of a time function. The probability density function is the analogue of the function of time, while the quantization levels are treated as sampling periodically in time. A criterion similar to the Nyquist criterion is thus established 12. If the criteria are satisfied, the error from a quantization may be considered to have uniform distribution over $\pm \frac{1}{2}$ of the smallest quantization interval. This leads to a representation of a quantizer as shown in Figure 12 (from reference 13).

For the purposes of this paper, this will be an adequate description of the quantization process, since the dynamic range of the quantized variables will extend over a number of quantization levels resulting in "fine" rather than "rough" quantization. Widrow's criteria may be used to determine the word size, thus creating the number of quantization levels needed to achieve this condition.
6.3 Sampling Rate:

The sampling rate of a digitally-controlled system is often determined primarily by considerations such as the speed of the digital controller and the rate at which digital commands are received. However, when a set of specifications must be met with respect to system errors, overriding factors may develop, which would require the choice of a higher rate. It may be shown\textsuperscript{14} that the errors due to quantization previously discussed are increased by a lowering of the sampling rate. This operates as a trade-off between the word size and speed of the digital controller.

The use of a digital actuator in the system requires special consideration be given to the amount of ripple or hidden oscillation which can occur in the system output. The degree of filtering that the system can provide to discrete inputs is reduced when the digital-analog interface is moved into the mechanical responder part of the system. This is of major concern in the choice of a sample rate.

Analysis methods which describe the output of the system between samples may be used to provide a check on the output ripple (modified Z-transform). However, the initial choice needed for use in the system design in best approached by using an approximate method such as that proposed by R. Saucedo and E. Schirling\textsuperscript{15}.
A term known as the ripple factor, $K_r$, is defined. The ripple factor is the square root of the mean squared steady-state error expressed as a percentage of the desired output when the input is a sine wave of radiant frequency ($\omega$).

$$K_r = \sqrt{\frac{\varepsilon(t)^2}{C_d(t)^2}}$$

$\varepsilon$ = steady state error

$C_d(t)$ = desired output of system

Assuming that the digital portion of the system has little effect on the amount of ripple in the output, the expression derived by Ref. 15 may be modified for analysis of the system. Thus

$$K_r^2 = \sum \frac{|G_1G_2[i(\omega-n\omega_s)]|^2}{|G_1G_2(i\omega)|^2}$$

$\omega$ = input frequency, rad/sec

$\omega_s$ = sampling frequency, rad/sec
The maximum allowable ripple factor in this system can be determined from the system specifications (as it has been previously determined)

\[ K_r(\text{max}) = 0.67 \times 10^{-2} \text{ at } \omega = 0.695 \text{ rad/sec.} \]

\[ K_r(\text{max}) = 4.67 \times 10^{-2} \text{ at } \omega = 1.39 \text{ rad/sec.} \]
The constants which have been used in deriving the mathematical model are listed below, together with their numerical values (as they have been determined in the preliminary design).

\( \tau \) = solenoid valve transport lag = 0.01 sec.

\( CV \) = scale factor of digital actuator = 0.0456 in/sec error unit

\( KP \) = pump constant = 308 in\(^2\)/sec.

\( \omega_n \) = \( Km/Kpm \) = natural frequency of transmission

\( \omega_n \) = 74.5 ad/sec.

\( \delta \) = \( (LJL/Dm Kpm) /2\omega_n \) = damping ratio of transmission = 0.255

Using the above constants, a transfer function can be written for the analog portion of the system

\[
G_1(s) \cdot G_2(s) = \frac{311(1-e^{-s\tau})e^{-0.01s}}{s^3(s^2+37.9s+5520)} \tag{24}
\]
RIPPLE FACTOR CALCULATION

The information rate for the computer output which supplies the digital commands to the system has been chosen as 10 samples/sec because of the considerations of computer size and speed. A ripple factor calculation based on (23) will be made to determine whether this is an adequate system sampling rate. Using the transfer function given in (24) the results shown in Table I were computed for two input frequencies \( \omega = 0.695 \) rad/sec, and \( \omega = 1.39 \) rad/sec respectively \( (\omega_s = 62.83 \text{ rad/sec}) \).

| \( n \) | \( \left| G[i(\omega - n\omega_s)] \right|^2 \) | \( n \) | \( \left| G[i(\omega - n\omega_s)] \right|^2 \) |
|---|---|---|---|
| 0 | 2.6001 \times 10^{-4} | 0 | 4.064 \times 10^{-2} |
| 1 | 18.696 \times 10^{-8} | 1 | 19.597 \times 10^{-8} |
| -1 | 16.978 \times 10^{-8} | -1 | 16.164 \times 10^{-8} |
| 2 | 1.884 \times 10^{-10} | 2 | 2.008 \times 10^{-10} |
| -2 | 1.661 \times 10^{-10} | -2 | 1.56 \times 10^{-10} |
| 3 | 2.175 \times 10^{-12} | 3 | 2.26 \times 10^{-12} |
| -3 | 2.011 \times 10^{-12} | -3 | 1.935 \times 10^{-12} |

Assuming that the higher order terms are negligible, ripple factors may be computed from the above data.
\[ K_r = 3.61 \times 10^{-6} \quad \text{at} \quad w = 0.695 \text{ rad/sec.} \]
\[ K_r = 2.46 \times 10^{-5} \quad \text{at} \quad w = 1.39 \text{ rad/sec.} \]

The computed ripple factor fall within the limits previously specified; therefore 10 samples / sec. is adequate for the system sampling rate.
THE SAMPLED-DATA SYSTEM

9.1 The Z-transform:

In References 15 and 16, the Z-transform technique as used in analyzing sampled-data feedback control systems is thoroughly explored. In this method the sampler is assumed to be a device whose output is a series of impulse functions the values of which are equal to the amplitude of the input at the sampling instants. This approximation of a sampler leads to a useful system model, especially when digital data transfers are to be represented by the sampler.

All sampling is assumed to be periodic in this analysis. By means of a transformation equation which results directly from the above representation of the sampler, the transfer functions are derived as a function of \( Z = e^{\frac{-s}{T}} \). \( T \) is the period between samples or data transfers.

9.2 The Closed-loop Model:

Use of the Z-transform method allows the hybrid control system to be studied as a sampled-data system with the understanding that information on the system output thus obtained can be used only to describe the output at the precise sampling instants. The equation relating the output of the system to the input in terms of the Z-variables is:
\[ G(z) = G_1(s)G_2(s) \]

\[ D(z) = \text{Z-transform of digital controller} \]
\[ G(z) = \text{Z-transform of analog portion of system } G_1(s)G_2(s) \]
\[ K_d = \text{controller scale factor (to be determined)} \]
\[ \theta_o(z), \theta_l(z) = \text{Z-transforms of the output and input to the system, respectively.} \]

9.3 Computation of Z-transforms:

Use of this method requires the derivation of the Z-transform of \( G_1(s) \cdot G_2(s) \)

The basic equation in the computation is the evaluation of a complex convolution integral using the method of residues, thus:

\[ G(z) = \sum \text{residues of } \frac{z^{-1} G(x)}{1 - z^{-1} e^{T x}} \text{ at } x = S_k \quad (26) \]

where \( S_k \) are the poles of \( G(s) \)

Due to the presence of a transport lag term \( e^{-T s} \) in \( G(s) \), modification of the above relationship is necessary. Also it is much simpler to transform the zero-order hold transfer function separately since it is easily shown that:

\[ \mathcal{Z} \left[ (1 - e^{-sT}) G(s) \right] = \left( 1 - z^{-1} \right) \mathcal{Z} \left[ G(s) \right] \quad (27) \]
the modified residue calculation becomes:

\[
G(z) = (1-z^{-1})z^{-1}\sum_{k}^\text{residues of} \frac{G_m(x)}{1-z^{-1}e^{tx}} \quad \text{at} \quad x=s_k \quad (28)
\]

where \( G_m(s) \) is equal to \( G(s) \) with the term \((1-e^{-st})\) removed and \( e^{-st} \) replaced by \( e^{-s(T-t)} \).
10.1 Choice of Synthesis Method:

Transformation of the expression in (24) results in the following sampled-data transfer function \( T=0.1 \) sec.

\[
G(2) = \frac{1.863 \times 10^{-4}(z-0.0889+j0.105)(z-0.0889-j0.105)(z+1.99)}{(z-1)^2(z-0.093+j0.117)(z-0.093-j0.117)}
\]

A number of purely analytical approaches to the design of digital compensators are described in the references 15, 18. These synthesis techniques share the common disadvantage that they are optimum only for a single type of test input (i.e. ramp, step, parabola) and they usually require a cancellation of some of the plant characteristics for implementation.

In this paper, the Root locus method 15, 18 will be used to arrive at the compensator configuration. Root locus is a plot of the roots of the characteristic equation of the closed-loop system as a function of the gain. The principle is based upon the fact that the poles of \( C(s)/R(s) \) (transient-response modes) are related to the zeros and poles of the open-loop transfer function \( G(s) \ H(s) \) and to the gain. An important advantage of the root locus method is that the roots of the characteristic equation of the system can be obtained directly; this results in a complete and accurate solution of the transient and steady-state response of the controlled variable. Another important
feature is that an approximate solution may be obtained with a reduction of the work required.

The uncompensated system will be plotted and a set of poles and zeros will be introduced to reshape the plot and place the closed loop poles into the proper position for a closed-loop response which is within the specification.

The necessity for reshaping the locus is easily seen from the root locus plot of the uncompensated system (figure 13). The plot shows closed-loop poles on or outside the unit circle for all values of system loop gain.

It can be shown\textsuperscript{15, 18} that the introduction of a lead compensator will reshape the locus so that a stable closed-loop configuration results. A transfer function of the form given below represents a lead term:

$$D(z) = \frac{Kd(z-a)}{(z+b)}$$

where "a" and "b" are positive real numbers less than unity.

If "a" and "b" are both chosen as 0.7 the resulting root locus plot is shown in Figure 14.

$$G_{\text{compen.}} = G(z)D(z)$$
Figure 13. Root locus plot of the uncompensated system.
The frequency response of the closed-loop system in Figure 15 shows that we are within the limits previously specified, and is obtained by plotting $\frac{G(z)}{1+G(z)}$.
\[ N = \text{numerator} \]
\[ D = \text{denom} \]

\[ N = (z^2 - 2z + 1)(z - 0.093 + j0.117)(z - 0.093 - j0.117)(z + 0.7) + 
   \]
\[ + 0.504(2 - 0.0889 + j0.105)(2 - 0.0889 - j0.105)(2^2 + 1.29z - 1.39) \]
\[ N = (2^3 - 0.093z^2 + z^2j0.117 - 2z + 0.186z - 2j0.234 + z - 0.093 + j0.117) \cdot 
   \]
\[ \cdot (z^2 + 0.7z - 0.093z - 0.0651 - zj0.117 - j0.0819) + \]
\[ + (0.504)(2 - 0.0889 + j0.105)(2^3 - 0.0889z^2 - 0.115z + 0.124 + 1.29z^2 - 
   - 1.39z - z^2j0.105 - 2j0.135 + j0.146) \]
\[ N = \left[ 2^3 - 2.093z^2 + z^2j0.117 + 1.186z - 2j0.234 - 0.093 + j0.117 \right] \cdot 
   \]
\[ \cdot \left[ z^2 + 0.607z - zj0.117 - 0.0651 - j0.0819 \right] + \]
\[ + (0.504)(2 - 0.0889 + j0.105)(2^3 - 0.0889z^2 - 0.115z + 0.124 + 1.29z^2 - 
   - 1.39z - z^2j0.105 - 2j0.135 + j0.146) \]
\[ N = \left[ 2^5 + 2^40.607 - 2^4j0.117 - 2^30.065 - 2^3j0.082 - 2^4 - 0.093 - 2^31.27 + \right] \]
\[ + z^3 \frac{1}{2} 0.245 + z^2 0.136 + z^2 \frac{1}{3} 0.171 + z^2 \frac{4}{3} 0.117 + z^3 \frac{1}{3} 0.071 + z^3 \frac{3}{2} 0.014 - \]
\[- z^2 \frac{1}{2} 0.008 + z^2 0.01 + z^3 1.186 + z^2 0.72 - z^2 \frac{3}{2} 0.139 - z 0.071 - z \frac{3}{2} 0.097 - \]
\[- z^3 \frac{1}{2} 0.234 - z^2 \frac{1}{2} 0.142 - z^2 0.027 + z \frac{1}{2} 0.015 - 20.019 - 2^2 0.093 - \]
\[- 20.056 + 2z 0.01 + 0.006 + 2z \frac{1}{2} 0.007 + 2z \frac{1}{3} 0.117 + 2z \frac{1}{3} 0.071 + 2z 0.014 - \]
\[- j 0.007 + 0.01 \] 
\[+ \left[ z^4 0.504 - z^3 0.045 - z^2 0.058 + z 0.063 - \right] \]
\[- z^3 \frac{1}{2} 0.053 - z^2 0.068 + 2z 0.074 - z^3 0.044 + z^3 0.004 + 20.005 - 0.006 + \]
\[+ z^2 \frac{1}{2} 0.005 + 2z 0.006 - j 0.006 + z^2 \frac{1}{3} 0.053 - z^2 \frac{1}{2} 0.005 - 2z 0.006 + \]
\[+ j 0.006 + z^2 0.005 + z 0.007 - 0.008 + z^3 0.56 - 2^2 0.70 - z^2 0.057 + \]
\[+ 20.06 + z^3 \frac{5}{2} 0.068 - 2z \frac{3}{2} 0.074 \]

\[N = z^4 0.504 + z^3 0.56 - z^2 0.81 + 20.14 - 0.014\]

\[
\frac{\Theta(z)}{G(z) + 1} = \frac{z^4 0.504 + z^3 0.56 - z^2 0.81 + 20.14 - 0.014}{z^5 - z^4 0.98 + z^3 0.43 - z^2 0.06 - 20.002 + 0.002} 
\]
Checking the velocity constant specification, substitution into the formula for the velocity constant of a unity feedback system, gives the following expression:

\[
K_{VS} = \frac{1}{T} \lim_{s \to 1} \Theta_{COMP}(s)
\]

Using the dominant poles and zeroes, the expression becomes

\[
K_{VS} = \frac{0.5041(2-0.7)(2+1.99)}{(2+0.7)(2-1)}
\]

It is easily seen that the theoretical \(K_{VS}\) for this system is infinite, which means that since \(e_{ss} = \frac{1}{K_{VS}}\), the error is zero.

The gain associated with the controller can be computed as follows:

\[
K_d = K_L / K = \frac{0.5041}{1.863 \times 10^{-4}} = 2.699 \times 10^3
\]

\[
D_2 = \frac{2.699 \times 10^3 - 1.889 \times 10^3 s^{-1}}{1 + 0.7 s^{-1}}
\]

(34)
**TRANSIENT RESPONSE AND INTERSAMPLING RIPPLE**

11.1 Transient Response Calculations:

Keeping in mind the inherent saturation characteristics of the plant, it is desired to compute the transient response of the system when driven by test input signals. While accomplishing this end, it is also possible to compute the intersample response of the system by using a modification of the Z-transformation analysis method, the modified Z-transform.

As is well known in the use of the Z-transform for numerical analysis, it is possible to describe the output of a system, utilizing the Z-transformation method, by an infinite series. This series is obtained through a division process applied to the Z-transform of the system output. Thus, in Equation (25),

\[
Q_0(z) = \frac{K_d D(z)G(z)Q_i(z)}{1 + K_d D(z)G(z)} = \alpha_1 z^{-1} + \alpha_2 z^{-2} + \ldots + \alpha_n z^{-n} + \ldots
\]

The coefficients \(\alpha_1, \alpha_2, \ldots, \alpha_n\) etc. are obtained by the division of the denominator polynomial into the numerator polynomial. Evaluation of the above response in terms of an infinite series for a unit step and a ramp input is as follows, and the results are plotted in figures 16 & 17.
\[
G_o(z) = \frac{(z^40.504 + z^30.561 - z^20.867 + z0.136 + 0.014)}{(z^5-2.062z^30.426-2^20.061-20.002+0.002)} G_o(z)
\]

For a step input, we have with \( T = 1/10 \) sec.

\[
G_o(z) = G(z) \frac{z}{z-1} \quad G_o(z) = G(z)(1-2^{-1})
\]

\[
G_o(z) = 0.505 z^{-1} + 1.15 z^{-2} + 1.51 z^{-3} + 1.28 z^{-4} +
\]

\[
+ 1.19 z^{-5} + 1.09 z^{-6} + 1.03 z^{-7}
\]

For a ramp input with \( T = 1/10 \) Sec.

\[
G_o(z) = \frac{G(z) T(z)}{(z-1)^2}
\]

\[
G_o(z) = 0.002 z^{-1} + 0.078 z^{-2} + 0.25 z^{-3} + 0.45 z^{-4} +
\]

\[
+ 0.52 z^{-5} + 0.61 z^{-6} + 0.70 z^{-7} + 0.80 z^{-8}
\]

These responses show clearly the effect of the plant saturation characteristics. The step response is much slower, due to the digital actuator velocity limits; the system reacts as a bang-bang system until effects of the sudden increase in system error produced by the step input is reduced.
The inability of the system to respond properly to large step inputs was predicted earlier in this paper. The solution to the problem appears to lie in the application of a "coarse" mode of control when the system error exceeds specified limits. This aspect of controller design is not included in this paper, although methods of time optimal controller design using a switching type of controller are covered extensively in the literature.

The response of the system to ramp inputs compares well with the results of the Z-transform analysis. Some saturation of the digital actuator occurs, however, it is not enough to seriously affect the transient response curve of the output.

It is pertinent at this time to point out that \( U_i(\text{max}) \), the flow limit (thus the velocity limit) in the digital actuator is a quantity chosen at the discretion of the system designer. In this design, it has been chosen deliberately to provide the main limitation on the system response. This is primarily to protect the other components of the system from the deleterious effects of hard saturation.

11.2 Intersample Response:

A method which requires the introduction of fictitious samples operating at a multiple of the basic sampling rate can
be used to generate an expression for the output of points between samples. This method is outlined in classroom notes (18), and application of it to this system results in the following expression:

\[ G_0(2)^n = \frac{K_d D_n(2) D_n^{in}(2) G(2)^n}{1 + K_d D_n(2) G_n(2)} \]  

(36)

where \( D_m(2), G_m(2), D_n^{in}(2) = D(2), G(2), D_n(2) \) respectively with \( 2 \) replaced by \( 2_m^n \) in each expression.

\( G(2)^m \) = Z-transform of system at the higher sample rate

\( G_0(2)^m \) = output of system given at points between samples.
Figure 16. SYSTEM OUTPUT FOR A UNIT STEP INPUT (Z-TRANSFORM)

Figure 17. SYSTEM OUTPUT FOR A UNIT RAMP INPUT (Z-TRANSFORM)
TIME DOMAIN ANALYSIS

12.1 The State Equations:

The preceding analysis was based almost entirely on a linearized model of the hybrid control system. As a check of this linear model, a set of time domain equations will be written and used to solve for the response of the system to test inputs.

The following set of equations is derived in what is known as the state-variable formulation. Included with the state-variable equations will be a set of limits, which are applied to certain of the state-variables. These represent the important non-linearities of the plant, which are its saturation characteristics.

The state equations may be written from equations (9), (15), (10), and (18) by making the following appropriate substitutions.

\[ x_1 = \dot{q}_m \]
\[ x_2 = \ddot{q}_m \]
\[ x_3 = p_1 - p_2 \]
\[ x_4 = x_p \]
\[ u_1 = e_o (t - \tau) \]
\[ K_T = \frac{1}{2}(K_e + \sqrt{\beta}) \]

\[ e_o = \text{The processed error in the feedback control system} \]

The state-variable equations, together with the limits are
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \left(\frac{D_m}{J_L}\right)x_3 \\
\dot{x}_3 &= \left(-\frac{D_m}{K_T}\right)x_2 - \left(\frac{L}{K_T}\right)x_3 + \left(\frac{K_p}{K_T}\right)x_4 \quad (37) \\
\dot{x}_4 &= c_v u_1 \\

x_3 &\leq x_3(\text{max}) \quad \text{pressure limits} \\
x_4 &\leq x_4(\text{max}) \quad \text{pump stroke limits} \\
x_3x_4 &\leq H.P(\text{max})/K_P \quad \text{horsepower limits} \\
u_1 &\leq u_1(\text{max}) \quad \text{flow limit in digital actuator.}
\end{align*}
\]

The preceding state equation can be written in the following matrix form:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & \left(\frac{D_m}{J_L}\right) & 0 \\
0 & \left(-\frac{D_m}{K_T}\right) & \left(-\frac{L}{K_T}\right) & \left(\frac{K_p}{K_T}\right) \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} u_1 \\
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
c_v
\end{bmatrix} u_1 
\]
12.2 Numerical Substitutions:

The following numbers have been obtained from the characteristics of existing plants:

\[ J_L = 1.285 \text{ lb.ft.sec.} \]
\[ D_m = 2.24 \times 10^{-3} \text{ ft}^3/\text{rad} \]
\[ K_T = 0.7 \times 10^{-9} \text{ ft}^5/\text{lb} \]
\[ K_P = 0.168 \text{ ft}^3/\text{sec/in} \]
\[ L = 3.655 \times 10^{-8} \text{ ft}^5/\text{lb.sec} \]
\[ C_V = 0.0456 \text{ in/sec/error unit} \]
\[ X_3(\text{max}) = 2.88 \times 10^5 \text{ lb/ft}^2 \]
\[ X_4(\text{max}) = 3.14 \text{ in} \]
\[ H.P.(\text{max}) = 2.2 \times 10^4 \text{ ft.lb/sec} \]
\[ U_1(\text{max}) = 127 \]

Substituting into the state equations results in the following set of first-order differential equations.

\[ \dot{X}_1 = X_2 \]
\[ \dot{X}_2 = (1.74 \times 10^{-3}) X_3 \]
\[ \dot{X}_3 = (-3.2 \times 10^{-6}) X_2 - 52.2 X_3 + (2.54 \times 10^8) X_4 \quad (40) \]
\[ \dot{X}_4 = 0.0456 U_1 \]

Also

\[ U_1 = e_0 (t - 0.01) \]
In a matrix form we have

\[
\begin{bmatrix}
\dot{X}_1 \\
\dot{X}_2 \\
\dot{X}_3 \\
\dot{X}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1.74 \times 10^{-3} & 0 \\
0 & -3.2 \times 10^{-6} & -52.2 & 2.54 \times 10^8 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix}
\]

The limits are

\[
X_3 \leq 2.88 \times 10^5 \\
X_4 \leq 3.14 \\
X_3 X_4 \leq 1.235 \times 10^6 \\
u_1 \leq 127
\]

A solution to this set of state-variable equations can be found through the use of numerical integration calculation on a digital computer. This type of solution is valid even though limits are placed on some of the variables as in relation (42).
CHOICE OF WORD SIZE AT THE OUTPUT OF THE DIGITAL CONTROLLER

13.1 Introduction:

Much of the following will be based on the earlier discussion of word size considerations in a digital system. However, since a sampled-data model of the system has now been derived, it is possible to make a quantitative determination of the number of binary digits which should be used at the output to the digital actuator.

The criteria for selection of a word size is to be based on the mean-squared error at the output of the system due to the quantization in the digital controller, it will be assumed that the mean-squared steady-state error is required to be two minutes of arc or less.

13.2 Mean Squared Error Due to a Quantization Error Input:

It can be shown that the mean-square of the output of a system having a discrete transfer function, $F_c(z)$, and with a random input of zero mean and second moment $\sigma_Q^2$, can be found by using the expression

$$E^2(\eta_T) = \frac{\sigma_Q^2}{2\pi} \int F(z) F(z^{-1}) \frac{dz}{z}$$

(43)

where $\sigma_Q^2$ is the variance of the random error due to a quantization at the input to the system.
In Figure 18, the block diagram of the system showing the quantization error input has been drawn. The term \( F(Z) \) in equation (43) can be derived from this block diagram.

\[
F(Z) = \frac{G(Z)}{1 + D(Z) G(Z)} \tag{44}
\]

Substituting the values for \( D(Z) \) and \( G(Z) \) from Equations (33) and (34):

\[
F(Z) = \frac{(1.863 \times 10^{-4}) z^2 + (5 \times 10^{-4}) z + (2.6 \times 10^{-4})}{z^2 - 0.80 z^2 + 0.25 z - 0.0047} \tag{45}
\]

Evaluation of the integral in Equation (43) can be accomplished through the use of an integral table

\[
\frac{1}{2\pi i} \oint_{\gamma} F(Z) F(Z^{-1}) \frac{dZ}{Z} = 1.19 \times 10^{-6} \tag{46}
\]

Also from Reference 11, it has been shown that given the probability distribution in Figure 12(B) for the quantizer error, the variance can be expressed in terms of the quantization levels used. Thus:

\[
\sigma_q^2 = \frac{h^2}{12} \tag{47}
\]
Figure 18. BLOCK DIAGRAM SHOWING THE ERROR INTRODUCED INTO THE SYSTEM FROM THE EFFECTS OF QUANTIZATION.
Where $h$ is the size of the smallest quantization interval.

13.3 Calculation of Mean-Squared Error:

The importance of selecting quantizer levels which satisfy Widrow's criterion has been stressed previously in practice, it is extremely difficult to satisfy the criteria exactly. However, if the choice of quantization levels is such that the dynamic range of the quantized variable is an order of magnitude greater than the smallest quantization interval, the flat-topped error distribution as shown in Figure 12(b) can be assumed with a small resulting error in the analysis.

In equation (42), the dynamic range of the output of the controller has been set at $U_{i}(\text{max}) = 127$. This limit reflects a choice of the maximum velocity of the digital actuator, and the word size is established at seven binary digits (plus the sign bit).

Since $h (2^{N} - 1) = 127$, where $N = \text{number of binary digits in the word}, N = 7$

$$h = \frac{127}{2^{7} - 1} = 1$$

the size of the smallest quantization interval.

Substitution of this value into Equation (43) and using the value for the integral given in Equation (46) results in the
following mean-squared error.

\[ \varepsilon^2(nT) = \frac{1.19}{12} \times 10^{-6} \approx 10^{-7} \text{ rad.} \]

From the criteria previously established,

\[ \varepsilon_2^2(nT)_{\text{max.}} = 2 \text{ minutes of arc} = 5.81 \times 10^{-4} \text{ rad.} \]

\[ 10^{-7} < 5.81 \times 10^{-4} \]

Thus it has been shown that the word size requirement for meeting the mean-squared error criteria for the system has been satisfied in the system design. Also Widrow's criterion has been met in the system design by selecting the smallest quantization interval to be \(1/128\) of the maximum value of the quantized variable.
It has been shown that the use of actuators which respond directly to digital signals without A/D conversion is within the state-of-the-art of control system design. The use of binary inputs without conversion to other discrete forms of driving signals is a more difficult requirement to meet; however, this too can be achieved in a practical system using the actuators described in this paper.

The design problems which result from the use of digital actuators can be placed in the following broad categories:

1. Those concerned with the practical difficulties and disadvantages of complex actuator design.

2. Those concerned mainly with matching the digital actuator to the system requirement (as represented by the specifications) and analyzing the results of using a digital actuator as part of a larger system.

This project has been devoted to consideration of the second class of problems. The results of the analysis has shown that the typical digital actuator can be modeled as an analog transfer function. The digital analog interface between the controller and the actuator, thus remains in its traditional location and the analysis proceeds as for any hybrid system. The system carried out in this project illustrated clearly the choices which
the system designer must make. Choice of the plant variable to be controlled is very important and probably contributed more to the performance characteristics of this design than any other factor. Choice of sample rate and work size are both equally important in their effect on the system error. If care is taken in specifying these quantities, as illustrated in this design study, the resulting system performance can closely approach that achieved in analog systems.

As to the larger question of the practical and economic aspects of digital actuation, no attempt has been made in this paper to reach any firm conclusion. Certainly, the difficulties of D/A conversion are not ordinarily so great as to justify the addition of expensive components to the mechanical portion of the system. Digital actuation is worth preserving as a concept, however, since it does represent a step in the direction of system simplification.
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