CALIFORNIA STATE UNIVERSITY NORTH RIDGE

AN INVENTORY MODEL CONSIDERING EXTERNAL RESUPPLY
AND INTERNAL RETURNS FROM THE REPAIR PIPELINE

A Thesis submitted in partial satisfaction of the requirements for the degree of Master of Science in
Engineering

by

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I wish to extend my gratitude to the Chairman of my thesis Committee, Dr. Harish Vaish, for his interest to this work and to my wife, Marina, for her comprehension.

Dedicated to my Grandmother Iole.
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AN INVENTORY MODEL CONSIDERING EXTERNAL RESUPPLY AND INTERNAL RETURNS FROM THE REPAIR PIPELINE

by

Pier Luigi Romagnoli

Master of Science in Engineering

This work derives the measures of performance of an inventory model in which returns from the repair pipeline are considered together with the external resupply system. The ordering policy is the one known in the inventory literature as the (S-1,S) policy: to order whenever demand occurs.

The lead times and the repair times are assumed constant. First general system measures have been derived for the case in which demand and failures processes are independent: expected number of backorders, expected service time, expected stockout time and expected number of on-hand inventory units.

All the measures have been specialized for Poisson processes, the expression for the expected total cost has been found and a performance sample has been studied. The state-
ment of the problem for the correlation case has also been provided at the end in order to derive, if it is needed, all the previous measures as a function of the initial stock, $S$, and the correlation coefficient, $c$. 
Chapter 1.
THE MODEL

Introduction

Many papers have been written on inventory control theory, but among them very few are involved with repairable items. Because the model here described is an attempt to adapt an (S-1,S) inventory model to the repairable items situation, to create the necessary introductory background this section will review the most recent works on the repairable items and then give a quick overview of the work completed on the (S-1,S) policy.

It seems quite appropriate to begin with the work of F. R. Richard\(^1\) which is the most advanced step in the solution of repairable item inventory theory. The work assumes random demands and random lead times; the policy studied is a continuous review policy of the (Q,r) type suitable and modified in such a manner to cover a repairable item inventory.

The merit of this work is to discover an expression for the stationary distribution of on-hand inventory, obtained by the proof of a theorem that considers such a distribution as the convolution of the repair inventory and the net inventory.

Following the Hadley and Within\(^2\) example, from the distribution of on-hand inventory the paper describes various measures
of performance as the probability of stockout, the expected number of backorders and the expected number of on-hand stock units. This important result has been obtained assuming that the processes generating repairable failures and nonrepairable failures are independent.

An interesting point to note is that the previous work of Allen and D'Esopo\(^3\) considered the same system with Poisson process and repair time constant, the work of Simon and D'Esopo\(^4\) examined the system with compound Poisson process and the Sherbrook\(^5\) study considered the complete recoverability of the failed units, and all were shown to be special cases of the results derived by Richard\(^1\).

All works previously mentioned have considered continuous review inventory policy; Haber and Sitgreaves\(^6\), instead dealt with periodic review policy. The problem solved is peculiar to the Polaris project and could conceivably be suitable for naval military applications where situations of periodic resupply tender to fleet in service are usual.

The framework of the discussion considers a sequence of alternating time periods; the first periods represent a fixed length of time during which an end-consumer activity uses repair parts; the following periods represent a different fixed length of time during which the end-consumer activity is resupplied by an intermediate supply echelon.

If an item fails during the first period, it is replaced from the main stock of supply; resupply to this stock occurs
only during the following period, at which time the failed items become candidates for repair by the intermediate echelon of supply. The function of the echelon is to replace, during the second periods, the units that failed during the first periods. This is achieved either by repair of the failed units or by transfer of units purchased by the intermediate echelon.

The problem faced by the echelon is to determine the optimal purchase quantity given by the probability of repairing none, one, ..., or all the units that failed and were returned to it during the second periods.

Having developed all the utility measures, a very useful analysis for the model sensitivity to failures and repair times is offered.

Several authors have addressed the (S-1,S) policy in several forms. Galliher, Morse and Simon" studied backorder case where demand is arbitrary and delivery time constant, and where demand is Poisson distributed and delivery times exponentially distributed. Hadley and Within extended the model for both the backorder and noncaptive cases when demand is Poisson distributed and the distribution of delivery time is arbitrary.

Feeney and Sherbrooke generalized the problem for the case of compound Poisson demand. Finally, Gross and Harris studied the problem when demand is compound Poisson and the delivery time is related to the number of backorders.
The model

The model describes an inventory system which contains a repair pipeline. The repairable failures are shipped to the repair facility which provides an input of returns for the inventory. The study is conducted on the assumption that the two random processes generating demands and failures are independent; later the means of modifying the model and deriving the system measures for the case in which correlation exists is indicated.

The inventory ordering policy considered in this context is of the (S-1,S) type: whenever a demand occurs a reorder is placed immediately even if there is a spare stock on-hand. Demands are always captured in the model, so that lost sales do not occur if a demand is generated when the stock on-hand is zero.

Under this policy, according with Hadley and Whitin, the inventory position is equal to the stock on-hand plus the stock on order minus the backorders.

The assumptions in the model

The following assumptions have been made to derive the system measures:

1. Units are demanded one at a time.
2. At the beginning of the first day of the horizon the inventory position consists of S units.
3. Each demand causes an order to be placed immediately
by the supply system, even if the demand could be filled from stock on-hand.

4. The model lead time is equal to $L$ days.

5. The repair facility turn-around-time is $R$ days.

6. The values $S$, $L$ and $R$ are known nonnegative constants.

7. Demand on different days are independent and identically distributed.

The model flow chart

Referring to figure 1, the following random variables are defined:

- $d$ daily demands
- $h$ daily satisfied demands
- $y_R$ daily repairable failures.
FIGURE 1.

MODEL FLOW CHART
Chapter 2.

DERIVATION OF THE SYSTEM MEASURES

Number of backorders

Suppose the number of backorders at the beginning of the day \( t=1 \) is equal to zero, then at the end of an arbitrary day \( t \) the number of backorders, denoted by \( b(t) \), is given by the difference between the number of units demanded in \( t \) days, denoted by \( d(t) \), less the number of filled demands in \( t \) days, \( h(t) \):

\[
b(t) = d(t) - h(t) = \sum_{j=1}^{t} d_j - \sum_{j=1}^{t} h_j
\]  

(1)

If \( t \) is chosen greater than \( R \), the system has reached the steady state conditions and all the units demanded between \( t=1 \) and \( t=t-L \) are filled by \( t \) either from resupply and/or repair pipeline returns; if the mean of the process generating demands is assumed to be much greater than the mean of the process generating failures, it can be stated that at the beginning of day \( t-L+1 \) the inventory position still consists of \( S \) units, but it is going to increase, at the end of \( t \), to \( S \) plus the number of units returned from the repair pipeline between \( t-L+1 \) and \( t \).

If the number of units demanded between \( t-L+1 \) and \( t \) exceeds the reached inventory position of \( t \), then \( S + \sum_{j=t-L+1}^{t} (y_R)_j \) addi-
tional demands are filled by \( t \), and the expression for \( h(t) \) will be:

\[
h(t) = \sum_{j=1}^{t-L} d_j + (S + \sum_{j=t-L+1}^{t} (y_R)_j)
\]  

(2)

If the number of units demanded between \( t-L+1 \) and \( t \) does not exceed the quantity \( S + \sum_{j=t-L+1}^{t} (y_R)_j \), the alternative is:

\[
h(t) = \sum_{j=1}^{t-L} d_j + \sum_{j=t-L+1}^{t} d_j
\]  

(3)

Combining equations (2) and (3), the following expression is obtained:

\[
h(t) = \sum_{j=1}^{t-L} d_j + \min(S + \sum_{j=t-L+1}^{t} (y_R)_j; \sum_{j=t-L+1}^{t} d_j)
\]  

(4)

Finally, combining equations (1) and (4), the expression for the number of backorders is:

\[
b(t) = \sum_{j=1}^{t} d_j - \sum_{j=1}^{t-L} d_j - \min(S + \sum_{j=t-L+1}^{t} (y_R)_j; \sum_{j=t-L+1}^{t} d_j)
\]  

(5)

rearranging equation (5):

\[
b(t) = \sum_{j=t-L+1}^{t} d_j - \min(S + \sum_{j=t-L+1}^{t} (y_R)_j; \sum_{j=t-L+1}^{t} d_j)
\]  

(6)

To express \( b(t) \) in a compact form, \( z \) is defined as the random variable "minimum between the two random variables \( S + \sum_{j=t-L+1}^{t} (y_R)_j \) and \( \sum_{j=t-L+1}^{t} d_j " :\)

\[
z = \min(S + \sum_{j=t-L+1}^{t} (y_R)_j; \sum_{j=t-L+1}^{t} d_j)
\]  

(7)
The number of backorders can then be expressed as:

\[ b(t) = \sum_{j=t-L+1}^{t} d_j - z \]  \quad (8)

It should be noticed that the right-hand side of equation (8) is independent of t, consequently equation (8) represents the steady state expression for the number of backorders in the system and \( b(t) = b \).

Expected number of backorders

First, let \( l \) be defined as the random variable representing the number of units demanded over the lead time \( L \):

\[ l = \sum_{j=t-L+1}^{t} d_j \]  \quad (9)

and \( r \) as the random variable representing the number of units returned in stock from the repair pipeline over the lead time \( L \):

\[ r = \sum_{j=t-L+1}^{t} (y_R)_j \]  \quad (10)

With those positions the random variable \( z \), previously defined, can be expressed:

\[ z = \min(S + r; l) \]  \quad (11)

The expected number of backorders is now equal to the sum of the expected values of the terms on the right-hand side of equation (8).

Because \( L \) is a constant, the expected value of the first term simply reduces to \( LE(d) \), where \( E(d) \) is the expected daily demand rate; the expected number of backorders can be written as:
\[ E(b) = LE(d) - \sum_{z=1}^{\infty} z \, p(z) \]  

(12)

using as the definition of \( z \) expression (11) and defining \( p(z) \) as the Probability Distribution Function of \( z \).

Final step to go through is to derive the expression of \( p(z) \); equation (12) then fully describes \( E(b) \). Let \( P(\bar{z}) \) be defined as the Cumulative Distribution Function of \( z \) for \( z=\bar{z} \).

Considering discrete random variables, it is known that:

\[ p(\bar{z}) = P(\bar{z} + 1) - P(\bar{z}) \]  

(13)

Examining either one of the cumulative distribution functions on the right-hand side of equation (13), it can be expressed in the following form:

\[ P(\bar{z}) = P(z \leq \bar{z}) = 1 - P(z \geq \bar{z} + 1) \]  

(14)

and, referring to equation (11), being \( z \) the minimum between \( S + r \) and \( l \), it can be also written:

\[ P(z \geq \bar{z} + 1) = P(S+r \geq \bar{z} + 1; l \geq \bar{z} + 1) = P(r \geq \bar{z} + 1 - S; l \geq \bar{z} + 1) \]  

(15)

It was stated, as the basic assumption for this study, the independence between the two random variables \( r \) and \( l \); hence:

\[ P(r \geq \bar{z} + 1 - S; l \geq \bar{z} + 1) = P(r \geq \bar{z} + 1 - S) \, P(l \geq \bar{z} + 1) \]  

(16)

In the same manner:

\[ P(z \geq \bar{z} + 2) = P(r \geq \bar{z} + 2 - S) \, P(l \geq \bar{z} + 2) \]  

(17)

Equation (13) can be rewritten as:

\[ p(\bar{z}) = [1 - P(z \geq \bar{z} + 2)] - [1 - P(z \geq \bar{z} + 1)] \]  

(18)
Finally, substituting equations (16) and (17) into equation (18), the probability function of \( p(\bar{z}) \) is obtained:

\[
p(\bar{z}) = [P(r \geq \bar{z}+1-S; P(1 \geq \bar{z}+1)) - P(r \geq \bar{z}+2-S; P(1 \geq \bar{z}+2))] (19)
\]

Equation (12), together with the expression of \( p(\bar{z}) \), given by equation (19), gives the expected number of backorders.

**Expected service time**

The expected service time in this model is equivalent to the expected number of backorder days per demand. Hence, calling \( v \) the random variable representing the service time, the mean time a unit spends in the system, \( E(v) \), is equal to the product of the mean number of backorders in the system, \( E(b) \), and the mean time between arrivals, as shown by Little\(^{10}\); because the arrivals in the model are composed by demands and returns, the mean time between arrivals can be expressed by \( 1/[E(d)+E(y_R)] \) and, recalling equation (12):

\[
E(v) = E(b)/[E(d)+E(y_R)] = [LE(d) - \sum_{\bar{z}=1}^{\infty} \bar{z}p(\bar{z})]/[E(d)+E(y_R)] (20)
\]

**Fraction of time out of stock**

This time, \( P(v \geq 0) \), is given by the probability that the demand over the lead time \( L \) is greater than \( S+r \). Calling \( w \) the random variable representing the difference between \( l \) and \( r \):

\[
P(v \geq 0) = \sum_{w=S}^{\infty} p(w,L) (21)
\]

where \( p(w,L) \) is the probability of \( w \) demands over the lead time \( L \).
Expected number of on-hand stock units

The on-hand inventory, represented by the random variable \( i \), is given by the inventory position minus the stock on order plus the backorders:

\[
i = s + \sum_{j=t-L+1}^{t} (y_R)_j - \sum_{j=t-L+1}^{t} d_j + b \tag{22}
\]

The expected value of \( i \), \( E(i) \), is the sum of the expected values of the four terms on the right-hand side of equation (22):

\[
E(i) = S + LE(y_R) - LE(d) + E(b) \tag{23}
\]
Probability Distribution Functions in the model

The P.D.F. of the random variables involved in the model will now be defined.

The demand process is Poisson with parameter $D$:

$$p(d) = \frac{\exp(-D)D^d}{d!}$$

The failures process is Poisson with parameter $F$:

$$p(y) = \frac{\exp(-F)F^y}{y!}$$

If $p$ is finally defined as the probability that a failed unit can be repaired, all the parameters in the system have been determined. In fact, the repairable failures process is Poisson with parameter $F_R = pF$; the nonrepairable failures process is also Poisson with parameter $F_{NR} = (1-p)F = pF$.

Last, two other parameters are defined, combinations of previous ones, that will be useful in the derivation of the specialized system measures:

$$A = F_R L$$

and:

$$B = DL$$

It is noted that $A$ and $B$, expressed by equations (26) and (27), are parameters derived from Poisson distribution parameters hence, themselves parameters of Poisson distributions.
Expected numbers of backorders

Equation (19) for \( p(z) \) can be specialized:

\[
p(z) = \sum_{y_R=Z+1-S}^{\infty} \left[ \frac{\exp(-A) \cdot A^y_R}{y_R!} \right] \sum_{d=Z+1}^{\infty} \left[ \frac{\exp(-B) \cdot B^d}{d!} \right] + \sum_{y_R=Z+2-S}^{\infty} \left[ \frac{\exp(-A) \cdot A^y_R}{y_R!} \right] \sum_{d=Z+2}^{\infty} \left[ \frac{\exp(-B) \cdot B^d}{d!} \right]
\]

Dealing with discrete random variables, it is very easy to verify that equation (28) can be also written as:

\[
p(z) = \exp(-A-B) \left[ \frac{A^{z-S}}{(z-S)!} \right] \left[ \frac{B^z}{z!} \right]
\]

Substituting equation (29) into equation (12), the expected number of backorders is obtained for the case under study:

\[
E(b) = DL - \exp(-A-B) \sum_{z=1}^{\infty} \left[ \frac{A^{z-S}}{(z-S)!} \right] \left[ \frac{B^z}{z!} \right]
\]

Expected service time

Substituting equation (29) into equation (20) for \( E(v) \), it is found:

\[
E(v) = \frac{[DL - \exp(-A-B) \sum_{z=1}^{\infty} \left[ \frac{A^{z-S}}{(z-S)!} \right] \left[ \frac{B^z}{z!} \right]]}{(D+y_R)}
\]

Fraction of time out of stock

To specialize expression (21), \( p(w,L) \) can be expressed as:

\[
p(w,L) = p(d=w,L)p(y_R=0,L) + p(d=w+1,L)p(y_R=1,L) + \ldots
\]

Substituting to all the terms at the right-hand side of equation (32) the appropriate density functions defined at the beginning of this chapter and remembering the meaning of the parameters
A and B, the following expression are obtained:

\[ p(w,L) = \exp(-B)B^w/w!\exp(-A)A^0/0! + \]
\[ + \exp(-B)B^{w+1}/(w+1)!\exp(-A)A^1/1! + \ldots = \]
\[ = \exp(-A-B)B^w[(AB)^{0}/w!0!] + [(AB)^{1}/(w+1)!1!] + \ldots \] (33)

also:

\[ p(w,L) = \exp(-A-B)B^w \sum_{j=0}^{\infty} [(AB)^{j}/(w+j)!j!] \] (34)

Finally:

\[ P(v \geq 0) = \sum_{w=S}^{\infty} \exp(-A-B)B^w \sum_{j=0}^{\infty} [(AB)^{j}/(w+j)!j!] \] (35)

**Expected number of on-hand stock units**

Expression (23) assumes for this case the form:

\[ E(i) = S+A-B-B\exp(-A-B) \sum_{z=1}^{\infty} z[A(z-S)/(z-S)!][B^z/z!] = \]
\[ = S+A-B\exp(-A-B) \sum_{z=1}^{\infty} z[A(z-S)/(z-S)!][B^z/z!] \] (36)
Chapter 4.

COST CONSIDERATIONS

Referring once again to Hadley and Within$^2$, the following symbols are defined:

- $\pi$ fixed cost for unit in backorder, in dollars.
- $\hat{\gamma}$ cost per unit in backorder per day, in dollars per day.
- $UC$ unit cost of the items in inventory, in dollars.
- $I$ inventory carrying charge, where $0 \leq I \leq 1$.

The average daily cost for this model consists of the backorder cost plus the inventory carrying cost; the average daily ordering cost is independent of $S$ and hence need not be included in the average daily system cost, which can be written as:

$$K(S) = I \cdot UC \cdot E(i) + \hat{\gamma} \cdot E(d) \cdot P(v \geq 0) + \pi \cdot E(b)$$  \hspace{1cm} (37)

The first term on the right-hand side of equation (37) represents the average daily carrying cost; the second term is the average daily backorder cost component dependent on time out of stock and the third term is the fixed backorder cost.

Specializing equation (37), it is obtained:

$$K(S) = I \cdot UC \cdot S + A \cdot \exp(-A-B) \sum_{z=1}^{\infty} z[A^{(z-S)}/(z-S)!][B^z/z!] +$$

$$+ \hat{\gamma} \cdot D \sum_{w=S}^{\infty} \exp(-A-B)B^w \sum_{j=0}^{\infty} [(AB)^j/(w+j)!j!] +$$

$$+ \pi \cdot B \cdot \exp(-A-B) \sum_{z=1}^{\infty} z[A^{(z-S)}/(z-S)!][B^z/z!].$$  \hspace{1cm} (38)

16
The problem of finding the optimal value of \( S, S^* \), which minimizes \( K(S) \) consists of satisfying the following two relationships:

\[
\Delta K(S^*) < 0; \Delta K(S^* + 1) \geq 0
\]  \hspace{1cm} (39)

To satisfy conditions (39) is rather complex in this situation; it is preferred to show how the daily average cost, together with the system measures derived previously, vary with different choices of \( S \). Table 1 illustrates the results of the FORTRAN program computing the system performance for \( S \) varying from one to thirty, with the choice of a lead time of four days, a repair turn-around-time of two days.

The daily new demand arrival rate is 1.2 units/day, the daily failure rate is 0.4 unit/day and the probability to repair failures is \( p=70\% \). The optimal choice of \( S, S^* \), comes out to be equal to ten; the computer program to achieve the results is listed in appendix A.
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**TABLE 1.**

**SYSTEM PERFORMANCE RESULTS**
Chapter 5.
ALTERNATIVE CASE WITH CORRELATION
BETWEEN DEMAND AND FAILURES

To illustrate the case with correlation between demand and failures, the demand process can be thought as composed of two random processes: the failures process and the new demand from customer process. The model flow chart for this case is illustrated in figure 2, where $x$ is the random variable representing daily new demands.

The items involved in this particular application of the model could be any installed in a weapon system: for this reason it is being considered that, any time an item fails in the field, even if the failure is repairable, one additional unit is added on the total demand in order to save the weapon system efficiency and the failures link, flowing into the total demand, has been added.

The total demand process is no longer Poisson distributed with parameter $D$ as was considered in the previous chapters, instead a bivariate Poisson process, as described by Haight\textsuperscript{11} should be considered.

If $c$ is defined as the correlation coefficient between the two random variables $x$ and $y$, the probability density function for $d$, given by the bivariate Poisson process, is:
FIGURE 2.
MODEL FLOW CHART FOR THE CORRELATION CASE
\[ p(d) = p(x, y) = \exp(-X-F-c) \sum_{q=0}^{\min(x,y)} [(x-c)^{x-q}/(x-q)!][(F-c)^{y-q}/(y-q)!][c^q/q!] \]  

(40)

First, it is proved that the two marginal distribution of \( x \) and \( y \) are Poisson distributed with parameters \( X \) and \( F \) respectively, as it was in the previous chapters for \( d \) and \( y \). It will be proved for the new demand process; the proof for the other random variable \( y \) is similar.

By definition of marginal distribution:

\[ p(x) = \sum_{y=-\infty}^{\infty} p(x, y) \]  

(41)

First, it should be noted that negative values for \( y \) are not allowed, because they are not applicable to a variable representing the daily number of failures; then, if the structure of equation (40) is analyzed, it should also be noted that \( y \) cannot assume values less than those assumed by the dummy variable \( q \), otherwise the term \((y-q)!\) looses its significance.

Hence equation (41) can be written as:

\[ p(x) = \sum_{y=q}^{\infty} p(x, y) = \exp(-X-F-c) \sum_{q=0}^{\min(x,y)} [(x-c)^{x-q}/(x-q)!][(F-c)^{y-q}/(y-q)!][c^q/q!] \]  

\[ \sum_{y=q}^{\infty} [(F-c)^{y-q}/(y-q)!] \]  

(42)

The second summation on the right-hand side of equation (42) is the Taylor series for the exponential function \( \exp(F-c) \); substituting this result into equation (42):
\[ p(x) = \exp(-X) \sum_{q=0}^{x} \left[ \frac{(x-c)^{q}/(x-q)!}{q!/q!} \right] cq^{q} \]  

(43)
multiplying and dividing by \( x! \) the right-hand side of equation (43):
\[ p(x) = \left[ \frac{\exp(-X)}{x!} \right] \sum_{q=0}^{x} \left[ \frac{x!/(x-q)!q!}{(x-c)^{x-q}c^{q}} \right] \]  

(44)
The summation on the right-hand side of expression (44) is the series obtained from the Newton binomial:
\[ [(X-c) + c]^{x} = x^{x} \]  

(45)
Substituting equation (45) into equation (44), the Poisson distribution for \( x \) is finally obtained:
\[ p(x) = \exp(-X) \frac{x^{x}}{x!} \]  

(46)
This step completes the proof.

To continue this case of correlation, all the system measures should be recomputed considering equation (40) whenever it is referred to the total demand process. The computational effort will result in a more tedious and longer one than in the previous case without correlation, but, at this point based on the positions previously stated, it will be conceptually the same.

The results will be function of \( S \) and \( c \), the correlation coefficient and eventual trade-off studies could be conducted to check how the two factors interact and influence the average system cost.
Bibliography


Appendix A

FORTRAN PROGRAM TO COMPUTE

THE SYSTEM PERFORMANCE SAMPLE

DIMENSION SPA(120)
READ(5,2) IL,D,FR
READ(5,5) EE1
READ(5,16) RATE,IUC,IPI,IPI1
16 FORMAT(F4.2,3(I4))
READ(5,6) (SPA(J),J=1,120)
6 FORMAT(60A1)
2 FORMAT(I3,2(F4.2))
5 FORMAT(F8.6)
13 FORMAT(////,8X,'TABLE 1 - SYSTEM PERFORMANCE RESULTS',///)
A=FR*IL
B=D*IL
PRINT 4,(SPA(J),J=1,120)
PRINT 8
8 FORMAT(SX,'S',4X,'E(B)',6X,'E(V)',6X,'P(V O)',5X,'E(I)',
15X,'COST/DAY')
PRINT 4,(SPA(J),J=1,120)
4 FORMAT(120A1)
DO 10 II=1,30
IS=II
JK=40+II
S=0
TSUM=0
DO 20 I=1,JK
N=I-IS+2
IF(I.LE.(IS-2)) GO TO 1
FACT1=1
DO 30 J=1,N
30 FACT2=FACT2*JJ
C=(A**N)
E=(B**M)
P=C/FACT1
G=E/FACT2
S=S+((I*F*G)*EE1)
IF(S.NE.TSUM) GO TO 21
I=JK 
GO TO 20 
21 TSUM=S 
20 CONTINUE 
S1=0 
TSUM1=S1 
DO 50 III=IS,40 
S2=0 
TSUM2=S2 
DO 60 MM=1,40 
FACT3=1 
NN=MM+III 
DO 70 MMM=1,NN 
70 FACT3=FACT3*MMM 
FACT4=1 
NNN=MM 
DO 80 JJJ=1,NNN 
80 FACT4=FACT4*JJJ 
H=(A**MM)/FACT4 
R=(B**MM)/FACT3 
S2=S2+(H*R) 
IF(S2.NE.TSUM2) GO TO 22 
MM=40 
GO TO 60 
22 TSUM2=S2 
60 CONTINUE 
S1=S1+(B**III)*S2 
IF(S1.NE.TSUM1) GO TO 23 
III=60 
GO TO 50 
23 TSUM1=S1 
50 CONTINUE 
EB=B-S 
PS=S1*EE1 
ES=(EB/(D+FR)) 
EI=IS+IL*FR-IL*D+EB 
TC=((RATE*IUC*EI))+IPI*PS*D+IPI1*EB 
10 PRINT 12,II,EB,ES,PS,EI,TC 
PRINT 4,(SPA(J),J=1,120) 
PRINT 13 
12 FORMAT(4X,I2,4(2X,F8.5),4X,'$',1X,F6.2) 
STOP 
END