FINITE ELEMENT ANALYSIS OF
NONLINEAR GEOMETRIC STRUCTURES

A project submitted in partial satisfaction of the requirements for the degree of Master of Science in Engineering

by

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June, 1979
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ACKNOWLEDGEMENT

Sincere gratitude and appreciation is expressed to my wife Josephine, for her patience, understanding and moral support, not only during the course of this research but also throughout the course of completing all the requirements for the degree.
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ABSTRACT

FINITE ELEMENT ANALYSIS OF
NONLINEAR GEOMETRIC STRUCTURES

by

Romeo Gene Suarez

Master of Science in Engineering

For some structures, linear analysis may not be adequate, and nonlinear analysis should be applied. Here, an incremental stiffness procedure which employs the finite element displacement technique, is presented for nonlinear analysis of planar trusses. Elastic and geometric stiffness matrices are derived for application to large deflection and structural stability problems. Several examples, including a five-element truss, have been solved. For comparison, both linear and nonlinear analyses are included. The numerical results are in good agreement with known analytical results. The computer program written for the nonlinear analysis appears in Appendix C.
1.0 INTRODUCTION

The incremental stiffness procedure embodying the finite element displacement method is presented for nonlinear analysis of planar trusses.

1.1 The Finite Element Method

The problem in structural mechanics is that of finding the forces and displacements (or the stresses and strains) in a continuous system. In the finite element method, the system is discretized by a number of elements which are joined at selected node points. Each element has a finite number of force and/or displacement parameter at the nodes. Relationships between nodal force and displacements are obtained by applying variational principles. The response of the system is obtained by an assemblage of the individual elements by requiring displacement continuity at the nodal points. The assemblage of elements results in a system of equations (with nodal displacements as the dependent variables) which govern the structural behavior for the entire structure. After applying the necessary boundary conditions to the equations, the system of equations is solved for the nodal displacements.
1.2 Nonlinear Problems

Nonlinearity is introduced into the theory of structures in three ways:

1. Through the strain-displacement equations.
2. Through the equilibrium equations.
3. Through the stress-strain relations.

The first two sources are termed geometric nonlinearities while the third is due to material behavior and is, therefore, classified as physical nonlinearity or plastic behavior.

Four general types of problems arise as a result of the considerations listed above. These are:

1. Problems which are geometrically and physically linear.
2. Problems which are geometrically linear and physically nonlinear.
3. Problems which are physically linear and geometrically nonlinear.
4. Problems which are physically and geometrically nonlinear.

The analyses presented are limited to physically linear and geometrically nonlinear structures.
1.3 Scope of Present Work

A finite element method of analysis for structural problems which are physically linear and geometrically nonlinear is presented and illustrated. Since large strains are encountered only occasionally during plastic deformation of ordinarily materials and in rubber-like materials which are seldom used in engineering structures, only the important sub-class of geometric nonlinearity in which the displacements are large but the strains are small will be discussed.

Method of analysis is illustrated for load-deflection response of a truss structure. The incremental stiffness solution procedure embodying the finite element method is presented in Section 2. Comparison between the linear and nonlinear analysis is presented along with the effect of the number of incremental loading used in analysis. Section 3 presents in detail the analysis to determine the critical load at which a simple truss becomes unstable utilizing the nonlinear stiffness matrix derived in Appendix A. Results of this analysis are compared to that obtained by the classical method for substantiation.
The same simple truss problem is also analyzed for maximum deflections for the applied load of same magnitude as the critical buckling load. The computer program developed to analyze nonlinear load-deflection response in a planar truss system is presented in Section 4. This computer program is open-ended in that any element can be added into it without affecting the existing computational procedure for load-deflection response of a truss structure. The nonlinear stiffness matrix derivation for a truss and beam element is presented in Appendix A and in Appendix B respectively. Listing for the computer program is presented in Appendix C.
2.0 THE NONLINEAR GEOMETRIC SOLUTION PROCEDURE

2.1 The Incremental Stiffness Method

The solution procedure presented is the incremental stiffness solution procedure embodying the finite element displacement method. In this method, the load is applied as a sequence of sufficiently small increments so that the structure can be assumed to respond linearly during each incremental loading. For each increment of load, increments of displacements and corresponding increments of strain are computed. These incremental quantities are used to compute various corrective stiffness matrices which serve to take into account the deformed geometry of the structures. A subsequent increment of load is applied and the process continued until the desired number of load increments have been applied. The net effect is to solve a sequence of linear problems wherein the stiffness properties are computed based on the current geometry prior to each load increment. The solution procedure takes the following mathematical form:

\[
\left( \begin{bmatrix} K \\ L \end{bmatrix} + \begin{bmatrix} K(q) \\ I \end{bmatrix} \right)_{i-1} \begin{bmatrix} \Delta q \end{bmatrix}_i = \begin{bmatrix} \Delta Q \end{bmatrix}. \tag{1}
\]
Where $[K_L]$ is the linear stiffness matrix, $[K_i(q_i)]$ is an incremental stiffness matrix which corrects for the deformed geometry and is based on displacements at the previous load step $i-1$, $\Delta q$ is the increment of displacement due to the $i^{th}$ load increment, and $\Delta Q$ is the increment of load applied. After the application of the $i^{th}$ load increment, the total displacement is given by

$$q_i = \sum_{j=1}^{i} \Delta q_j$$  \hspace{1cm} (2)$$

The general incremental stiffness solution procedure is presented symbolically in Table 1.
<table>
<thead>
<tr>
<th>Step No.</th>
<th>Equations To Solve</th>
<th>Incr. Displ.</th>
<th>Total Displ.</th>
<th>Total Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \left[ K \right] + \left[ K(q) \right] ) [ \Delta q ] = [ \Delta Q ]</td>
<td>( \Delta q )</td>
<td>( q = \Delta q )</td>
<td>( Q = \Delta Q )</td>
</tr>
<tr>
<td>2</td>
<td>( \left[ K \right] + \left[ K(q) \right] ) [ \Delta q ] = [ \Delta Q ]</td>
<td>( \Delta q )</td>
<td>( q = \Delta q + q )</td>
<td>( Q = \Delta Q + Q )</td>
</tr>
<tr>
<td>3</td>
<td>( \left[ K \right] + \left[ K(q) \right] ) [ \Delta q ] = [ \Delta Q ]</td>
<td>( \Delta q )</td>
<td>( q = \Delta q + q )</td>
<td>( Q = \Delta Q + Q )</td>
</tr>
<tr>
<td>( n )</td>
<td>( \left[ K \right] + \left[ K(q) \right] ) [ \Delta q ] = [ \Delta Q ]</td>
<td>( \Delta q )</td>
<td>( q = \sum_{i=1}^{n} \Delta q_i )</td>
<td>( Q = \sum_{i=1}^{n} \Delta Q_i )</td>
</tr>
</tbody>
</table>

where \( \left[ K(q) \right] = 0 \)
2.2 Comparison Between Linear And Nonlinear Analysis

Consider the single degree-of-freedom elementary truss problem shown in Figure 1.

Due to the support conditions of the truss, the displacements $q_1$, $q_2$, and $q_3$ are all zero. These boundary conditions will reduce the matrix expression, Equations (A26) and (A27) of Appendix A, into the simpler forms:

$$\begin{bmatrix} K \end{bmatrix} = \frac{EA}{L} \sin \theta; \quad (3)$$

and

$$\begin{bmatrix} K_{NL} \end{bmatrix} = \frac{EA}{L} (2 \theta \sin \theta \cos \theta + \cos^2 \theta). \quad (4)$$
From Equation (A25):

\[ e = \frac{1}{L} (q_4 \sin \gamma) ; \]
\[ \Phi = \frac{1}{L} (q_4 \cos \gamma) ; \]  
\[ \lambda = \frac{1}{L} (q_4 \sin \gamma) + \left( \frac{3}{2L^2} \right) (q_4^2 \cos^2 \gamma) . \]  

(5)

By substituting Equation (5) into Equation (4), the nonlinear stiffness matrix is obtained as:

\[ [K_{NL}] = \frac{EA}{L} \left[ 3 \frac{q_4}{L} \sin \gamma \cos^2 \gamma + 1.5 \left( \frac{q_4}{L} \right)^2 \cos^2 \gamma \right] . \]  

(6)

The system geometric matrix, Eqn. (A28)

\[ [K_G] = [K_L] + [K_{NL}] \]
\[ = \frac{EA}{L} \left[ \sin^2 \gamma + \frac{q_4}{L} \cos^2 \gamma \left( 6 \sin \gamma + 3 \frac{q_4}{L} \right) \right] . \]  

(7)

LINEAR SOLUTION

The linear solution is obtained by applying or incrementing the applied load only once.

\[ [K_G] = [K_L] + [K(0)] = [K_L] . \]

The vertical deflection at node 2 is

\[ \{q_4\} = [K_L]^{-1} \{Q\} . \]
where:

\[ q_4 = \frac{L}{EA \sin^2 \frac{Q}{4}} \]

\[ Q = P = -2.0 \text{ pounds}; \]
\[ E = 10^7 \text{ psi}; \]
\[ L = 100.005 \text{ inches}; \]
\[ \sin^2 \gamma = 1/L^2, \]

therefore,

\[ q_4 = -0.20 \text{ inch.} \]

NONLINEAR ANALYSIS

The computational procedure as presented in Section 2.1 and as tabulated in Table 1 is used for this purpose. The results are shown in Table 2 and in Table 3. Table 2 shows the results for 4 equal incremental loadings and Table 3 shows the results for 20 equal incremental loadings. These results are plotted in Figure 2 alongside a plot for the linear analysis for convenience in comparing the two methods of analysis.
Table 2

Result Of Incremental Stiffness Method Of Analysis To The Simple Truss Of Figure 1 For FOUR Incremental Loading

<table>
<thead>
<tr>
<th>Incr. No.</th>
<th>$[K_L]$</th>
<th>$[K_{NL}]$</th>
<th>$[K_G]$</th>
<th>$[K_{FL}]$</th>
<th>$\Delta Q$</th>
<th>$Q$</th>
<th>$\Delta q$</th>
<th>$q$</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>9.99850</td>
<td>0</td>
<td>9.99850</td>
<td>0.10002</td>
<td>-0.50</td>
<td>-0.50</td>
<td>-0.05001</td>
<td>-0.05001</td>
</tr>
<tr>
<td>2</td>
<td>9.99850</td>
<td>-1.46236</td>
<td>8.53614</td>
<td>0.11705</td>
<td>-0.50</td>
<td>-1.00</td>
<td>-0.05857</td>
<td>-1.0858</td>
</tr>
<tr>
<td>3</td>
<td>9.99850</td>
<td>-3.07986</td>
<td>6.91864</td>
<td>0.14454</td>
<td>-0.50</td>
<td>-1.50</td>
<td>-0.07227</td>
<td>-1.8085</td>
</tr>
<tr>
<td>4</td>
<td>9.99850</td>
<td>-4.93367</td>
<td>5.06483</td>
<td>0.19744</td>
<td>-0.50</td>
<td>-2.00</td>
<td>-0.09872</td>
<td>-2.7957</td>
</tr>
</tbody>
</table>
Table 3
Result of Incremental Stiffness Method of Analysis to the Simple Truss of Figure 1 for Twenty Incremental Loading.

<table>
<thead>
<tr>
<th>Incr. No.</th>
<th>([K_L])</th>
<th>([K_{NL}])</th>
<th>([K_g])</th>
<th>([K_g]^{-1})</th>
<th>(\Delta Q)</th>
<th>(Q)</th>
<th>(\Delta q)</th>
<th>(q)</th>
</tr>
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<tr>
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<td>9.998500</td>
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<td>-0.10</td>
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<td>-0.01000</td>
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<tr>
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<td>9.9985</td>
<td>-0.298470</td>
<td>9.700030</td>
<td>1.030997</td>
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<td>-0.20</td>
<td>-0.01031</td>
<td>-0.02031</td>
</tr>
<tr>
<td>3</td>
<td>9.9985</td>
<td>-0.602984</td>
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<td>-0.30</td>
<td>-0.01064</td>
<td>-0.03095</td>
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<tr>
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<td>-0.40</td>
<td>-0.40</td>
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<td>8.766360</td>
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<td>6</td>
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<td>-1.55796</td>
<td>8.440540</td>
<td>1.123360</td>
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<td>-0.60</td>
<td>-0.01185</td>
<td>-0.06522</td>
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<tr>
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<td>-1.89223</td>
<td>8.106270</td>
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<td>-0.70</td>
<td>-0.01234</td>
<td>-0.07755</td>
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<td>-2.23579</td>
<td>7.762710</td>
<td>1.134970</td>
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<td>-1.20</td>
<td>-0.01595</td>
<td>-0.14908</td>
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<tr>
<td>13</td>
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<tr>
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<td>-4.56912</td>
<td>5.429380</td>
<td>1.184180</td>
<td>-1.40</td>
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<td>-0.18456</td>
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<tr>
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<td>9.9985</td>
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<td>-1.50</td>
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<td>-0.20467</td>
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<td>-8.02481</td>
<td>1.973690</td>
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<td>-2.00</td>
<td>-2.00</td>
<td>-0.05067</td>
<td>-3.6885</td>
</tr>
</tbody>
</table>
Comparison of load-displacement curves for linear and nonlinear analysis along with effect of the number of incremental loading.

**Figure 2**

Comparison of load-displacement curves for linear and nonlinear analysis along with effect of the number of incremental loading.
3.0 APPLICATION OF THE INCREMENTAL STIFFNESS METHOD

The solution procedure for stability and load-deflection response of a planar truss structure is presented in this section. Consider the simple truss shown in Figure 3.

The simple truss is first analyzed to determine the critical buckling load, $P$, and using this load, the load-deflection response of the structure is determined. From Figure 3, displacements $q_1$, $q_2$, $q_3$, and $q_4$ are zero. These boundary conditions along with the tabulated data from Figure 3 above, will be used in the stability and load-deflection analysis to this problem.

<table>
<thead>
<tr>
<th>Member</th>
<th>$L$</th>
<th>$\theta$</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>Area</th>
<th>$E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100.50</td>
<td>5.71°</td>
<td>0.09950</td>
<td>0.9950</td>
<td>1.0</td>
<td>$10^7$</td>
</tr>
<tr>
<td>2</td>
<td>10.00</td>
<td>90.00°</td>
<td>1.00000</td>
<td>0.0000</td>
<td>1.0</td>
<td>$10^7$</td>
</tr>
</tbody>
</table>
3.1 Stability Analysis

Applying Equation (A34) and the boundary conditions to the simple truss problem of Figure 3:

\[
\begin{bmatrix}
    K^* \\
    G
\end{bmatrix} = \frac{EA}{L} \begin{bmatrix}
    c^2 & cs \\
    cs & s^2
\end{bmatrix} + \frac{P}{L} \begin{bmatrix}
    s^2 & -sc \\
    -sc & c^2
\end{bmatrix}.
\]

The element forces for members 1 and 2 are \( F_1 = 0 \) and \( F_2 = P \). Substituting data from Figure 3, the element stiffness matrices are

\[
\begin{bmatrix}
    k^*_1 \\
    G_1
\end{bmatrix} = \frac{EA}{G} \begin{bmatrix}
    0.0099 & 0.0010 \\
    0.0010 & 0.0001
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
    k^*_2 \\
    G_2
\end{bmatrix} = \frac{EA}{G} \begin{bmatrix}
    0 & 0 \\
    0 & 0.10
\end{bmatrix} + \frac{P}{G} \begin{bmatrix}
    0.10 & 0 \\
    0 & 0
\end{bmatrix}.
\]

The stability matrix of the system is therefore

\[
\begin{bmatrix}
    K^* \\
    G
\end{bmatrix} = \begin{bmatrix}
    k^*_1 \\
    G_1
\end{bmatrix} + \begin{bmatrix}
    k^*_2 \\
    G_2
\end{bmatrix}.
\]
\[
\begin{bmatrix}
K^*
\end{bmatrix} = EA\begin{bmatrix}
.0099 & .0010 \\
.0010 & 1.001
\end{bmatrix} + P\begin{bmatrix}
.10 & 0 \\
0 & 0
\end{bmatrix}.
\]

From Equation (A35) and (A36)

\[
\begin{bmatrix}
K^*
\end{bmatrix} = \begin{bmatrix}
.0099E & .0010E \\
.0010E & 1.001E
\end{bmatrix} + \lambda\begin{bmatrix}
.10 & 0 \\
0 & 0
\end{bmatrix}
\]

from which

\[
\begin{vmatrix}
.0099E + .10 & .0010E \\
.0010E & 1.001E
\end{vmatrix} = 0.
\]

The lowest root of \( \lambda \) gives the critical buckling load.

\[
P_{\text{critical}} = -9.8420 \times 10^{-2} E.
\]

A detailed classical solution of this simple truss is found from Reference 5, page 147. The critical value is found to be

\[
P_{\text{critical}} = \frac{A_1 E \sin \gamma \cos^2 \gamma}{1 + \frac{A_1}{A_2} \sin^3 \gamma} = -9.8420 \times 10^{-2} E.
\]

This result is identical to that obtained by using Eqn. (A34). This substantiates the formulation of Equation (A34).
3.2 Load-Deflection Analysis

The load-deflection response of the simple truss of Figure 1 is determined for an applied load equal to the critical buckling load obtained in Section 3.1. The linear solution is first determined followed by the nonlinear solution utilizing the incremental stiffness method discussed in Section 2.0.

Applying the boundary conditions of the problem to Equation(A26), Appendix A, the linear stiffness matrix of the system is

\[
[K_L] = \frac{EA}{L} \begin{bmatrix}
\cos^2 \gamma & \sin \gamma \cos \gamma \\
\sin \gamma \cos \gamma & \sin^2 \gamma
\end{bmatrix},
\]

and from Equation(A27) the nonlinear stiffness matrix of the system is

\[
K_{NL} = \frac{EA}{L} \begin{bmatrix}
-2sc \Theta + s^2 \alpha & (\text{symm}) \\
-s^2 \Theta + c^2 \Theta - sc \alpha & 2sc \Theta + c^2 \alpha
\end{bmatrix}
\]

where

\[
\begin{align*}
s &= \sin \gamma \\
c &= \cos \gamma \\
\Theta &= \text{rotational strain} \\
e &= \text{axial strain} \\
\alpha &= e + 3/2 \Theta^2
\end{align*}
\]
From Equation (A25):

\[ e = \frac{1}{L_1} \left[ (q - q_1) \cos \gamma_1 + (q - q_2) \sin \gamma_1 \right] \]

\[ e = \frac{1}{L_2} \left[ (q - q_3) \cos \gamma_2 + (q - q_4) \sin \gamma_2 \right] \]

\[ \phi = \frac{1}{L_1} \left[ (q - q_1) \sin \gamma_2 + (q - q_2) \cos \gamma_2 \right] \]

\[ \phi = \frac{1}{L_2} \left[ (q - q_3) \sin \gamma_1 + (q - q_4) \cos \gamma_1 \right] \]

\[ \lambda_1 = e + \frac{3}{2} \phi_2 \]

\[ \lambda_2 = e + \frac{3}{2} \phi_2 \]

From Section 3.1, the applied load \( P \) is 984.20 kips. The number of incremental loading, \( \text{NINCR} \), for our problem is 5 increments.
Linear Analysis

For the linear solution, $[K_{NL}] = 0$ and the load $P$ is incremented only once.

The linear stiffness matrix, $[K_L]$ is

$$[k_L] = EA \begin{bmatrix} .0099 & .0010 \\ .0010 & .0001 \end{bmatrix},$$

$$[k_L^2] = EA \begin{bmatrix} 0 & 0 \\ 0 & .1000 \end{bmatrix},$$

$$[K_L] = [k_L] + [k_L^2] = E \begin{bmatrix} .0099 & .0010 \\ .0010 & .1001 \end{bmatrix}.$$

The nodal deflections

$$\begin{bmatrix} q_5 \\ q_6 \end{bmatrix} = [K_L] \begin{bmatrix} P_{5x} \\ P_{5y} \end{bmatrix}$$

$$\begin{bmatrix} q_5 \\ q_6 \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 101.605 & -1.000 \\ -1.000 & 10.000 \end{bmatrix} \begin{bmatrix} 0 \\ -984.20 \end{bmatrix}$$

$$= \begin{bmatrix} .098420 \\ -.984200 \end{bmatrix} \text{ inch}.$$
Nonlinear Analysis

For the incremental load, $\Delta Q$

$$\{\Delta Q\} = \begin{pmatrix} 0 \\ -984.20 / 5 \end{pmatrix} = \begin{pmatrix} 0 \\ -196.84 \end{pmatrix}.$$  

The linear stiffness matrix, $\begin{bmatrix} K_L \end{bmatrix}$

$$\begin{bmatrix} K_L \end{bmatrix} = E \begin{bmatrix} .0099 & .0010 \\ .0010 & 1001 \end{bmatrix}.$$  

Note that the load matrix $\Delta Q$ and the linear stiffness matrix $\begin{bmatrix} K_L \end{bmatrix}$ are constant throughout the whole iterational process.

For the FIRST INCREMENT

$$\begin{bmatrix} K_{NL} \end{bmatrix} = \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} q \end{bmatrix} = 0$$

The total system geometric matrix and its inverse is

$$\begin{bmatrix} K_G \end{bmatrix} = \begin{bmatrix} K_L \end{bmatrix} = \begin{bmatrix} 98.5185 & 9.85185 \\ 9.85185 & 1000.99 \end{bmatrix},$$

$$\begin{bmatrix} K_G \end{bmatrix} = \begin{bmatrix} .01016 & -.000100 \\ -.00010 & .001000 \end{bmatrix}.$$
The total deflection after the first increment is equal to the first incremental deflection

\[
\begin{pmatrix}
\Delta q_5 \\
\Delta q_6
\end{pmatrix} = \begin{pmatrix}
\Delta q_5 \\
\Delta q_6
\end{pmatrix} = \left[ K \right]^{-1} \begin{pmatrix}
\Delta q_5 \\
\Delta q_6
\end{pmatrix} = \begin{pmatrix}
0.019684 \\
-0.196840
\end{pmatrix}.
\]

By substituting the first incremental deflections into the linear and rotational strain equations (ref. page 18), the following results are obtained for the second increment.

\[\varepsilon_1 = 0 \quad \text{and} \quad \varepsilon_2 = -0.019684\]

\[\phi_1 = \phi_2 = -0.0019684\]

\[\alpha_1 = 0.000006\]

\[\alpha_2 = -0.0196782\]

The nonlinear stiffness matrix

\[
\begin{bmatrix}
K_{NL1} \\
K_{NL2}
\end{bmatrix}_2 = \begin{bmatrix}
0.03879 & -19.204 \\
-19.204 & -0.03818
\end{bmatrix},
\]

\[
\begin{bmatrix}
K_{NL1} \\
K_{NL2}
\end{bmatrix}_2 = \begin{bmatrix}
-19.6782 & 1.9684 \\
1.9684 & 0
\end{bmatrix},
\]

\[
\begin{bmatrix}
K_{NL1} \\
K_{NL2}
\end{bmatrix}_2 = \begin{bmatrix}
-19.63941 & 1.77636 \\
1.77636 & -0.03818
\end{bmatrix}.
\]
The system geometric matrix and its inverse

\[
\begin{bmatrix}
K \\
G
\end{bmatrix}_2 = \begin{bmatrix}
K_L \\
K_{NL}
\end{bmatrix}_2
= \begin{bmatrix}
78.8791 & 11.6282 \\
11.6282 & 1000.95
\end{bmatrix}
\]

\[
\begin{bmatrix}
K \\
G
\end{bmatrix}_2^{-1} = \begin{bmatrix}
.0126990 & -.000148 \\
-.000148 & .001001
\end{bmatrix}
\]

The nodal deflection for the second incremental loading is

\[
\begin{bmatrix}
\Delta q_5 \\
\Delta q_6
\end{bmatrix}_2 = \begin{bmatrix}
.02904 \\
-.19699
\end{bmatrix}
\]

The total deflection after the second increment is

\[
\begin{bmatrix}
q_5 \\
q_6
\end{bmatrix}_2 = \begin{bmatrix}
\Delta q_5 \\
\Delta q_6
\end{bmatrix}_2 + \begin{bmatrix}
q_5 \\
q_6
\end{bmatrix}_1
\]

\[
= \begin{bmatrix}
.048724 \\
-.393830
\end{bmatrix}
\]

This process is repeated for all loading incrementation.
For the **THIRD INCREMENT**

\[
e_1 = 9.24846 \times 10^{-4}
\]

\[
e_2 = -3.93831 \times 10^{-1}
\]

\[
\theta_1 = -3.94756 \times 10^{-2}
\]

\[
\theta_2 = -4.87241 \times 10^{-2}
\]

\[
\alpha_1 = -1.15859 \times 10^{-3}
\]

\[
\alpha_2 = -3.93475 \times 10^{-1}
\]

The nonlinear stiffness matrix

\[
\begin{bmatrix}
k_{NL_1} & = & \begin{bmatrix}
-19.5615 & 1.3902 \\
1.3902 & -1.0458
\end{bmatrix}
\\
\end{bmatrix}
\]

\[
\begin{bmatrix}
k_{NL_2} & = & \begin{bmatrix}
-39.3475 & 4.8724 \\
4.8724 & 0
\end{bmatrix}
\\
\end{bmatrix}
\]

\[
\begin{bmatrix}
k_{NL} & = & \begin{bmatrix}
-58.9090 & 6.2626 \\
6.2626 & -1.0458
\end{bmatrix}
\\
\end{bmatrix}
\]
The system geometric matrix and its inverse

\[
\begin{bmatrix}
K \\
G
\end{bmatrix}_3 = \begin{bmatrix} K \\
L \\
NL \end{bmatrix}_3
\]

\[
\begin{bmatrix}
K \\
G
\end{bmatrix}_3 = \begin{bmatrix}
39.6095 & 16.1145 \\
16.1145 & 1000.88
\end{bmatrix}
\]

\[
\begin{bmatrix}
K \\
G
\end{bmatrix}_3 \begin{bmatrix}
.0254130 \\
-.000409
\end{bmatrix} = \begin{bmatrix}
.001006
\end{bmatrix}
\]

The nodal deflection for the third incremental loading is

\[
\begin{bmatrix}
\Delta q_5 \\
\Delta q_6
\end{bmatrix} = \begin{bmatrix}
.0805381 \\
-.1979630
\end{bmatrix}
\]

The total deflection after the third increment is

\[
\begin{bmatrix}
q_5 \\
q_6
\end{bmatrix} = \begin{bmatrix}
.12926 \\
-.59179
\end{bmatrix}
\]
For the \textbf{FOURTH INCREMENT}

\begin{align*}
e_1 &= 0.633888 \times 10^{-3} \\
e_2 &= -0.591795 \times 10^{-1} \\
\varphi_1 &= -0.598734 \times 10^{-2} \\
\varphi_2 &= -0.129262 \times 10^{-1} \\
\lambda_1 &= 0.747660 \times 10^{-3} \\
\lambda_2 &= -0.589288 \times 10^{-1}
\end{align*}

The nonlinear stiffness matrix is

\[
\begin{bmatrix}
k_{NL,1} \\
k_{NL,2}
\end{bmatrix}_4 = \begin{bmatrix}
-58.7903 & 5.671270 \\
5.67127 & -1.48894
\end{bmatrix}
\]

\[
\begin{bmatrix}
k_{NL,1} \\
k_{NL,2}
\end{bmatrix}_4 = \begin{bmatrix}
-58.9290 & 12.92623 \\
12.92623 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
k_{NL}
\end{bmatrix}_4 = \begin{bmatrix}
-117.719 & 18.59750 \\
18.5975 & -1.4889
\end{bmatrix}
\]
The system geometric matrix and its inverse is

\[
\begin{bmatrix}
K \\ G
\end{bmatrix}_4 = \begin{bmatrix}
K \\ L
\end{bmatrix}_4 + \begin{bmatrix}
K \\ NL
\end{bmatrix}_4
\]

\[
= \begin{bmatrix}
-19.2006 & 28.4493 \\ 28.4493 & 1000.84
\end{bmatrix}
\]

\[
\begin{bmatrix}
K \\ G
\end{bmatrix}_4^{-1} = \begin{bmatrix}
-0.0499768 & 0.00142062 \\ 0.1422062 & 0.00095878
\end{bmatrix}
\]

The nodal deflection for the fourth incremental loading is

\[
\begin{align*}
\Delta q_5 \\ \Delta q_6
\end{align*} = \begin{bmatrix}
-0.279635 \\ -0.188727
\end{bmatrix}
\]

The total deflection after the fourth increment is

\[
\begin{align*}
q_5 \\ q_6
\end{align*} = \begin{bmatrix}
-0.15037 \\ -0.78052
\end{bmatrix}
\]
For the **FIFTH INCREMENT**

\[
\begin{align*}
e_1 &= -2.26163 \times 10^{-2} \\
e_2 &= -7.80521 \times 10^{-1} \\
\Theta_1 &= -7.57905 \times 10^{-2} \\
\Theta_2 &= 1.50372 \times 10^{-1} \\
\kappa_1 &= -2.17547 \times 10^{-2} \\
\kappa_2 &= -7.80521 \times 10^{-1}
\end{align*}
\]

The nonlinear stiffness matrix is

\[
\begin{bmatrix}
k_{NL_1}^5 \end{bmatrix} = \begin{bmatrix}
-117.5720 & 17.8797 \\
17.8797 & -0.51255
\end{bmatrix}
\]

\[
\begin{bmatrix}
k_{NL_2}^5 \end{bmatrix} = \begin{bmatrix}
-77.71300 & -15.03724 \\
-15.03724 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
K_{NL}^5 \end{bmatrix} = \begin{bmatrix}
-195.285 & 2.84246 \\
2.84246 & -0.512553
\end{bmatrix}
\]
The system geometric matrix and its inverse is

\[
[K_{G}] = [K_{L}] + [K_{NL}]
\]

\[
= \begin{bmatrix}
-96.7664 & 12.6943 \\
12.6943 & 1000.47
\end{bmatrix}
\]

\[
[K_{G}]^{-1} = \begin{bmatrix}
-10317 \times 10^{-1} & 130905 \times 10^{-3} \\
130905 \times 10^{-3} & 997867 \times 10^{-3}
\end{bmatrix}
\]

The nodal deflection for the \textit{\textsuperscript{5\textsuperscript{th}}} incremental loading is

\[
\begin{bmatrix}
\Delta q_5 \\
\Delta q_6
\end{bmatrix} = \begin{bmatrix}
0.0257674 \\
-1.96420
\end{bmatrix}
\]

The total deflection after the \textit{\textsuperscript{5\textsuperscript{th}}} increment is

\[
\begin{bmatrix}
q_5 \\
q_6
\end{bmatrix} = \begin{bmatrix}
-1.7614 \\
-0.97694
\end{bmatrix} \text{ inch}
\]
4.0 THE COMPUTER PROGRAM

4.1 Flow Chart For The Mechanized Incremental Stiffness Solution Procedure

Input Data

Generate Global Linear Stiffness Matrix

Generate Global Nonlinear Stiffness Matrix

Boundary Conditions

Assemble Reduced Linear Stiffness Matrix, $[K_L]$ (as $[K_G] = [K_L] + [K_{NL}]$)

Assemble Reduced Nonlinear Stiffness Matrix, $[K_{NL}]$
Compute Incremental Deflections
\[ \{\Delta q\}_n = [K_G]^{-1}\{\Delta Q\} \]

Compute Total Deflections After Each Incremental Loading, \( n \)
\[ \{q\}_n = \{q\}_{n-1} + \{\Delta q\}_n \]

Compute Sum Of All Incremental Loads,
\[ \sum_{i=1}^{n} \{\Delta q\}_i \]

If, \[ \sum_{i=1}^{n} \{\Delta q\}_i = \text{Applied Load } P \]

Print Input Data and Nodal Deflections

END
4.2 Input Instructions

The input data required to obtain the deflections of a truss structure using the program (Appendix C) developed for this analysis are described below along with the input data format.

One or more truss element can be analyzed. The present capacity of the program is for a maximum of 10 elements, 11 nodes and as many materials as there are elements in the system.

Input Data Cards

<table>
<thead>
<tr>
<th>Card No.</th>
<th>Columns</th>
<th>Format</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td></td>
<td></td>
<td>Parameter Card</td>
</tr>
<tr>
<td>1 - 5</td>
<td>I5</td>
<td></td>
<td>Total no. of nodes.</td>
</tr>
<tr>
<td>6 - 10</td>
<td>I5</td>
<td></td>
<td>Total no. of elements.</td>
</tr>
<tr>
<td>11 - 15</td>
<td>I5</td>
<td></td>
<td>Total no. of loads.</td>
</tr>
<tr>
<td>16 - 20</td>
<td>I5</td>
<td></td>
<td>No. of increments to divide loads.</td>
</tr>
<tr>
<td>(2)</td>
<td></td>
<td></td>
<td>Node specification card( The no. of cards equal to the no. of nodes and must be inputed in node order 1,2,3 ....,etc.).</td>
</tr>
<tr>
<td>1 - 5</td>
<td>I5</td>
<td></td>
<td>Number.</td>
</tr>
<tr>
<td>6 - 10</td>
<td>I5</td>
<td></td>
<td>Degree of freedom in x and y direction respectively. 0 = free and 1 = restrained.</td>
</tr>
<tr>
<td>11 - 15</td>
<td>I5</td>
<td></td>
<td>Coordinate in x axis.</td>
</tr>
<tr>
<td>16 - 25</td>
<td>F10:5</td>
<td></td>
<td>Coordinate in y axis.</td>
</tr>
</tbody>
</table>
Input data Cards (cont'd)

<table>
<thead>
<tr>
<th>Card No.</th>
<th>Columns</th>
<th>Format</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3)</td>
<td></td>
<td></td>
<td>Element Specification Card. (No. of cards equal to the number of elements and must be inputted in element order 1, 2, 3, etc.).</td>
</tr>
<tr>
<td></td>
<td>1 - 5</td>
<td>I5</td>
<td>Element number.</td>
</tr>
<tr>
<td></td>
<td>6 - 10</td>
<td>I5</td>
<td>The i\textsuperscript{th} node number.</td>
</tr>
<tr>
<td></td>
<td>11 - 15</td>
<td>I5</td>
<td>The j\textsuperscript{th} node number.</td>
</tr>
<tr>
<td></td>
<td>16 - 25</td>
<td>F10.5</td>
<td>Area of element.</td>
</tr>
<tr>
<td></td>
<td>26 - 40</td>
<td>F15.3</td>
<td>Young's modulus of elasticity.</td>
</tr>
<tr>
<td>(4)</td>
<td></td>
<td></td>
<td>Load Data Cards (Requires 2 cards for each load application at a node. The first card is for the node number and the second card is the magnitude of the load in the x and y direction. Sign of the load is positive in the +x and +y direction and negative on the opposite directions.).</td>
</tr>
<tr>
<td>(4a)</td>
<td>1 - 5</td>
<td>I5</td>
<td>Node number.</td>
</tr>
<tr>
<td>(4b)</td>
<td>1 - 15</td>
<td>F15.3</td>
<td>Node load in x-direction.</td>
</tr>
<tr>
<td></td>
<td>16 - 30</td>
<td>F15.3</td>
<td>Node load in y-direction.</td>
</tr>
</tbody>
</table>
4.3 Example Problems

Two nonlinear truss problems are analyzed utilizing the mechanized analysis for the solution procedure as presented in Section 4.1.

The first example problem is the same problem solved by longhand method in Section 3.2. Choice of this problem is to countercheck the results obtained by the longhand method. Further description of this problem is presented in Section 3.0.

The second example problem is a 5 member planar truss system of uniform material for each member but of different cross-sectional areas and length. This problem is solved for a single and for 20 loading incrementation to obtain a linear and nonlinear analysis result.
Example Problem No. 1

Determine the load-deflection response of the simple truss shown below for an applied load equal to its critical load at buckling, \( P = 984.20 \) kips. Use 5 equal incremental loading.

Notes:

1.) Numbers enclosed by \( \circ \) denotes Node number.
2.) Numbers enclosed by \( \Box \) denotes element number.
Computer Output For Problem No. 1

NUMBER OF NODES = 3

NUMBER OF ELEMENTS = 2

NUMBER OF NODES WITH LOAD = 1

NUMBER OF INCREMENTAL LOADING = 5

<table>
<thead>
<tr>
<th>NODE NUMBER</th>
<th>DEGREE OF FREEDOM</th>
<th>LOCATION OF COORDINATES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>U</td>
<td>V</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ELEMENT NUMBER</th>
<th>NODE POINT NUMBERS</th>
<th>AREA</th>
<th>YOUNGS MOD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 3</td>
<td>1.00000</td>
<td>0.10000E+05</td>
</tr>
<tr>
<td>2</td>
<td>2 3</td>
<td>1.00000</td>
<td>0.10000E+05</td>
</tr>
<tr>
<td>LOAD NODE</td>
<td>INCR NUMBER</td>
<td>DEFLECTIONS AT THE NODES</td>
<td></td>
</tr>
<tr>
<td>-----------</td>
<td>-------------</td>
<td>--------------------------</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.00000E+00 0.00000E+00</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.00000E+00 0.00000E+00</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.19684E-01 -0.19684E+00</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>LOAD NODE</th>
<th>INCR NUMBER</th>
<th>DEFLECTIONS AT THE NODES</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0.00000E+00 0.00000E+00</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.00000E+00 0.00000E+00</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.48724E-01 -0.39383E+00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>LOAD NODE</th>
<th>INCR NUMBER</th>
<th>DEFLECTIONS AT THE NODES</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>0.00000E+00 0.00000E+00</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.00000E+00 0.00000E+00</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.12626E+00 -0.59179E+00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>LOAD NODE</th>
<th>INCR NUMBER</th>
<th>DEFLECTIONS AT THE NODES</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>0.00000E+00 0.00000E+00</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.00000E+00 0.00000E+00</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>-0.15037E+00 -0.78032E+00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>LOAD NODE</th>
<th>INCR NUMBER</th>
<th>DEFLECTIONS AT THE NODES</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>0.00000E+00 0.00000E+00</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0.00000E+00 0.00000E+00</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>-0.17614E+00 -0.97694E+00</td>
</tr>
</tbody>
</table>
Example Problem No. 2

Determine the load-deflection response of the planar truss shown below, first by linear analysis and finally by nonlinear analysis using 20 incremental loading.
### Computer Output For Problem No. 2

**NODE NUMBER DEGREE OF FREEDOM LOCATION OF COORDINATES**

<table>
<thead>
<tr>
<th>Node</th>
<th>Degree of Freedom</th>
<th>Location of Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>U</td>
<td>X: 0.0000, Y: 0.0000</td>
</tr>
<tr>
<td>2</td>
<td>V</td>
<td>X: 0.0000, Y: 240.0000</td>
</tr>
<tr>
<td>3</td>
<td>U</td>
<td>X: 180.0000, Y: 240.0000</td>
</tr>
<tr>
<td>4</td>
<td>V</td>
<td>X: 180.0000, Y: 0.0000</td>
</tr>
</tbody>
</table>

**ELEMENT NUMBER NODE POINT NUMBERS AREA YOUNG'S MOD**

<table>
<thead>
<tr>
<th>Element</th>
<th>Node Point Numbers</th>
<th>Area</th>
<th>Young's Mod</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 2</td>
<td>1.3333</td>
<td>0.1000E+08</td>
</tr>
<tr>
<td>2</td>
<td>2 3</td>
<td>1.0000</td>
<td>0.1000E+08</td>
</tr>
<tr>
<td>3</td>
<td>4 2</td>
<td>1.6667</td>
<td>0.1000E+08</td>
</tr>
<tr>
<td>4</td>
<td>1 3</td>
<td>1.6667</td>
<td>0.1000E+08</td>
</tr>
<tr>
<td>5</td>
<td>4 3</td>
<td>1.3333</td>
<td>0.1000E+08</td>
</tr>
</tbody>
</table>
Problem No. 2, Part a) LINEAR ANALYSIS

NUMBER OF NODES = 4
NUMBER OF ELEMENTS = 5
NUMBER OF NODES WITH LOAD = 1
NUMBER OF INCREMENTAL LOADING = 1

\( P_{3x} = 30 \text{ kips} \)
\( P_{3y} = 20 \text{ kips} \)

<table>
<thead>
<tr>
<th>LOAD</th>
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<th>NUMBER</th>
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Problem No. 2, Part b) NONLINEAR ANALYSIS

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5.0 CONCLUSIONS AND RECOMMENDATIONS

The incremental stiffness procedure embodying the finite element displacement method is presented for nonlinear analysis of planar trusses.

The method of analysis is convenient and easy to use in that it only requires the specification of a load step size. Refining the load step size or stepping the number of load incrementation increases the degree of accuracy of the solution. It is evident, as shown in Figure 2, that the linear method of structural analysis does not adequately describe the behavior of a structure under loads. The deformation of a structure alters the directions of the initial internal forces. The nonlinear curves in Figure 2 reflect the effect of geometry changes as the truss deflects under load. Some structures are more nonlinear than others as illustrated by the example problems presented in Section 4.3. The first example problem is more nonlinear than the second problem. It is therefore necessary, to include nonlinear effects to obtain a more accurate result.

The solution procedure presented for a truss structure can be easily extended for beam analysis. The nonlinear stiffness matrix derived for a beam element (Appendix B)
can be easily included to extend the computer program presented in Appendix C. Further refinement or improvement of the computer program can also be accomplished, to include in the output, the corresponding load and deflection result for each incremental solution. The incremental and total strains and stresses along with element internal loads should also be included.
6.0 BIBLIOGRAPHY


APPENDIX A

DERIVATION OF THE NONLINEAR GEOMETRIC MATRIX FOR A TRUSS ELEMENT

When a uniform truss is loaded in simple tension, the forces do a certain amount of work as the member stretches. If the truss element shown in Figure A-1 is subjected to a normal stress $\sigma_x$ only, there exists a force $F_x = \sigma_x dy dz$ which does work on an extension $\varepsilon_x dx$. The relation between the force $F_x$ and the extension $\varepsilon_x dx$ during loading is shown as line OA in Figure A-1(c). The work done during deformation in the triangle OAB is

$$W = \frac{1}{2} F_x \varepsilon_x dx \quad (A1)$$

Let $W = \text{du}$, then

$$\text{du} = \frac{1}{2} \varepsilon_x \sigma_x dx dy dz \quad (A2)$$

Integrating Eqn. (A2) throughout the length of the element

$$U = \frac{1}{2} \int \int \int \sigma_x \varepsilon_x dx dy dz \quad (A3)$$

where $\int \int dy dz = \text{area of section}$, therefore the internal strain energy for the truss element is
The nonlinear relation between strain, $\varepsilon_x$, and the displacement components $u$ and $v$ in the direction of the truss element axis is

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right]. \tag{A5}$$

Since strain is small compared to unity, $\varepsilon_x$ can be taken as the true physical strain and since $(\partial u/\partial x)^2$ is small compared to $\partial u/\partial x$, therefore $(\partial u/\partial x)^2$ can be omitted. The term $(\partial v/\partial x)^2$ is the contribution of rotation to $\varepsilon_x$, therefore this term is retained.

Equation (A5) can be written as:
\[ \varepsilon_x = \frac{\partial u}{\partial x} + 1/2 \left( \frac{\partial v}{\partial x} \right)^2 \]  

(A6)

where the second term is nonlinear. Equation (A6) determines the strain in the direction of the element axis (Ref. 4).

Consider a pin jointed constant stress truss element with a uniform cross sectional area \( A \), modulus of elasticity \( E \), and length \( L \) as shown in Figure A-2. Under the action of an applied load, the truss is displaced from its original location \( AB \) to \( A'B' \). The displacement in the global \( x \) and \( y \) directions at ends \( A \) and \( B \) respectively are \( q_1, q_2 \) and \( q_3, q_4 \). The local displacements at ends \( A \) and \( B \) in the local coordinate system is \( x_1 \) and \( y_1 \) respectively. For the displacement model, let \( u(x) \) and \( v(x) \) be linear functions at a node (linear two term polynomial):

\[ u(x) = a_0 + a_1 x \]
\[ v(x) = b_0 + b_1 x \]  

(A7)

The choice of the displacement model, Equation (A7), provides the necessary constant strain along the length of the member and also at the same time provides as many constants as there are nodal degree of freedom. The first condition in Equation (A7) is the linear expression for strain and the second condition is necessary to
express the strain energy in terms of nodal displacement. Writing u and v displacements at each node of element in Figure A-2, gives

\[ a_0 = u_1 \quad \text{and} \quad b_0 = v_1, \]
\[ a_1 = \frac{\partial u}{\partial x} = (u_2 - u_1)/L, \] \hfill (A8)
\[ b_1 = \frac{\partial v}{\partial x} = (v_2 - v_1)/L. \]

By substituting Equation (A8) into Equation (A7), the following expressions are obtained:

\[ u(x) = u_1 + \frac{\partial u}{\partial x} / L, \]
\[ v(x) = v_1 + \frac{\partial v}{\partial x} / L. \] \hfill (A9)

The displacements u and v is therefore varying linearly along the length L. Equation (A9) can then be expressed as

\[ u = (1 - \frac{x}{L}) q_1 + q_3 \left( \frac{x}{L} \right), \]
\[ v = (1 - \frac{x}{L}) q_2 + q_4 \left( \frac{x}{L} \right). \] \hfill (A10)

In matrix form:

\[
\begin{bmatrix}
\{ u \\
\{ v \\
\end{bmatrix}
= \begin{bmatrix}
(1-x/L) & 0 & x/L & 0 \n0 & (1-x/L) & 0 & x/L \\
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4 \\
\end{bmatrix}, \hfill (A11)
\]

The axial energy stored in a truss with linear stress-strain law (Hookean elasticity) is

\[ U = \frac{1}{2} \int_0^L E \varepsilon_x^2 \, dx = \frac{AE}{2} \int_0^L \varepsilon_x^2 \, dx. \] \hfill (A12)
Substituting Equation (A6) into Equation (A12) yields,

\[ U = \frac{AE}{2} \int_{0}^{L} \left[ \frac{3}{3} + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2 \right] dx \]

\[ = \frac{AE}{2} \int_{0}^{L} \left[ \frac{v^3}{3} + \frac{3}{2} \frac{u}{x} \left( \frac{v}{x} \right)^2 + \frac{1}{4} \left( \frac{3v}{x} \right)^2 \right] dx \]  

(A13)

The first term inside the bracket in the right hand side of Equation (A13) represent the strain energy, \( U_L \), obtained by linear theory.

\[ U_L = \frac{AE}{2} \int_{0}^{L} \left( \frac{v^3}{3} \right) dx \]  

(A14)

The last two terms inside the bracket in the right hand side of Equation (A13) represent the strain energy, \( U_{NL} \), due to the nonlinear terms in the strain-displacement relations

\[ U_{NL} = \frac{AE}{2} \int_{0}^{L} \left[ \frac{3}{3} + \frac{1}{4} \left( \frac{3v}{x} \right)^2 \right] dx \]  

(A15)

The Linear Stiffness Matrix

The linear stiffness matrix is now obtained by substituting Equation (A8) into Equation (A14), integrating and then evaluating the second partial derivative of the resulting equation with respect to \( q_1 \) and \( q_j \).

\[ U_L = \frac{AE}{2} \int_{0}^{L} \left[ \frac{1}{L} \left( q_3 - q_1 \right) \right] \]  

(A16)
Integration of Equation (A16) yields

\[ U_L = \frac{AE}{2L} (q_3^2 - 2q_1 q_3 + q_1^2) \]

The partial derivatives of \( U_L \) with respect to \( q_1 \) and \( q_j \) are:

\[ \frac{\partial U_L}{\partial q_1} = \frac{AE}{L} (q_1 - q_3) \]

\[ \frac{\partial U_L}{\partial q_j} = \frac{AE}{L} (q_3 - q_1) \]

Therefore:

\[
\{ Q \} = \left\{ \frac{\partial U_L}{\partial q} \right\} = \frac{AE}{L} \begin{bmatrix} q_1 & -q_3 \\ -q_1 & q_3 \end{bmatrix}, \quad (A17)
\]

and

\[
\frac{\partial^2 U_L}{\partial q_i \partial q_j} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [K_L] \]

Equation (A19) is the linear stiffness matrix of the system in the local coordinates.
Nonlinear Stiffness Matrix

The nonlinear stiffness matrix is similarly obtained by determining $U_{NL}$ as a function of the displacements (in terms of strains), substituting assumed displacement functions and evaluating the second derivatives. From Equation (A15), let

$$\frac{\partial u}{\partial x} = e \quad \text{and} \quad \frac{\partial v}{\partial x} = \phi.$$

then, Equation (A15) can be written as

$$U_{NL} = \int_0^L \left[ e \frac{\partial^2}{4} + \frac{1}{4} \phi^4 \right] dx;$$

or

$$U_{NL} = \frac{AEL}{2} \left( e \frac{\partial^2}{4} + \frac{1}{4} \phi^4 \right). \quad \text{(A20)}$$

The nonlinear stiffness matrix is therefore

$$\frac{\partial^2 U_{NL}}{\partial q_i \partial q_j} = \frac{AEL}{2} \left[ 2e \frac{\partial \phi}{\partial q_i} \frac{\partial \phi}{\partial q_j} + 2e \frac{\partial \phi}{\partial q_i} \frac{\partial e}{\partial q_j} + 2e \frac{\partial e}{\partial q_i} \frac{\partial \phi}{\partial q_j} + 3\phi^2 \frac{\partial \phi}{\partial q_i} \frac{\partial \phi}{\partial q_j} \right]. \quad \text{(A21)}$$

Substituting partial derivatives into Equation (A21), the nonlinear stiffness matrix in the local coordinate system is obtained as follows:
\[
\begin{bmatrix}
\frac{K_{NL}}{L} = \frac{AE}{L} \\
\end{bmatrix}
\]

where:

\[\lambda = e + \frac{3}{2} \phi^2;\]

\[e = \frac{1}{L} (u_2 - u_1);\]

\[\phi = \frac{1}{L} (v_2 - v_1).\]
Nonlinear Geometric Matrix

The local to global transformation matrix is

\[
[T_m] = \begin{bmatrix}
\cos \gamma & \sin \gamma & 0 & 0 \\
-sin \gamma & \cos \gamma & 0 & 0 \\
0 & 0 & \cos \gamma & \sin \gamma \\
0 & 0 & -\sin \gamma & \cos \gamma \\
\end{bmatrix} \tag{A23}
\]

\(X, Y = \text{Global System}\)
\(x, y = \text{Undeformed Element System}\)

Figure A-2: Truss Element

From Figure A-2, the displacement at the nodes are

\[
\begin{align*}
    u_1 &= -q_1 \cos \gamma + q_2 \sin \gamma \\
    v_1 &= -q_1 \sin \gamma + q_2 \cos \gamma \\
    u_2 &= q_3 \cos \gamma + q_4 \sin \gamma \\
    v_2 &= -q_3 \sin \gamma + q_4 \cos \gamma
\end{align*} \tag{A24}
\]
and
\[ e = \frac{1}{L} (u - u') \]
\[ = \frac{1}{L} (q \cos \gamma + q \sin \gamma) - (-q \cos \gamma + q \sin \gamma) \]
\[ = \frac{1}{L} (q + q) \cos \gamma + (q - q) \sin \gamma \]

\[ \Theta = \frac{1}{L} (v - v') \]
\[ = \frac{1}{L} (-q \sin \gamma + q \cos \gamma) - (-q \sin \gamma + q \cos \gamma) \]
\[ = \frac{1}{L} (q - q) \cos \gamma + (q - q) \sin \gamma \]
\[ \lambda = e + \frac{3}{2} \Theta^2. \]

The global linear and nonlinear stiffness matrices are therefore

\[ [K_L] = [T_m]^T [K] [T_m] \]

\[ [K_L] = \frac{AE}{L} \]
\[ \begin{bmatrix}
  c^2 & c^2 & s^2 & (\text{symm}) \\
  cs^2 & s^2 & c^2 \\
  -s^2 & cs & c^2 \\
  -cs & -s^2 & cs & s^2
\end{bmatrix} \]
and
\[
\begin{bmatrix}
K_{NL}
\end{bmatrix} = \begin{bmatrix}
T_m
\end{bmatrix}^T \begin{bmatrix}
K_{NL}
\end{bmatrix} \begin{bmatrix}
T_m
\end{bmatrix}
\]  \hspace{1cm} (A27),

\[
= \frac{EA}{L} \begin{bmatrix}
-2sc\theta + s^2\alpha \\
c^2\theta - s^2\theta - sc\alpha \\
2sc\theta + c^2\alpha \\
2sc\theta - s^2\alpha & s^2\theta - c^2\theta + sc\alpha & -2sc\theta + s^2\alpha \\
-c^2\theta + s^2\theta + sc\alpha & -2sc\theta - c^2\alpha & c^2\theta - s^2\theta - sc\alpha & 2sc\theta + c^2\alpha
\end{bmatrix}
\]

where the subscript 1 denotes local coordinate system
and \( s = \sin \gamma \) and \( c = \cos \gamma \).

Combining Equation(A26) and Eqn.(A27), the global nonlinear geometric matrix of the system is obtained

\[
\begin{bmatrix}
K
\end{bmatrix} = \begin{bmatrix}
K
\end{bmatrix} + \begin{bmatrix}
K_{NL}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
K_L
\end{bmatrix} + \begin{bmatrix}
K_{I (q)}
\end{bmatrix}
\]  \hspace{1cm} (A28)
Truss Element Nonlinear Stiffness Matrix For Stability Analysis

In determining the load at which the truss structure will become unstable, the nonlinear stiffness matrix \( [K_{NL}] \) must be derived to allow the inclusion of the nonlinear term \( \partial v/\partial x \) in Equation (A6) for the case of incrementing the applied load once only.

From Equation (A13)

\[
U = \frac{AE}{2} \int_0^L \left[ \left( \frac{\partial v}{\partial x} \right)^2 + \frac{\partial u}{\partial x} \left( \frac{\partial v}{\partial x} \right)^2 + \frac{1}{4} \left( \frac{\partial v}{\partial x} \right)^4 \right] \, dx
\]

Substituting Equation (A17) into the above expression and neglecting the higher order \( (\partial v/\partial x)^4 \), since this will only make \( [K_{NL}] \) as a function of displacement, and integrating yields

\[
U = \frac{AE}{2L} (u - 2u + u^2 + u^2) + \frac{AE}{2L^2} (u - u)(v - 2v + v^2) \tag{A29}
\]

Since

\[
AE(u - u) / L = AE \varepsilon = \text{The Applied Load, } P ; \tag{A30}
\]

therefore,

\[
U = \frac{AE}{2L} (u^2 - 2u_1u_2 + u_2^2) + P(v^2 - 2v_1v_2 + v_2^2) \tag{A30}
\]
From Castigliano's first theorem, the stiffness coefficient is

\[ K_{ij} = \frac{\partial^2 U}{\partial u_i \partial u_j} \]

Therefore, from Equation (A30)

\[ \frac{\partial U}{\partial u_1} = \frac{AE}{L} (u_1 - u_2) \]
\[ \frac{\partial U}{\partial u_2} = \frac{P}{L} (v_1 - v_2) \] \hspace{1cm} (A31)
\[ \frac{\partial U}{\partial u_3} = \frac{AE}{L} (-u_1 + u_2) \]
\[ \frac{\partial U}{\partial u_4} = \frac{P}{L} (-v_1 + v_2) \]

Equation (A31) can be written in matrix form as

\[
\begin{bmatrix}
K^{*}
\end{bmatrix}
= \frac{AE}{L} \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
+ \frac{P}{L} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}
\]
\[ \begin{bmatrix}
K^{*}
\end{bmatrix}
= \begin{bmatrix}
K \\
L
\end{bmatrix}
+ \begin{bmatrix}
K_{NL}
\end{bmatrix}. \] \hspace{1cm} (A32)

\[ \begin{bmatrix}
K^{*}
\end{bmatrix}
= \begin{bmatrix}
K \\
L
\end{bmatrix}
+ \begin{bmatrix}
K_{NL}
\end{bmatrix}. \] \hspace{1cm} (A33)
Note that Equation(A32) is no longer a function of displacement unlike Equation(A27).

Transforming Equation(A32) into the global coordinate system by use of the transformation equation, Eqn.(A23), yields

\[
\begin{bmatrix}
    c^2 & cs^2 & s^2 \\
    cs^2 & s^2 & \text{(symm)} \\
    -s^2 & -cs & c^2 \\
    cs & -s^2 & cs \\
    -s^2 & sc & -cs \\
    sc & -c^2 & sc \\
\end{bmatrix}
\]

\[
\left( \begin{array}{c}
    \frac{EA}{L} \\
    \frac{P}{L} \end{array} \right) 
\]

\[
\begin{bmatrix}
    s^2 & \text{(symm)} \\
    -sc & c^2 \\
    -s^2 & sc \\
    sc & -c^2 \\
\end{bmatrix}
\]

From Equation(A32), the classical stability or eigenvalue problem for buckling is formulated by considering a single increment without external loading (Ref. 2).

Assigning a scaling factor \( \lambda \) and evaluating \( [K_{NL}] \) for a unit value of \( \lambda \), Equation(A33) can be written as

\[
\left( \begin{bmatrix} K_L \end{bmatrix} + \left[ K_{NL}^* \right] \lambda \right) \{ \Delta q \} = 0 .
\]  

(A35)

Equation(A35) has nontrivial solutions only if the determinant

\[
\left| \begin{bmatrix} K_L \end{bmatrix} + \lambda \left[ K_{NL}^* \right] \right| = 0 .
\]

(A36)
APPENDIX B

DERIVATION OF NONLINEAR GEOMETRIC MATRIX FOR A BEAM ELEMENT

Consider the beam element shown in Figure B-1

The nodal displacements $q_1$ and $q_4$ are axial displacements, $q_2$ and $q_5$ are transverse displacements, and $q_3$ and $q_6$ are rotations. From Ref. 6, the strain energy $U$, for a beam is

$$ U = \frac{1}{2} \int_0^L E \varepsilon^2 \, dV \quad \text{(B1)} $$

and the total strain $\varepsilon$, for moderate rotation is taken as
\[ \varepsilon = e + 1/2 \theta^2 - xy \]  \hspace{1cm} (B2)

where
\begin{align*}
e &= \frac{du}{dx} \\
\theta &= \frac{dv}{dx} \hspace{1cm} (B3) \\
x &= \frac{d^2v}{dx^2}
\end{align*}

and \( u \) and \( v \) are the mid-axial and transverse displacements respectively within the beam, \( e \) is the linear strain component, \( \theta \) is the rotation of the element and \( x \) is the linear curvature. It is assumed that moderate rotations occur without excessive deformations (curvature changes) so that only the linear component of the curvature, \( x \), is considered.

Substituting Equations (B2) and (B3) into Equation (B1) and integrating over the cross-sectional area yields
\[ U = \frac{AE}{2} \int_0^L e \, dx + \frac{EI}{2} \int_0^L x \, dx + \frac{AE}{2} \int_0^L \left( e\theta^2 + \frac{1}{4} \theta^4 \right) \, dx \]  \hspace{1cm} (B4)

where \( A \) is the cross-sectional area and \( I \) is the moment of inertia about the \( z \)-axis. The strain energy Equation (B4), can be rewritten to separate the strain energy into a contribution due to linear terms, \( U_L \), and due to nonlinear terms in the strain displacement relations, \( U_{NL} \).
\[ U = U_L + U_{NL} \]  \hspace{1cm} (B5)
where $U_L$ represents the first two terms on the right
hand side of Equation (B4) and the last term represents
$U_{NL}$. Therefore

$$
U_L = \frac{AE}{2} \int_0^L e^{2} \, dx + \frac{EI}{2} \int_0^L x^{2} \, dx,
$$

and

$$
U_{NL} = \frac{AE}{2} \int_0^L \left(e^{\theta^{2}} + \frac{1}{4} \theta^{4}\right) \theta \, dx.
$$

From Equation (B6), the linear stiffness matrix, $[K_L]$, 
is obtained in the local coordinate system as

$$
[K_L] = \frac{EI}{L^3} 
\begin{bmatrix}
AL^2/I & -AL^2/I \\
0 & 0 & 0 & AL^2/I \\
0 & -12 & -6L & 0 & 12 \\
0 & 6L & 2L^2 & 0 & -6L & 4L^2
\end{bmatrix}
$$

where the subscript $l$ denotes local coordinate system.
The nonlinear stiffness matrix, $K_{NL}$, is obtained by determining $U_{NL}$, Equation (B7), as a function of displacements (in terms of strains), substituting assumed displacement functions and evaluating the second derivatives in similar manner as for the truss $K_{NL}$ in Appendix A.

$$
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
= \frac{EA}{L} \begin{bmatrix}
\phi & \alpha \\
-\phi & 0 \\
0 & -\alpha & \alpha \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

(B9)

where $e$, $\sigma$ and $\lambda$ are the same notations as presented in Appendix A derivation for the truss element $[K_{NL}]$.

The local to global transformation matrix for the linear and nonlinear stiffness matrix is

$$
\begin{bmatrix}
c & s & 0 & 0 & 0 & 0 & 0 \\
-s & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & s & 0 & 0 \\
0 & 0 & 0 & -s & c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
$$

(B10)
where \( c = \cos \theta \) and \( s = \sin \theta \).

The local to global transformation for the linear and nonlinear stiffness matrix is given by

\[
\begin{bmatrix} K \end{bmatrix}_{\text{global}} = \begin{bmatrix} T \end{bmatrix}^T \begin{bmatrix} K \end{bmatrix}_{\text{local}} \begin{bmatrix} T \end{bmatrix}
\]

(B11)

and the system geometric matrix is similarly obtained as previously shown in Appendix A as

\[
\begin{bmatrix} K^G \end{bmatrix} = \begin{bmatrix} K^L \end{bmatrix} + \begin{bmatrix} K^NL \end{bmatrix}
\]

(B12)

\[
\begin{bmatrix} K^L \end{bmatrix} + \begin{bmatrix} K^I(q) \end{bmatrix}
\]
APPENDIX C

COMPUTER PROGRAM LISTING

This computer program determines the load-deflection characteristics of a truss structure utilizing the incremental stiffness solution procedure presented in Section 2.1. Refer to Section 4.2 for input instructions.
DIMENSION SL(20,20),SNL(20,20),SKG(20,20),SL(21,21)
DIMENSION E(20),IDF(20),X(20),Y(20),B(20,4),LENGTH(10)
DIMENSION DISP(20),U(20),V(20),SNP(21,20),SKI(21,20)
DIMENSION NN1(20),NNJ(20),AREA(10),SGK(20,20),SNL(20,20)

C READ TOTAL NUMBER OF NODES, TOTAL NUMBER OF ELEMENTS,
C TOTAL NUMBER OF NODES WITH LOADS AND NUMBER OF INCREMENTS TO DIVIDE LOAD
C READ NODE NUMBER, DEGREE OF FREEDOM IN THE U, V DIRECTION. 13 FOR FREE,
C 1 FOR FIXED, AND THE X, Y LOCATION FOR THE PARTICULAR NODE. (NOTE, THE
C NODES MUST BE READ IN NODE ORDER 1, 2, 3, . . . . .)
C READ ELEMENT NUMBER, ITH NODE NUMBER, JTH NODE NUMBER,
C ELEMENT CROSS SECTION AREA AND YOUNGS MODULUS
C READ LENGTH OF ELEMENTS.
C COMPUTE NUMBER OF DEGREE OF FREEDOM.
WORKING ON THE THIRD ORDER SHEAR (0,0)

C ZERO OUT NODAL DISPLACEMENTS
DO 64 I=1,NDF
   DO 64 DISP(I)=0.
64 DISP(I)=0.
C ZERO OUT LOADS
DO 56 I=1,NDF
   P(I)=0.
56 P(I)=0.
C READ NODE NUMBER AND NODE LOADS IN X,Y DIRECTIONS
DO 74 I=1,NLOAD
   READ(60,640) NN
640 FORMAT(15)
   N1=NN+2-2
74 READ(60,650) P(N1+1),P(N1+2)
650 FORMAT(2F15.3)
C ASSEMBLE FECTED P COLUMN MATRIX
DO 78 I=1,NDF
   IF(DF(I).NE.0) GO TO 78
   N0=0.N+1
78 CONTINUE
WRITE(61,600) NNODE,NELEM,NLOAD,NINCR
600 FORMAT(* NUMBER OF NODES=*,IS15,\,,) *
   * NUMBER OF ELEMENTS=*,IS15,\,,) *
   3* NUMBER OF NODES WITH LOAD=*,IS15,\,,) *
WRITE(61,601)
601 FORMAT(1)
   1* NODE DEGREE OF FREEDOM LOCATION OF COORDINATES*,IS15,\,,)
   2* NUMBER X U
DO 76 I=1,NNODE
   N2=I+2-2
76 WRITE(61,602) I,DF(N2+1),DF(N2+2),X(I),Y(I)
602 FORMAT(15,21R,2(8X,F10.5))
WRITE(61,603)
603 FORMAT(*,,)
   1* ELEMENT NUMBER NODE POINT NUMBERS AREA YOUNGS MOD*,IS15,\,,)
INTEGER SIZE = 1

C ZERO OUT LINEAR AND NONLINEAR STIFFNESS MATRIX

C CALCULATE LINEAR GLOBAL STIFFNESS OF THE SYSTEM

C ASSEMBLE REDUCED LINEAR STIFFNESS MATRIX OF THE SYSTEM
DO 130 J=1,NEF
IF(IDF(J) .NE. 0) GO TO 130
IF(IDF(I) .NE. 0) GO TO 130
IA=1
SLR(ISR,JSR)=SL(I,J)
JSR=JSR+1
CONTINUE
130 CONTINUE

I COUNT=I
DO 130 L=1,NELEM
NI=NI(L)
NJ=NJ(L)
COS=(X(NJ)-X(NI))/LENGTH(L)
SIN=(Y(NJ)-Y(NI))/LENGTH(L)
NL1=NI*2-1
NL2=NJ*2-1
U(NI)=DISP(NL1)
U(NJ)=DISP(NL2)
V(NI)=DISP(NL1+1)
V(NJ)=DISP(NL2+1)
ESTRAIN=E(U(NI)+U(NJ))/LENGTH(L)*COS+(V(NJ)-V(NI))/LENGTH(L)*SIN
KSTRAIN=E(U(NI)-U(NJ))/LENGTH(L)*SIN+(V(NJ)-V(NI))/LENGTH(L)*COS
ALPHA=ESTRAIN+(3.0/2.0)*(RSTRAIN**2)
SNP(NL1,NL1)=(-2)*SIN*COS*ESTRAIN+2*ALPHA
SNP(NL1,NL1+1)=(-SIN**2)*RSTRAIN+(COS**2)*RSTRAIN-SIN*COS*ALPHA
SNP(NL1+1,NL1+1)=(-SNP(NL1,NL1))
SNP(NL1+1,NL1+2)=(-SNP(NL1+1,NL1))
SNP(NL1+2,NL1+1)=(-SNP(NL1+1,NL1))
SNP(NL1+1,NL1+1)=(-SNP(NL1+1,NL1))
SNP(NL1+2,NL1+1)=(-SNP(NL1+1,NL1))
SNP(NL2,NL1)=-SNP(NL1,NL1+1)
SNP(NL2,NL1+1)=(-SNP(NL1+1,NL1))
SNP(NL2,NL2+1)=SNP(NL1,NL1+1)
SNP(NL2,NL2)=SNP(NL1,NL1)
INTEGER WORD SIZE = 1, * OPTION IS OFF, 0 OPTION

C ASSEMBLE TOTAL SYSTEM GEOMETRIC MATRIX

DO 160 I = 1, 2
DO 160 J = 1, 2
N1 = N1 + 2 - 2 + J
N2 = N1 + 2 - 2 + J
N3 = N1 + 2 - 2 + J
SN (N1, N2) = SNL (N1, N2) + SNP (N1, N2) * E (L) * AREA (L) / TLENGTH (L)
SN (N3, N4) = SNL (N3, N4) + SNP (N3, N4) * E (L) * AREA (L) / TLENGTH (L)
SN (N1, N4) = SNL (N1, N4) + SNP (N1, N4) * E (L) * AREA (L) / TLENGTH (L)
SN (N3, N2) = SNL (N3, N2) + SNP (N3, N2) * E (L) * AREA (L) / TLENGTH (L)
160 CONTINUE

C ASSEMBLE REDUCED SYSTEM GEOMETRIC MATRIX

IA = C
ISF = 1
CO 136 I = 1, NDF
JSF = 1
ISF = ISR + IA
IA = C
DO 136 J = 1, NDF
IF (IDF (J) .NE. 0) GO TO 136
IF (IDF (I) .NE. 0) GO TO 136
IA = C
JSR (ISR, JSR) = SNL (I, J)
JSF = JSR + 1
136 CONTINUE

C ASSEMBLE TOTAL SYSTEM NONLINEAR STIFFNESS MATRIX

DO 170 I = 1, NR
DO 170 J = 1, NR
SKGR (I, J) = SLR (I, J) + SNLR (I, J)
170 CONTINUE
C SIZE OF REDUCED MATRIX = SKGR(NR, NR)
C CALCULATE THE INVERSE OF THE REDUCED MATRIX, SKGR(NR, NR)
CALL INVERT(NR, SKGR, SKI)
C COMPUTE NODAL INCREMENTAL DISPLACEMENTS AT EACH INCR. LOADS
DO 175 J=1, NDF
DSP(I)=DSP(I)+SKI(I, J)*EF(J)/NINCR
175 CONTINUE
C CALCULATE TOTAL DISPLACEMENTS
IDX=1
DO 176 I=1, NDF
IF(IDX(I) > NDF), GO TO 177
DSP(I)=DSP(IDX)
IDX=IDX+1
GO TO 176
177 OUTPUT_DSP(1)=0.
176 CONTINUE
DO 815 I=1, NDF
DSP(I)=DSP(I)+EDSP(I)
OUTPUT_DSP(I)=DSP(IDX)
815 CONTINUE
655 FORMAT(4*H 'DEFLECTIONS AT THE NODES (U, V)' )
2* INCR NUMBER
I=1
DO 715 J=1, NDF
IMPUT_DSP(I, J)=DSP(I)+DISP(I), DISP(I+1)
715 CONTINUE
662 FORMAT(15, 4X, 2(4X, 516.5))
I=I+2
715 CONTINUE
C COMPUTE ELEMENT STRAINS AND STRESSES
IF(ICONT.EQ. NINCR), GO TO 680
DO 79 I=1, NDF
DO 79 J=1, NDF
SNF(I, J)=
79 CONTINUE
680 ICONT=INCR
END
SUBROUTINE INVERT(N, XK, XKI)

DIMENSION XK(20,20), XKI(20,20)

DO 10 I=1,N
D(I)=0.
DO 10 J=1,N
U(I,J)=0.
10 U(I,J)=0.
DO 60 I=1,N
D(I)=D(I)+XK(I,I)
U(I,I)=1.
MK=I-1
K=1
40 IF(MK.LT.K) GO TO 30
D(I)=D(I)-U(K,I)*U(K,I)*D(K)
K=K+1
GO TO 40
30 JN=I+1
DO 50 J=JN,N
60 IF(MK.LT.K) GO TO 50
U(I,J)=U(I,J)+XK(I,J)/D(I)
K=K+1
GO TO 60
50 U(I,J)=U(I,J)+XK(I,J)/D(I)
D(I)=D(I)-U(K,I)*U(K,I)*D(K)
JN=J+1
DO 70 J=JN,N
MK=J-1
K=1
90 IF(MK.LT.K) GO TO 70
U(I,J)=U(I,J)-U(I,K)*U(K,J)
K=K+1
GO TO 90
70 CONTINUE
DO 100 I=1,N
DO 100 J=1,N
100 U(I,J)=U(I,J)/D(J)
CC 112 I=1,N
DO 100 J=1,N
XKI(I,J)=J.
SC 112 J=1,N
112 XKII(I,J)=XKI(I,J)+UID(I,K)*UI(J,K)
RETURN
END