GENERALIZED DARBOUX TRANSFORMATION AND N'TH ORDER ROGUE WAVE SOLUTION OF A GENERAL COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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August 2016
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Acknowledgements

I would like to thank my thesis advisor Dr. Yomba for all the support and help throughout my thesis preparation. I also wish to thank my committee members Dr. Zakeri and Dr. Panferov.
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Nonlinear partial differential equations (NLPDEs) are extremely hard to solve analytically because no specific general algorithm exists for this task. As is well known, integrability of the NLPDEs plays an important role in soliton theory (when the solutions exist, they help to understand the phenomena modeled by these NLPDEs), which can be regarded as a pretest and the first step of its exact solvability. Among the properties that can characterize the integrability of NLPDEs are: the bilinear representation, Bäcklund transformation (BT), Lax pair, infinitely many conservation laws, infinite symmetries, Hamiltonian structure, Painlevé test and so on. A NLPDE is said to be completely integrable in the sense of Lax if it can be written as a system of linear PDEs in an auxiliary function under the condition that the original NLPDE satisfies a compatibility condition. In this thesis, Lax-pair and generalized Darboux transformation (GDT) method are constructed for the generalized coupled nonlinear Schrödinger equations (GCNLSEs). Using the GDT method, the recursive formula for the Nth-order rogue wave solution for the GCNLSEs is derived and the Nth-order determinant representation for these equations is given. From the recursive formula, the first-, second-, and third-order rogue wave solutions with certain free parameters are obtained. Based on the Darboux transformation, breather solutions for the GCNLSE are derived as well. Moreover, the dynamical features of these solutions are graphically discussed and the modulation instability of GNLSes is investigated.
Chapter 1

Introduction

Nonlinear PDEs are extremely hard to solve because it seems that no particular general algorithm exists for it. However, any completely integrable (in the sense of Lax) nonlinear PDE can be written as a system of linear PDEs in an auxiliary function; on the condition that the original nonlinear PDE satisfies the compatibility condition. This system of compatible linear PDEs is called a Lax pair. In 1968 Peter Lax introduced the concept of linearizing PDEs, which is known as Lax pair. Lax pair is a pair of time dependent matrices that satisfy a given differential equation. Lax pair converts a high order, completely integrable, nonlinear PDE into a system of linear equations [1]. Each PDE can be linearized in different ways, depending on the choices of Lax pair. Hence, if one can find an arbitrary Lax pair for a given PDE, then that PDE will be integrable by means of some transformation [2]. Such transformations are: inverse scattering transformation, Reimann-Hilbert scattering method, Darboux transformation (DT) and so on [3]. DT is a well known and very powerful method for solving nonlinear PDEs. DT finds solutions for nonlinear integrable systems based on their corresponding Lax pairs [3]. This method was introduced by Gaston Darboux in 1882 to solve the Sturm-Liouville differential equation, which is also called linear Schrödinger equation. However, this method was used not only to solve the Sturm-Liouville equation but also many other linear and nonlinear differential equations [3].

The purpose of this thesis is to derive the breather and rogue wave (RW) solutions for general coupled nonlinear Schrödinger equations (GCNLSEs). Also, we will construct $N$th order RW solutions for this system. In order to derive breather and RW solutions we will use DT. However, in order to obtain the $N$th order RW solutions we will consider generalized Darboux transformation (GDT). First, we will show in detail how to construct DT for GCNLSEs. Using this transformation we will obtain breather and RW solutions for GCNLS system and we will show that we obtain those solutions only when our complex parameter becomes a real constant. After obtaining those solutions we will also investigate the modulation instability (MI) of the GCNLS system and show the connection to RW solutions. Furthermore, using GDT we will show in detail the process for obtaining $N$th order RW solutions. We will obtain both recursive formula and determinant expressions for $N$th order RW solutions. GDT contains $N$ distinct eigenvalues and we will show how we can restrict to a single critical eigenvalue by applying an appropriate limit process. Applying this limit process we will obtain explicit form of first, second and third order RW solutions [18, 19].

The plan of this thesis is as follows. In chapter 1, we will show the detailed work for obtaining Lax pair for different nonlinear partial differential equations. In chapter 2, we will derive the explicit form of first, second and third order RW solutions and will show detail discussion for both recursive formula and determinant expressions of $N$th order RW solution using GDT. In chapter 3, we will derive general breather and RW solutions of GCNLS system using DT and investigate modulation instability of this system. Finally, in
chapter 4 we will draw our conclusion.

1.1 Nonlinear Schrödinger Equation

The NLS equation has been studied for a long time and many results have been obtained for it. For instance, the first-order rational solution [13, 14], the control of the rogue wave [15], the soliton solutions [16], and the Darboux transformation of the NLS equation [17]. The NLS equation is the most common mathematical description of RWs [19].

We consider the following form of nonlinear Schrödinger equation:

\[ i\Psi_t + \frac{1}{2}\Psi_{xx} + |\Psi|^2\Psi = 0, \quad (1.1) \]

where,

- \(|\Psi|^2 = \Psi\Psi^*\).
- \(\Psi(x, t)\) is the slowly varying pulse envelope.
- \(x\) and \(t\) are the spatial and temporal coordinates.
- The subscripts denote partial differentiation with respect to that variable.

This equation is integrable in the sense of Peter Lax if there is a vector function \(F(x, t)\), such that

\[
\begin{align*}
F_x &= UF, \\
F_t &= VF, \\
\end{align*}
\]

(1.2)

where,

- \(U(x, t)\) and \(V(x, t)\) are 2 by 2 matrices.

Also, \(F(x, t)\) is such that the compatibility condition \(F_{xt} = F_{tx}\) gives equation (1.1). Then,

\[
\begin{align*}
F_{xt} &= (UF)_t = U_t F + UF_t, \\
F_{tx} &= (VF)_x = V_x F + VF_x. \\
\end{align*}
\]

(1.3)

Substituting equation (1.2) into (1.3) we have the following:

\[
\begin{align*}
F_{xt} &= U_t F + UF_t = U_t F + UV F = (U_t + UV)F, \\
F_{tx} &= V_x F + VF_x = V_x F + VUF = (V_x + VU)F. \\
\end{align*}
\]

(1.4)

Then by compatibility condition we will have:

\[(U_t + UV)F = (V_x + VU)F.\]

(1.5)
So this implies that
\[(U_t - V_x + [U, V])F = 0_{2 \times 2}, \tag{1.6}\]
here \([U, V] = UV - VU\) is the operation commutator. Since the above equation should hold for all \(F\) thus we can eliminate \(F\) and we will obtain the matrix Lax equation,

\[U_t - V_x + [U, V] = 0_{2 \times 2}. \tag{1.7}\]

A pair of solutions \(U\) and \(V\) for equation (1.7) gives assurance of existence of vector function \(F\). Now using equation (1.7) we can test if there are any matrices \(U\) and \(V\) that are valid Lax pair. The most common choice for matrix \(U\) is in this form:

\[U = \begin{pmatrix} i\lambda & q \\ r & -i\lambda \end{pmatrix} = U_0 + \lambda U_1, \tag{1.8}\]

here, \(r\) and \(q\) are functions of \(x\) and \(t\), and \(\lambda\) is fixed parameter. Also,

\[U_0 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \tag{1.9}\]
\[U_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{1.10}\]

We seek \(V\) as a quadratic polynomial function of \(\lambda\):

\[V = V_0 + \lambda V_1 + \lambda^2 V_2. \tag{1.11}\]

Since \(U\) function is known we can find \(V_0, V_1\) and \(V_2\) using equation (1.7). Substituting Eqs. (1.8) and (1.11) into Eq. (1.7) we get

\[(U_0 + \lambda U_1)_t + (V_0 + \lambda V_1 + \lambda^2 V_2)_x + (U_0 + \lambda U_1)(V_0 + \lambda V_1 + \lambda^2 V_2) - (V_0 + \lambda V_1 + \lambda^2 V_2)(U_0 + \lambda U_1) = 0_{2 \times 2}. \tag{1.12}\]

Next, separate into several equations corresponding to powers of \(\lambda\):

\[\lambda^0 : U_{0t} - V_{0x} + U_0V_0 - V_0U_0 = 0_{2 \times 2}, \tag{1.13}\]
\[\lambda^1 : U_{1t} - V_{1x} + U_0V_1 + U_1V_0 - V_0U_1 - V_1U_0 = 0_{2 \times 2}, \tag{1.14}\]
\[\lambda^2 : -V_{2x} + U_0V_2 + U_1V_1 - V_1U_1 - V_2U_0 = 0_{2 \times 2}, \tag{1.15}\]
\[\lambda^3 : U_1V_2 - V_2U_1 = 0_{2 \times 2}. \tag{1.16}\]

Now we will consider equations (1.13)-(1.16) one at a time and find our desired matrices. We will start from equation (1.16). Let matrix \(V_2\) be as following:

\[V_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}. \tag{1.17}\]
First, we substitute Eq. (1.10) and Eq. (1.17) into (1.16):

\[
\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} - \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]  \hspace{1cm} (1.18)

\[
\Rightarrow \begin{pmatrix} i a_2 & i b_2 \\ -i c_2 & -i d_2 \end{pmatrix} - \begin{pmatrix} i a_2 & -i b_2 \\ i c_2 & -i d_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]  \hspace{1cm} (1.19)

\[
\Rightarrow \begin{pmatrix} 0 & 2i b_2 \\ -2i c_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]  \hspace{1cm} (1.20)

Eq. (1.20) implies that \( b_2 = 0 \) and \( c_2 = 0 \). However, \( a_2 \) and \( d_2 \) can be any arbitrary constant not equal to zero. Let \( a_2 = i \) and \( d_2 = -i \). Hence, \( V_2 \) matrix becomes as following:

\[
V_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]  \hspace{1cm} (1.21)

Now we will consider equation (1.15), that is: \(-V_2 + U_0 V_2 + U_1 V_1 - V_1 U_1 - V_2 U_0 = 0_{2 \times 2} \). First, we need to substitute matrices \( U_0, U_1 V_2 \) and \( V_1 \). Let matrix \( V_1 \) be as following:

\[
V_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.
\]  \hspace{1cm} (1.22)

Substitute equations (1.9), (1.10), (1.21) and (1.22) into equation (1.15), we get

\[
\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} - \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]  \hspace{1cm} (1.23)

\[
\Rightarrow \begin{pmatrix} 0 & -iq \\ ir & 0 \end{pmatrix} + \begin{pmatrix} ia_1 & ib_1 \\ -ic_1 & -id_1 \end{pmatrix} - \begin{pmatrix} ia_1 & -ib_1 \\ ic_1 & -id_1 \end{pmatrix} - \begin{pmatrix} 0 & iq \\ -ir & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]  \hspace{1cm} (1.24)

\[
\Rightarrow \begin{pmatrix} 0 & -2iq + 2ib_1 \\ 2it - 2ic_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]  \hspace{1cm} (1.25)

Hence, from Eq. (1.15) we have that \(-2iq + 2ib_1 = 0 \) and \( 2it - 2ic_1 = 0 \), which implies that \( b_1 = q \) and \( c_1 = r \). From the above equation we can also conclude that \( a_1 \) and \( d_1 \) can be any arbitrary constants, choose \( a_1 = 0 \) and \( d_1 = 0 \). Thus, \( V_1 \) matrix becomes as following:

\[
V_1 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}.
\]  \hspace{1cm} (1.26)

Next we can consider Eq. (1.14): \( U_{1t} - V_{1x} + U_0 V_1 + U_1 V_0 - V_0 U_1 - V_1 U_0 = 0_{2 \times 2} \). Let the matrix \( V_0 \) be in the following form:

\[
V_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}.
\]  \hspace{1cm} (1.27)
Also we need to calculate \( U_{1t} \) and \( V_{1x} \). So

\[
U_1 = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \Rightarrow U_{1t} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

and

\[
V_1 = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \Rightarrow V_{1x} = \begin{pmatrix} 0 & q_x \\ r_x & 0 \end{pmatrix}.
\]

Substituting matrices into equation (1.14) we get the following

\[
- \begin{pmatrix} 0 & q_x \\ r_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \left( \begin{pmatrix} i a_0 \\ -i c_0 \end{pmatrix} \begin{pmatrix} b_0 \\ -i d_0 \end{pmatrix} \right) - \begin{pmatrix} i a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
\Rightarrow - \begin{pmatrix} 0 & q_x + 2ib_0 \\ -r_x - 2ic_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

From equation (1.30) we obtain that \( 2ib_0 = q_x \) and \( -2ic_0 = r_x \) OR \( b_0 = \frac{-iq_x}{2} \) and \( c_0 = \frac{ir_x}{2} \).

Also, consider \( a_0 \neq 0 \) and \( d_0 \neq 0 \). Hence, \( V_0 \) matrix becomes as following:

\[
V_0 = \begin{pmatrix} a_0 & -\frac{-iq_x}{2} \\ \frac{ir_x}{2} & d_0 \end{pmatrix},
\]

and

\[
V_{0x} = \begin{pmatrix} a_0 & \frac{-iq_x}{2} \\ \frac{ir_x}{2} & d_0 \end{pmatrix}.
\]

Now we will consider our last equation, Eq. (1.13): \( U_{0t} - V_{0x} + U_0V_0 - V_0U_0 = 0_{2x2} \). Substitute \( U_0, V_0, U_{0x} \) and \( V_{0x} \) into Eq. (1.13) and solve for remaining parameters.

\[
\begin{pmatrix} 0 & q_t \\ r_t & 0 \end{pmatrix} - \begin{pmatrix} a_{0x} & -\frac{-iq_{xx}}{2} \\ \frac{ir_{xx}}{2} & d_{0x} \end{pmatrix} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \begin{pmatrix} a_0 & -\frac{-iq_x}{2} \\ \frac{ir_x}{2} & d_0 \end{pmatrix} - \begin{pmatrix} a_0 & -\frac{-iq_x}{2} \\ \frac{ir_x}{2} & d_0 \end{pmatrix} \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
\Rightarrow \begin{pmatrix} 0 & q_t \\ r_t & 0 \end{pmatrix} - \begin{pmatrix} a_{0x} & -\frac{-iq_{xx}}{2} \\ \frac{ir_{xx}}{2} & d_{0x} \end{pmatrix} + \begin{pmatrix} \frac{iqr_x}{r_a} & qd_0 \\ -\frac{-iq_{xx}}{2} & rd_0 \end{pmatrix} - \begin{pmatrix} \frac{-iq_{xx}}{2} & qa_0 \\ \frac{ir_{xx}}{2} & \frac{iqr}{r} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
\Rightarrow \begin{pmatrix} -a_{0x} + \frac{iqr_x}{r_a} + \frac{iq_{xx}}{2} \\ r_t - \frac{ir_{xx}}{2} + ra_0 - rd_0 \end{pmatrix} - a_{0x} + \frac{iqr_x}{r_a} + \frac{iq_{xx}}{2} + qd_0 - qa_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
Then from above Eq. (1.35) we have $-a_{0x} + \frac{iqr}{2} + \frac{iq_r}{2} = 0$ and $-d_{0x} - \frac{iq_r}{2} + \frac{iqr}{2} = 0$, which implies that $a_0 = \frac{iqr}{2}$ and $d_0 = -\frac{iqr}{2}$. Since we found all terms of matrix $V_0$, then the following will be the form of matrix $V_0$:

$$V_0 = \begin{pmatrix} \frac{iqr}{2} & -\frac{iqr}{2} \\ \frac{iq_r}{2} & -\frac{iq_r}{2} \end{pmatrix}.$$ 

Moreover, we have that

$$q_t + \frac{iq_{xx}}{2} + qd_0 - qa_0 = 0,$$

$$r_t - \frac{ir_{xx}}{2} + ra_0 - rd_0 = 0.$$ 

Since we already obtained the exact value for $a_0$ and $d_0$, thus we can substitute them in the above equation. Then we will have the following equation:

$$q_t + \frac{iq_{xx}}{2} - iq^2r = 0,$$  

$$r_t - \frac{ir_{xx}}{2} + iqr^2 = 0.$$  

(1.36)  

(1.37)

Let $r = i\Psi$ and $q = i\Psi^*$, then substitute in Eqs. (1.36) and (1.37):

$$(i\Psi^*)_t + \frac{i(i\Psi^*)_xx}{2} - i(i\Psi^*)^2(i\Psi) = 0,$$  

$$(i\Psi)_t - \frac{i(i\Psi)_{xx}}{2} + i(i\Psi^*)(i\Psi)^2 = 0.$$  

(1.38)  

(1.39)

After simplifying this, we get

$$-i\Psi_t^* + \frac{1}{2}\Psi_{xx}^* + |\Psi|^2\Psi^* = 0,$$  

$$i\Psi_t + \frac{1}{2}\Psi_{xx} + |\Psi|^2\Psi = 0.$$  

(1.40)  

(1.41)

Equation (1.41) represents a nonlinear Shrödinger equation and Eq. (1.40) is the complex conjugate of the first one. Therefore we have found the Lax pair representation for nonlinear Schrödinger equations.

1.2 Hirota Equation

Hirota equation is important in optics to illustrate the transmission when pulse lengths become comparable to the wavelength, while in this case the simple Manakov model is inadequate, and the high-order nonlinear effects must be considered. Some important results have been obtained for this equation such as the Lax pair, the classical Darboux transformation, the Painlevé analysis, the bright- and dark- soliton solutions [5].
Consider the following form of Hirota equation:

\[ i\Psi_t + \frac{1}{2}\Psi_{xx} + |\Psi|^2 \Psi - i\alpha (\Psi_{xxx} + 6|\Psi|^2\Psi) = 0, \]  

(1.42)

where,

- \( |\Psi|^2 = \Psi\Psi^*. \)
- \( \Psi(x, t) \) is the slowly varying pulse envelope.
- \( x \) and \( t \) are the spatial and temporal coordinates.
- The subscripts denote a partial differentiation with respect to that variable.

Using the same methodology as in the previous section, we will have exactly the same equations as Eqs. (1.2)-(1.6). Consider the same form for \( U \) as in the previous section,

\[ U = \begin{pmatrix} i\lambda & i\Psi^* \\ i\Psi & -i\lambda \end{pmatrix} = U_0 + \lambda U_1. \]

(1.43)

Hence, \( U_0 \) and \( U_1 \) matrices are as following:

\[ U_0 = \begin{pmatrix} 0 & i\Psi^* \\ i\Psi & 0 \end{pmatrix}, \]

(1.44)

\[ U_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \]

(1.45)

We seek \( V \) as a cubic polynomial function of \( \lambda \):

\[ V = V_0 + \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3. \]

(1.46)

Substitute Eq. (1.43) and Eq. (1.46) into (1.7), we have:

\[ (U_0 + \lambda U_1)_t - (V_0 + \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3)_x + (U_0 + \lambda U_1)(V_0 + \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3) - (V_0 + \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3)(U_0 + \lambda U_1) = 0_{2\times2}. \]

(1.47)

Simplify and split into several equations corresponding to power of \( \lambda \):

\[ \lambda^0 : U_{0t} - V_{0x} + U_0 V_0 - V_0 U_0 = 0_{2\times2}, \]

(1.48)

\[ \lambda^1 : U_{1t} - V_{1x} + U_0 V_1 + U_1 V_0 - V_0 U_1 - V_1 U_0 = 0_{2\times2}, \]

(1.49)

\[ \lambda^2 : -V_{2x} + U_0 V_2 + U_1 V_1 - V_1 U_1 - V_2 U_0 = 0_{2\times2}, \]

(1.50)

\[ \lambda^3 : -V_{3x} + U_0 V_3 + U_1 V_2 - V_2 U_1 - V_3 U_0 = 0_{2\times2}, \]

(1.51)

\[ \lambda^4 : U_1 V_3 - V_3 U_1 = 0_{2\times2}. \]

(1.52)
Again, we will consider each equation separately and solve for $V_0$, $V_1$, $V_2$ and $V_3$.

Consider Eq. (1.52) and let matrix $V_3$ be as following:

$$V_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}. \tag{1.53}$$

Substitute Eq. (1.45) and Eq. (1.53) into Eq. (1.52):

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} - \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.54}$$

$$\Rightarrow \begin{pmatrix} ia_3 & ib_3 \\ -ic_3 & -id_3 \end{pmatrix} - \begin{pmatrix} ia_3 & -ib_3 \\ ic_3 & -id_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.55}$$

$$\Rightarrow \begin{pmatrix} 0 & 2ib_3 \\ -2ic_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.56}$$

Hence, from above matrix, we can conclude that $b_3 = 0$ and $c_3 = 0$. Also, $a_3$ and $d_3$ are arbitrary parameters. So matrix $V_3$ becomes as following:

$$V_3 = \begin{pmatrix} a_3 & 0 \\ 0 & d_3 \end{pmatrix}. \tag{1.57}$$

Now we will consider Eq. (1.51), that is: $-V_{3x} + U_0 V_3 + U_1 V_2 - V_2 U_1 - V_3 U_0 = 0_{2 \times 2}$. After we substitute the required matrices into Eq. (1.51) and simplify it, we obtain the following matrix:

$$\begin{pmatrix} -a_{3x} & -\Psi d_3 - i\Psi^* a_3 + 2ib_2 \\ i\Psi a_3 - i\Psi d_3 - 2ic_2 & -d_{3x} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.58}$$

Then from the above matrix we have $a_{3x} = 0$ and $d_{3x} = 0$. This implies that $a_3 = f(t)$ and $d_3 = g(t)$. Moreover, from the above matrix we have that $b_2 = \frac{\Psi^*}{2}(a_3 - d_3)$ and $c_2 = \frac{\Psi^*}{2}(a_3 - d_3)$, or $b_2 = \frac{\Psi^*}{2}(f(t) - g(t))$ and $c_2 = \frac{\Psi^*}{2}(f(t) - g(t))$. Thus, matrices $V_3$ and $V_2$ become as following:

$$V_3 = \begin{pmatrix} f(t) & 0 \\ 0 & g(t) \end{pmatrix}; \quad V_2 = \begin{pmatrix} \frac{\Psi}{2}(f(t) - g(t)) & a_2 \\ \frac{\Psi}{2}(f(t) - g(t)) & d_2 \end{pmatrix}. \tag{1.58}$$

Next we will consider Eq. (1.50), that is: $-V_{2x} + U_0 V_2 + U_1 V_1 - V_1 U_1 - V_2 U_0 = 0_{2 \times 2}$. As before, after substituting the required matrices and simplifying it we have:

$$\begin{pmatrix} -a_{2x} & 2ib_1 - ia_2 \Psi + id_2 \Psi - \frac{1}{2}(f(t) - g(t))\Psi^* \\ -2ic_1 + ia_2 \Psi - id_2 \Psi + \frac{1}{2}(f(t) - g(t))\Psi^* & -d_{2x} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.59}.$$
From the above matrix, we have that \( a_{2x} = 0 \) and \( d_{2x} = 0 \) this implies that \( a_{2x} = l(t) \) and \( d_{2x} = m(t) \). Moreover, we have

\[
b_1 = \frac{\Psi^*}{2}(a_2 - d_2) - \frac{i\Psi^*}{4}(f(t) - g(t)) \quad \text{and} \quad c_1 = \frac{\Psi}{2}(a_2 - d_2) - \frac{i\Psi}{4}(f(t) - g(t)),
\]

or

\[
b_1 = \frac{\Psi^*}{2}(l(t) - m(t)) - \frac{i\Psi^*}{4}(f(t) - g(t)) \quad \text{and} \quad c_1 = \frac{\Psi}{2}(l(t) - m(t)) - \frac{i\Psi}{4}(f(t) - g(t)).
\]

Hence, \( V_2 \) and \( V_1 \) matrices will have the following form

\[
V_2 = \begin{pmatrix}
\frac{\Psi}{2}(f(t) - g(t)) & \frac{\Psi^*}{2}(f(t) - g(t)) \\
\frac{\Psi}{2}(l(t) - m(t)) & m(t)
\end{pmatrix},
\]

\[
V_1 = \begin{pmatrix}
\frac{\Psi}{2}(l(t) - m(t)) & \frac{\Psi^*}{2}(f(t) - g(t)) \\
\frac{\Psi}{2}(l(t) - m(t)) & \frac{\Psi^*}{2}(l(t) - m(t)) - \frac{i\Psi^*}{4}(f(t) - g(t)) \\
\frac{\Psi}{2}(f(t) - g(t)) & \frac{\Psi}{2}(l(t) - m(t)) - \frac{i\Psi}{4}(f(t) - g(t)) \\
\end{pmatrix}.
\]

Consider Eq. (1.49): \( U_1 - V_1 + U_0 V_1 + U_1 V_0 - V_0 U_1 - V_1 U_0 = 0_{2 \times 2} \). Next we need to substitute required matrices and simplify. However, the matrix that we get is very long that is why we will present the entries of the matrix as four functions:

\[
- a_{1x} - \frac{1}{4}(\Psi^* \Psi_x + \Psi \Psi_x^*)(f(t) - g(t)) = 0, \tag{1.60}
\]

\[
2ib_0 - i\Psi^*(a_1 - d_1) - \frac{\Psi^*}{2}(l(t) - m(t)) - \frac{i\Psi^*}{4}(f(t) - g(t)) = 0, \tag{1.61}
\]

\[
- 2ic_0 - i\Psi(a_1 - d_1) - \frac{\Psi}{2}(l(t) - m(t)) - \frac{i\Psi}{4}(f(t) - g(t)) = 0, \tag{1.62}
\]

\[
- d_{1x} + \frac{1}{4}(\Psi^* \Psi_x + \Psi \Psi_x^*)(f(t) - g(t)) = 0. \tag{1.63}
\]

Then from the above we have:

\[
a_{1x} = - \frac{1}{4}(\Psi^* \Psi_x + \Psi \Psi_x^*)(f(t) - g(t)) \quad \Rightarrow \quad a_1 = - \frac{1}{4}\Psi^* \Psi(f(t) - g(t)),
\]

\[
d_{1x} = \frac{1}{4}(\Psi^* \Psi_x + \Psi \Psi_x^*)(f(t) - g(t)) \quad \Rightarrow \quad d_1 = \frac{1}{4}\Psi^* \Psi(f(t) - g(t)).
\]

Note that the constants of integrations are take to be zero. Hence, the matrix \( V_1 \) will be in the following form:

\[
V_1 = \begin{pmatrix}
\frac{\Psi}{2}(l(t) - m(t)) & \frac{\Psi^*}{2}(f(t) - g(t)) \\
\frac{\Psi}{2}(l(t) - m(t)) & \frac{\Psi}{2}(l(t) - m(t)) - \frac{i\Psi}{4}(f(t) - g(t)) \\
\frac{\Psi}{2}(f(t) - g(t)) & \frac{\Psi}{2}(l(t) - m(t)) - \frac{i\Psi}{4}(f(t) - g(t))
\end{pmatrix}.
\]
Moreover, we have that:

\[ 2ib_0 - i\Psi^*(a_1 - d_1) - \frac{\Psi^*}{2}(l(t) - m(t)) - \frac{i\Psi^*_{xx}}{4}(f(t) - g(t)) = 0, \quad (1.65) \]

and

\[ -2ic_0 - i\Psi(a_1 - d_1) - \frac{\Psi}{2}(l(t) - m(t)) - \frac{i\Psi_{xx}}{4}(f(t) - g(t)) = 0. \quad (1.66) \]

We can solve Eq. (1.65) for \( b_0 \) and Eq. (1.66) for \( c_0 \).

\[ b_0 = \frac{\Psi^*}{2}(a_1 - d_1) - \frac{i\Psi^*}{2}(l(t) - m(t)) + \frac{\Psi^*_{xx}}{8}(f(t) - g(t)), \]

\[ c_0 = -\frac{\Psi}{2}(a_1 - d_1) + \frac{i\Psi}{2}(l(t) - m(t)) - \frac{\Psi_{xx}}{8}(f(t) - g(t)). \quad (1.67) \]

Substitute \( a_1 \) and \( d_1 \) and simplify, we have

\[ b_0 = -\frac{1}{8}(f(t) - g(t))(2|\Psi|^2\Psi^* + \Psi^*_{xx}) - \frac{i\Psi^*_{xx}}{4}(l(t) - m(t)), \quad (1.68) \]

\[ c_0 = \frac{1}{8}(f(t) - g(t))(2|\Psi|^2\Psi + \Psi_{xx}) - \frac{i\Psi_{xx}}{4}(l(t) - m(t)). \quad (1.69) \]

This implies that \( V_0 \) matrix will be in this form:

\[ \begin{pmatrix} 
\frac{1}{8}(f(t) - g(t))(2|\Psi|^2\Psi + \Psi_{xx}) & a_0 \\
-\frac{i\Psi_{xx}}{4}(l(t) - m(t)) & d_0 
\end{pmatrix} \]

\[ \begin{pmatrix} 
\frac{1}{8}(f(t) - g(t))(2|\Psi|^2\Psi^* + \Psi^*_{xx}) & -\frac{i\Psi^*_{xx}}{4}(l(t) - m(t)) \\
\frac{i\Psi_{xx}}{4}(l(t) - m(t)) & d_0 
\end{pmatrix} \].

\[ \quad (1.70) \]

Lastly, we will consider Eq. (1.48): \( U_{0u} - V_{0x} + U_0V_0 - V_0U_0 = 0_{2 \times 2} \). After substituting \( U_0, V_0, U_{0x} \) and \( V_{0x} \) into this equation and simplifying it we get the following:

\[ a_0 = \frac{1}{8}((-2l(t) + 2m(t))(|\Psi|^2 - i(f(t) - g(t))(\Psi^*\Psi_x - \Psi^*_x)), \quad (1.71) \]

\[ d_0 = \frac{1}{8}((2l(t) - 2m(t))(|\Psi|^2 + i(f(t) - g(t))(\Psi^*\Psi_x - \Psi^*_x)). \quad (1.72) \]

Moreover, we obtain the following two equations:
\[ i\Psi^*(d_0 - a_0) + i\Psi_x + \frac{1}{8}((f(t) - g(t))(2(\Psi^*)^2\Psi_x + 4|\Psi|^2\Psi^* + \Psi_{xxx}^*) \\
+ 2i(l(t) - m(t))\Psi_{xx}^*) = 0, \quad (1.73) \]
\[ i\Psi(a_0 - d_0) + i\Psi_x + \frac{1}{8}((f(t) - g(t))(2(\Psi^*)^2\Psi_x^* + 4|\Psi|^2\Psi_x + \Psi_{xxx}^*) \\
+ 2i(l(t) - m(t))\Psi_{xx}^*) = 0. \]

Next we can substitute \( a_0 \) and \( d_0 \) and simplify:
\[ \frac{1}{4}(m(t) - i(t))(2|\Psi|^2\Psi + i\Psi_{xx}) + i\Psi_t + \frac{3}{4}(f(t) - g(t))|\Psi|^2\Psi_x \\
+ \frac{1}{8}(f(t) - g(t))\Psi_{xxx} + i\Psi_x = 0, \quad (1.74) \]
\[ \frac{1}{4}(l(t) - m(t))(2|\Psi|^2\Psi^* + i\Psi_{xx}^*) + i\Psi_t^* + \frac{3}{4}(f(t) - g(t))|\Psi|^2\Psi_x^* \\
+ \frac{1}{8}(f(t) - g(t))\Psi_{xxx}^* + i\Psi_{xx}^* = 0. \]

If we do more simplification and also substitute \( g(t) = f(t) - 8i\alpha \) and \( m(t) = l(t) - 2i \), we will obtain the following equations:
\[ |\Psi|^2\Psi + i\Psi_t + 6i\alpha|\Psi|^2\Psi_x + \frac{1}{2}\Psi_{xx} + i\alpha\Psi_{xxx} = 0, \quad (1.75) \]
\[ |\Psi|^2\Psi^* + i\Psi_t^* + 6i\alpha|\Psi|^2\Psi_x^* + \frac{1}{2}\Psi_{xx}^* + i\alpha\Psi_{xxx}^* = 0. \quad (1.76) \]

We can see that Eq. (1.75) represents the Hirota equation and Eq. (1.76) is the complex conjugate of the first one. Therefore we have found the Lax pair representation for the Hirota equation.

### 1.3 General Coupled nonlinear Schrödinger (GCNLS) Equations

Consider the following form of GCNLS system:
\[ ip_t + p_{xx} + (a|p|^2 + d|q|^2 + bp^* + b^*qp^*)p = 0, \quad (1.77) \]
\[ iq_t + q_{xx} + (a|p|^2 + d|q|^2 + bp^* + b^*qp^*)q = 0, \quad (1.78) \]

where,
- \( p(x, t) \) and \( q(x, t) \) are functions of \( x \) and \( t \).
- \( a \) and \( d \) are real constants.
- \( b \) is a complex constant and \( * \) denotes complex conjugation.

Again, we will use the same methodology as before, thus equations (1.2) - (1.7) satisfy this
function. However, for this equation $U$ and $V$ are $3 \times 3$ matrices. Consider the following form for matrix $U$:

$$U = \begin{pmatrix} i\lambda & 0 & p \\ 0 & i\lambda & q \\ r_1 & r_2 & -i\lambda \end{pmatrix} = U_0 + \lambda U_1. \quad (1.79)$$

Hence, matrices $U_0$ and $U_1$ are in following form:

$$U_0 = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & q \\ r_1 & r_2 & 0 \end{pmatrix}, \quad (1.80)$$

$$U_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad (1.81)$$

where, $r_1 = -(ap^* + bq^*)$ and $r_2 = -(b^*p^* + dq^*)$[18]. We seek $V$ as a quadratic polynomial function of $\lambda$:

$$V = V_0 + \lambda V_1 + \lambda^2 V_2. \quad (1.82)$$

When we substitute Eq. (1.79) and Eq. (1.82) into (1.7) we will obtain Eq. (1.12). Next we simplify Eq.(1.12) and split it into several equations corresponding to powers of $\lambda$:

$$\lambda^0 : U_{0t} - V_{0x} + U_0 V_0 - V_0 U_0 = 0_{3 \times 3}, \quad (1.83)$$

$$\lambda^1 : U_{1t} - V_{1x} + U_0 V_1 + U_1 V_0 - V_0 U_1 - V_1 U_0 = 0_{3 \times 3}, \quad (1.84)$$

$$\lambda^2 : -V_{2x} + U_0 V_2 + U_1 V_1 - V_1 U_1 - V_2 U_0 = 0_{3 \times 3}, \quad (1.85)$$

$$\lambda^3 : U_1 V_2 - V_2 U_1 = 0_{3 \times 3}. \quad (1.86)$$

First, consider Eq. (1.86) and substitute matrices for $U_1$ and $V_2$, then:

$$\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{pmatrix} - \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & k_2 \end{pmatrix} \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (1.87)$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 2ic_2 \\ 0 & 0 & 2if_2 \\ -2ig_2 & -2ih_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (1.88)$$

So from matrix (1.88) it implies that $c_2 = f_2 = g_2 = h_2 = 0$ and $a_2, b_2, d_2, e_2$ and $k_2$ can be anything. We choose $V_2$ to be diagonal matrix, that is $a_2 = e_2 = -2i$ and $k_2 = 2i$ [18]. Then the matrix $V_2$ is in the following form:
\[ V_2 = -2 \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \]  \hspace{1cm} (1.89)

Now consider Eq. (1.85): 
\[-V_2x + U_0V_2 + U_1V_1 - V_1U_1 - V_2U_0 = 0_{3 \times 3}. \]

Next we substitute the required matrices and simplify:
\[
\begin{pmatrix}
0 & 0 & 4ip + 2ic_1 \\
0 & 0 & 4iq + 2if_1 \\
-4ir_1 - 2iq_1 & -4ir_2 - 2ih_1 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]  \hspace{1cm} (1.90)

This implies that \( c_1 = -2p, f_1 = 2q, q_1 = -2r_1 \) and \( h_1 = -2r_2 \) and we will let all the other terms be equal to zero. Hence, \( V_1 \) matrix will be in the following form:
\[
V_1 = \begin{pmatrix}
0 & 0 & -2p \\
0 & 0 & -2q \\
-2r_1 & -2r_2 & 0
\end{pmatrix}.
\]  \hspace{1cm} (1.91)

Consider Eq. (1.84): 
\[
U_{1t} - V_{1x} + U_0V_1 + U_1V_0 - V_0U_1 - V_1U_0 = 0_{3 \times 3}. \]

Substitute matrices, then simplify and we obtain the following matrix:
\[
\begin{pmatrix}
0 & 0 & 2px + 2c_0 \\
0 & 0 & 2qx + 2f_0 \\
2r_{1x} - 2ig_0 & 2r_{2x} - 2ih_0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]  \hspace{1cm} (1.92)

From matrix (1.92) we have that \( c_0 = ip_x, f_0 = iq_x, g_0 = -ir_{1x} \) and \( h_0 = -ir_{2x}. \) Furthermore, we will not make any assumptions for the other terms of matrix \( V_0 \), instead we will use equation (1.83) in order to solve for missing terms. Thus matrix \( V_0 \) is in the following form:
\[
V_0 = \begin{pmatrix}
a_0 & b_0 & ip_x \\
d_0 & e_0 & iq_x \\
-ir_{1x} & -ir_{2x} & k_0
\end{pmatrix}.
\]  \hspace{1cm} (1.93)

So now we will consider Eq. (1.83): 
\[
U_{0t} - V_{0x} + U_0V_0 - V_0U_0 = 0_{3 \times 3}. \]

After substituting the matrices into Eq. (1.83) and simplifying it we obtain the following matrix:
\[
\begin{pmatrix}
-a_{0x} - ip_{1x} - ir_{1x}p_x & -b_{0x} - ip_{2x} - ir_{2x}p_x & p_t - ip_{xx} + pk_0 - pa_0 - qb_0 \\
-d_{0x} - iqr_{1x} - ir_{1x}q_x & -c_{0x} - iqr_{2x} - ir_{2x}q_x & q_t - iq_{xx} + qk_0 - pd_0 - qc_0 \\
r_{1t} + ir_{1xx} + r_2d_0 + r_1(a_0 - k_0) & r_{2t} + ir_{2xx} + r_1b_0 + r_2(e_0 - k_0) & -k_{0x} + ir_{1px} + ir_{2qx} + ip_{r1x} + iqr_{2x}
\end{pmatrix}.
\]  \hspace{1cm} (1.94)
Since each term of matrix (1.94) is equal to zero, this implies that:

\[
\begin{align*}
a_{0x} &= -i(pr_{1x} - r_{1p_x}) \quad \Rightarrow \quad a_0 = -i pr_1, \\
b_{0x} &= -i(pr_{2x} - r_{2p_x}) \quad \Rightarrow \quad b_0 = -i pr_2, \\
d_{0x} &= -i(qr_{1x} - r_{1q_x}) \quad \Rightarrow \quad d_0 = -i qr_1, \\
e_{0x} &= -i(qr_{2x} - r_{2q_x}) \quad \Rightarrow \quad e_0 = -i qr_2, \\
k_{0x} &= i(r_1p_x + r_2q_x + pr_{1x} + qr_{2x}) \quad \Rightarrow \quad k_0 = i(pr_1 + qr_2).
\end{align*}
\]

Note that the constants of integration are all taken to be zero. Since we found all missing terms of matrix \( V_0 \) and the following will be the form of matrix \( V_0 \):

\[
V_0 = \begin{pmatrix}
-ip_{r_1} & -ip_{r_2} & ip_x \\
-iq_{r_1} & -iq_{r_2} & iq_x \\
-ir_{1x} & -ir_{2x} & i(pr_1 + qr_2)
\end{pmatrix}.
\] (1.95)

Moreover, in matrix (1.94) we have the following two equations:

\[
\begin{align*}
p_t - ip_{xx} + pk_0 - pa_0 - qb_0 &= 0, \\
q_t - iq_{xx} + qk_0 - pd_0 - qe_0 &= 0.
\end{align*}
\] (1.96)

Substitute \( a_0, b_0, d_0, e_0 \) and \( k_0 \) into Eq. (1.96) and simplify, then:

\[
\begin{align*}
p_t - ip_{xx} + pi(pr_{1x} + qr_{2x}) - p(-ip_{r_1}) - q(-ip_{r_2}) &= 0, \\
q_t - iq_{xx} + qi(pr_{1x} + qr_{2x}) - p(-iq_{r_1}) - q(-iq_{r_2}) &= 0.
\end{align*}
\] (1.97)

Now we need to substitute \( r_1 = -(ap^* + bq^*) \), \( r_2 = -(b^*p^* + dq^*) \) and simplify, then we have:

\[
\begin{align*}
p_t - ip_{xx} - 2i(a|p|^2 + bpq^*)p - 2i(b^*p^*q + d|q|^2)p &= 0, \\
q_t - iq_{xx} - 2i(a|p|^2 + bpq^*)q - 2i(b^*p^*q + d|q|^2)q &= 0,
\end{align*}
\] (1.98)

\[
\Rightarrow \begin{align*}
&\begin{align*}
(ip_t + p_{xx} + 2(a|p|^2 + d|q|^2 + bpq^* + b^*p^*q)p &= 0, \\
iq_t + q_{xx} + 2(a|p|^2 + d|q|^2 + bpq^* + b^*p^*q)q &= 0.
\end{align*}
\end{align*}
\] (1.99)

Here Eq. (1.99) represents a General Coupled nonlinear Shrödinger (GCNLS) equation. Thus, we have found the Lax pair representation for GCNLS equations.
Chapter 2

Generalized Darboux transformation and $N$th order rogue wave solution of a general coupled nonlinear Schrödinger equations

2.1 Introduction

In recent years the study of rogue waves (RWs) attracted a significant deal of attention due to their phenomenal properties and their use in potential applications. Rogue waves were first observed in the arbitrary depth of the ocean, but now it has been shown that it appears in various areas of physics such as nonlinear optical fibers [4], BEC [6], super fluid HE [7], capillary waves [8], multi-component plasmas [9] and so on. RWs are also known as freak waves, monster waves, killer waves and extreme waves. A wave is called a rogue wave if its wave height reaches at least twice the significant wave height [10, 11, 12]. RWs appear from nowhere and disappear without a trace [12].

A well known mathematical model for rogue waves is the nonlinear Schrödinger (NLS) equation, namely

$$i\Psi_t + \Psi_{xx} + 2|\Psi|^2\Psi = 0,$$

where,

- $\Psi(x, t)$ is the slowly varying pulse envelope.
- $x$ and $t$ spatial and temporal coordinates.
- Subscripts denote partial differentiation with respect to that variable.

Rational solutions of a nonlinear Schrödinger equation can characterize many dynamical features of the RWs. The NLS equation has been studied for a long time and many results have been obtained for it. For instance, the first-order rational solution [13, 14], the control of the rogue wave [15], the soliton solutions [16], and the Darboux transformation of the NLS equation [17]. Besides constructing RW solutions for the NLS equation, researchers also investigated solutions for an alternative version of the nonlinear Schrödinger equation, such as general coupled nonlinear Schrödinger (GCNLS) system. In this chapter we will construct $N$th order RW solution for GCNLS system,

$$\begin{cases}
    ip_t + p_{xx} + (a|p|^2 + d|q|^2 + bqp^* + b^*qp^*)p = 0, \\
    iq_t + q_{xx} + (a|p|^2 + d|q|^2 + bqp^* + b^*qp^*)q = 0.
\end{cases}$$

Where,

- $p(x, t)$ and $q(x, t)$ are slowly varying pulse envelopes.
\* a and d are real constants, they describe self phase modulation and cross phase modulation respectively.

\* b is a complex constant and * denotes complex conjugation, describe the four wave mixing effects.

If we let \( a = d \) and \( b = 0 \) then Eq. (2.1) becomes Mankov system, and if we let \( a = -d \) and \( b = 0 \) the GCNLS system reduces to a mixed coupled nonlinear Schrödinger equation [18].

We will construct \( N \)th order RW solutions for system (2.1) using the Darboux transformation (DT) method. Since using modified DT will be hard to construct explicit \( N \)th order RW solutions of this system, thus we will consider the generalized Darboux transformation (GDT) method. It has been shown that higher order RW solutions only contain one critical eigenvalue \( \lambda_0 \). However, the N-fold DT has N-distinct eigenvalues. Therefore, using a suitable limit process we will restrict all the eigenvalues to a single critical eigenvalue [18]. Applying this limit process we will be able to construct a recursive formula for an \( N \)th order RW solution. Moreover, using this limit process on the determinant representation of N-fold DT we can get the determinant representation of the \( N \)th order RW solution. In this chapter we will present the exact form of first and second order RW solutions of the GCNLS system and we provide both the recursive formula and the determinant representation of \( N \)th order RW solution [18]. The Lax pair of (2.1) is the following:

\[
\Psi_x = U\Psi = (\Lambda J + p)\Psi, \quad (2.2a)
\]
\[
\Psi_t = V\Psi = (\Lambda^2 V_0 + \Lambda V_1 + V_2)\Psi, \quad (2.2b)
\]

where \( \Psi = (\psi(x,t), \phi(x,t), \varphi(x,t))^T \) is the vector eigenfunction and \( T \) denotes the transpose of the matrix. Where \( \Lambda = diag(\lambda, \lambda, \lambda) \) and the matrices \( J, P, V_0, V_1, \) and \( V_2 \) have been obtained in the previous section and are defined as follows:

\[
J = \begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i
\end{pmatrix}, \quad P = \begin{pmatrix}
0 & 0 & p \\
0 & 0 & q \\
0 & 0 & 0
\end{pmatrix}, \quad (2.3)
\]

\[
V_0 = -2 \begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i
\end{pmatrix}, \quad V_1 = -2 \begin{pmatrix}
0 & 0 & p \\
0 & 0 & q \\
0 & 0 & 0
\end{pmatrix}, \quad V_2 = \begin{pmatrix}
-ipr_1 & -ipr_2 & ip_x \\
-iqr_1 & -iqr_2 & iq_x \\
-ir_{1x} & -ir_{2x} & i(pr_1 + qr_2)
\end{pmatrix}
\]

where \( r_1 = -(ap^* + bq^*) \) and \( r_2 = -(b^* p^* + dq^*) \) and \( \lambda \) is the isospectral parameter.

The plan of this chapter is as following. In section 2.2, we will construct first, second, third iteration and present the \( N \)th iteration of DT for (2.1). In section 2.3 we present in detail the recursive formula and determinant expressions of the \( N \)th order RW solution using GDT. Next in section 2.4, we derive the explicit form of first, second and third order RW solutions. Finally, in section 2.5 we present our summary and conclusion [18].
2.2 Darboux Transformation for GCNLS system

2.2.1 First Iteration

Darboux transformation (DT) is a special transformation where $\Psi$ and $\Psi[1]$ are old and new eigenfunctions of (2.2),

$$\Psi[1] = T[1]\Psi = \Psi\Lambda - S[1]\Psi. \quad (2.4)$$

$T[1]$ is the Darboux matrix, $S[1]$ is a non singular $3 \times 3$ matrix. Equation (2.4) transforms the original Lax pair (2.2) into a new Lax pair.

$$\Psi[1]_x = U[1]\Psi[1] = (\Lambda J + P[1])\Psi[1],$$

$$\Psi[1]_t = V[1]\Psi[1] = (\Lambda^2 V_0[1] + \Lambda V_1[1] + V_2[1])\Psi[1], \quad (2.5)$$

$P[1], V_0[1], V_1[1]$ and $V_2[1]$ assume to have the same form as $P, V_0, V_1$ and $V_2$ except that $p$ and $q$ have obtained a new form of expression, namely $p[1]$ and $q[1]$ in $U[1]$ and $V[1]$. Now we can substitute DT (2.4) into Lax pair (2.2) and compare our results with (2.5) [18].

From Eq. (2.4) we have: $\Psi[1] = T[1]\Psi$, then

$$\begin{align*}
(\Psi[1])_x &= (T[1]_x \Psi + T[1]_t \Psi) = T[1]_x \Psi + T[1]_t U \Psi = (T[1]_x + T[1]_t U) \Psi, \\
(\Psi[1])_t &= (T[1]_t \Psi + T[1]_x \Psi) = T[1]_t \Psi + T[1]_x V \Psi = (T[1]_t + T[1]_x V) \Psi.
\end{align*} \quad (2.6)$$

Which implies that,

$$\begin{align*}
\end{align*}$$

where the first equality we got from equation (2.5) and the second equality is obtained from the fact that $\Psi[1] = T[1]\Psi$. By comparing Eq. (2.6) and (2.7) we get

$$\begin{align*}
(T[1]_x + T[1]_t U) \Psi &= U[1]T[1] \Psi, \\
(T[1]_t + T[1]_x V) \Psi &= V[1]T[1] \Psi \quad \Rightarrow \quad (T[1]_x + T[1]_t U) = U[1]T[1], \\
\end{align*}$$

Thus we get the following equations

$$\begin{align*}
V[1] &= (T[1]_t + T[1]_x V)T[1]^{-1}. \quad (2.8)
\end{align*}$$

Also we have,
On the other hand from Eq. (2.5) we have that:

\[
\begin{align*}
(\Psi[1])_x &= (\Lambda J + P[1])(I\Lambda - S[1])\Psi, \\
(\Psi[1])_t &= (\Lambda^2 V_0[1] + \Lambda V_1[1] + V_2[1])(I\Lambda - S[1])\Psi. 
\end{align*}
\]

(2.10)

Hence we have,

\[
\begin{align*}
(U\Lambda - S[1]_x - S[1]U)\Psi &= (\Lambda J + P[1])(I\Lambda - S[1])\Psi, \\
\end{align*}
\]

Since this is true for all \( \Psi \), then

\[
\begin{align*}
U\Lambda - S[1]_x - S[1]U &= (\Lambda J + P[1])(I\Lambda - S[1]), \\
V\Lambda - S[1]_x - S[1]V &= (\Lambda^2 V_0[1] + \Lambda V_1[1] + V_2[1])(I\Lambda - S[1]). 
\end{align*}
\]

(2.11)

Now we can substitute expression for \( U[1], V[1], U, V \) and \( T[1] \) into above equation and expand:

\[
\]

and

\[
\]

Equate the coefficients of various powers of \( \Lambda \) on both sides we get the following relationship between old and new potentials,

\[
\begin{align*}
\Lambda^3 : & \quad V_0 = V_0[1], \\
\Lambda^2 : & \quad J = J, \\
& \quad V_1 - S[1]V_0 = V_1[1] - V_0[1]S[1], \\
\Lambda^1 : & \quad P - S[1]J = P[1] - JS[1], \\
& \quad V_2 - S[1]V_1 = V_2[1] - V_1[1]S[1], \\
\Lambda^0 : & \quad - S[1]_x - S[1]P = - P[1]S[1], \\
\end{align*}
\]
Then,
\[ V_0[1] = V_0, \]

After simplification we have,
\[ V_0[1] = V_0, \quad (2.12a) \]
\[ V_1[1] = V_1 + [V_0, S[1]], \quad (2.12b) \]
\[ V_2[1] = V_2 + [V_1, S[1]] + [V_0, S[1]]S[1], \quad (2.12c) \]
\[ P[1] = P + [J, S[1]], \quad (2.12d) \]
\[ S[1]_x = [P, S[1]] + [J, S[1]]S[1], \quad (2.12e) \]
\[ S[1]_t = [V_2, S[1]] + [V_1, S[1]]S_1 + [V_0, S[1]]S[1]^2. \quad (2.12f) \]

If \( S[1] \) satisfies all equations of (2.12) then eigenvalue problem (2.2) will remain invariant under DT (2.4). We consider the following general form of matrix \( S[1] \)
\[
S[1] = \begin{pmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{pmatrix}
\quad (2.13)
\]

Substitute matrix (2.13) in Eq. (2.12d),
\[
\begin{pmatrix}
0 & 0 & p[1] \\
0 & 0 & q[1] \\
r[1]_1 & r[1]_2 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & p \\
0 & 0 & q \\
r_1 & r_2 & 0
\end{pmatrix}
+ \begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i
\end{pmatrix}
\begin{pmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{pmatrix}
- \begin{pmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{pmatrix}
\begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i
\end{pmatrix}.
\]

Then we have
\[ p[1] = p + iS_{13} + iS_{13}, \]
\[ q[1] = q + iS_{23} + iS_{23}. \quad (2.14) \]

In order to obtain the exact form of \( S[1] \), we consider \( S[1] \) to be as following [17].
\[ S[1] = H_1\Lambda H_1^{-1}, \quad (2.15) \]
where,
\[
H_1 = \begin{pmatrix}
\psi_1 & \varphi_1^* & 0 \\
\phi_1 & 0 & \varphi_1^* \\
\varphi_1 & -\psi_1^* & -\phi_1^*
\end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_1^* & 0 \\
0 & 0 & \lambda_1^*
\end{pmatrix}.
\]

In the matrix \( H_1 \), \( \Psi_1 = (\psi_1, \phi_1, \varphi_1)^T \) is an eigenfunction of the Lax Pair Eq. (2.2) associated with the eigenvalue \( \lambda = \lambda_1 \). From the orthogonality condition, it follows that \( (\varphi_1^*, 0, -\psi_1^*)^T \) and \( (0, \varphi_1^*, -\phi_1^*)^T \) are also the eigenfunctions of the Lax pair (2.2) at \( \lambda = \lambda_1^* \).

The first iterated DT is given in Eq. (2.4). If \( H_1 \) is the solution of \( \Psi[1] \) at \( \Lambda = \Lambda_1 \) then it should satisfy (2.15) \[18\]:
\[
\Psi[1]_1 = T[1]H_1 = 0 = H_1\Lambda_1 - S[1]H_1,
\]

and the matrix form of Eq. (2.18) is,
\[
\begin{pmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{pmatrix}
\begin{pmatrix}
\psi_1 & \varphi_1^* & 0 \\
\phi_1 & 0 & \varphi_1^* \\
\varphi_1 & -\psi_1^* & -\phi_1^*
\end{pmatrix}
= \begin{pmatrix}
\psi_1 & \varphi_1^* & 0 \\
\phi_1 & 0 & \varphi_1^* \\
\varphi_1 & -\psi_1^* & -\phi_1^*
\end{pmatrix}
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_1^* & 0 \\
0 & 0 & \lambda_1^*
\end{pmatrix}.
\]

From Eq. (2.19) we can generate the following system of equations:
\[
S_{11}\psi_1 + S_{12}\phi_1 + S_{13}\varphi_1 = \lambda_1\psi_1,
S_{11}\varphi_1^* - S_{13}\psi_1^* = \lambda_1^*\varphi_1^*,
S_{12}\varphi_1^* - S_{13}\phi_1^* = 0,
S_{21}\psi_1 + S_{22}\phi_1 + S_{23}\varphi_1 = \lambda_1\phi_1,
S_{21}\varphi_1^* - S_{23}\psi_1^* = 0,
S_{22}\varphi_1^* - S_{23}\phi_1^* = \lambda_1^*\phi_1^*,
S_{31}\psi_1 + S_{32}\phi_1 + S_{33}\varphi_1 = \lambda_1\varphi_1,
S_{31}\varphi_1^* - S_{33}\psi_1^* = -\lambda_1^*\psi_1^*,
S_{32}\varphi_1^* - S_{33}\phi_1^* = -\lambda_1^*\phi_1^*.
\]

Specially, from the algebraic equations given above, and using the first six equations, coefficients \( S_{13} \) and \( S_{23} \) are acquired by Cramer’s rule.
In order to determine \( S_{13} \) and \( S_{23} \) from Eq. (2.21) one need to know the explicit expression of \( \psi_1, \phi_1 \) and \( \varphi_1 \) which are solutions of the eigenvalue problem (2.2) [18]. That is

\[
\begin{bmatrix}
\psi_1 \\
\phi_1 \\
\varphi_1
\end{bmatrix}_x = (\lambda_1 J + P)
\begin{bmatrix}
\psi_1 \\
\phi_1 \\
\varphi_1
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\psi_1 \\
\phi_1 \\
\varphi_1
\end{bmatrix}_t = (\lambda_1^2 V_0 + \lambda_1 V_1 + V_2)
\begin{bmatrix}
\psi_1 \\
\phi_1 \\
\varphi_1
\end{bmatrix}
\]

\( \psi_{1x} = i\lambda_1 \psi_1 + p\varphi_1, \)

\( \phi_{1x} = i\lambda_1 \phi_1 + q\varphi_1, \)

\( \varphi_{1x} = r_1 \psi_1 + r_2 \phi_1 - i\lambda_1 \varphi_1, \)

\( \psi_{1t} = (-2i\lambda_1^2 - ipr_1)\psi_1 - ipr_2\phi_1 + (ip_x - 2p\lambda_1)\varphi_1, \)

\( \phi_{1t} = (-2i\lambda_1^2 - iqr_2)\phi_1 - iqr_1\psi_1 + (iq_x - 2q\lambda_1)\varphi_1, \)

\( \varphi_{1t} = (-ir_{1x} - 2r_1\lambda_1)\psi_1 + (-ir_{2x} - 2r_2\lambda_1)\phi_1 + (2i\lambda_1^2 + ipr_1 + iqr_2)\varphi_1. \)

Solving this system of equations given in (2.22) with appropriate seed solutions \( p \) and \( q \) one can obtain the explicit expression of \( \psi_1, \phi_1 \) and \( \varphi_1 \). We can fix \( S_{13} \) and \( S_{23} \) with known...
expressions of $\psi_1, \phi_1$ and $\varphi_1$ [18]. Substitute (2.20) into (2.14) we have

\[ p[1] = p + 2i \frac{(\lambda_1 - \lambda_1^*) \psi_1 \varphi_1^*}{|\psi_1|^2 + |\phi_1|^2 + |\varphi_1|^2}, \]

\[ q[1] = q + 2i \frac{(\lambda_1 - \lambda_1^*) \phi_1 \varphi_1^*}{|\psi_1|^2 + |\phi_1|^2 + |\varphi_1|^2}. \]  

(2.23)

We can also write equation (2.23) in the determinant form

\[
\begin{vmatrix}
\psi_1 & \varphi_1^* & 0 \\
\phi_1 & 0 & \varphi_1^* \\
\lambda_1 \psi_1 & \lambda_1^* \varphi_1^* & 0
\end{vmatrix}
= p[1] + 2i \begin{vmatrix}
\psi_1 & \varphi_1^* & 0 \\
\phi_1 & 0 & \varphi_1^* \\
\varphi_1 - \psi_1^* & -\phi_1^*
\end{vmatrix},
\]

(2.24)

\[
\begin{vmatrix}
\psi_1 & \varphi_1^* & 0 \\
\phi_1 & 0 & \varphi_1^* \\
\lambda_1 \psi_1 & \lambda_1^* \varphi_1^* & 0
\end{vmatrix}
= q[1] + 2i \begin{vmatrix}
\psi_1 & \varphi_1^* & 0 \\
\phi_1 & 0 & \varphi_1^* \\
\varphi_1 - \psi_1^* & -\phi_1^*
\end{vmatrix}.
\]

Using the formula (2.23) or (2.24) one can generate a class of solutions including solitary wave solutions, breather and RW solutions for Eq. (2.1) [18].

### 2.2.2 Second Iteration

The second iterated DT reads,


(2.25)

\(\Psi[2]\) is the second iterated eigenfunction and \(\Psi[1]\) is the first iterated eigenfunction. \(T[2]\) is the second iterated DT matrix. Moreover, the DT (2.25) transforms the Lax pair (2.5) into the second iterated Lax pair. That is :

\[
\Psi[2]_x = U[2] \Psi[2] = (\Lambda J + P[2]) \Psi[2],
\]

\[
\]  

(2.26)

where, the parameters \(p[1]\) and \(q[1]\) obtained new expressions \(p[2]\) and \(q[2]\) in matrices \(U[2]\) and \(V[2]\). However, the matrices \(P[2], V_0[2], V_1[2]\) and \(V_2[2]\) are in the same form as \(P[1], V_0[1], V_1[1]\) and \(V_1[1]\). Next, substitute Eq. (2.25) into (2.26), that is repeating same steps from Eq. (2.6)-(2.8) we get

\[
\]

\[
\]  

(2.27)
Also using same methodology as in Eqs. (2.9)-(2.10), we will get the following equation


(2.28)

Next substitute \( U[1], V[1], J, P[1], V_0[1], V_1[1], V_2[1] \) into Eq. (2.28) we get and equate the coefficients of powers of \( \Lambda \) on both sides we get

\[ V_0[2] = V_0[1], \]  

(2.29a)

\[ V_1[2] = V_1[1] + [J, S[2]], \]  

(2.29b)

\[ V_2[2] = V_2[1] + [V_1[1], S[2]] + [V_0[1], S[2]]S[2], \]  

(2.29c)

\[ P[2] = P[1] + [J, S[2]], \]  

(2.29d)

\[ S[2]_x = [P[1], S[2]] + [J, S[2]]S[2], \]  

(2.29e)

\[ S[2]_t = [V_2[1], S[2]] + [V_1[1], S[2]]S[1][1] + [V_0[1], S[2]]S[2]^2. \]  

(2.29f)

If \( S[2] \) satisfies all equations of (2.29) then eigenvalue problem (2.2) will remain invariant under DT (2.45). We consider the following general form of matrix \( S[1] \)


(2.30)

Substitute matrix (2.30) in Eq. (2.29d),

\[
\begin{pmatrix}
0 & 0 & p[2] \\
0 & 0 & q[2] \\
r_{[2]}_1 & r_{[2]}_2 & 0
\end{pmatrix}
=
\begin{pmatrix}
0 & 0 & p \\
0 & 0 & q \\
r_1 & r_2 & 0
\end{pmatrix}
+
\begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i
\end{pmatrix}
\begin{pmatrix}
S[2]_{11} & S[2]_{12} & S[2]_{13} \\
S[2]_{21} & S[2]_{22} & S[2]_{23} \\
S[2]_{31} & S[2]_{32} & S[2]_{33}
\end{pmatrix}
-
\begin{pmatrix}
S[2]_{11} & S[2]_{12} & S[2]_{13} \\
S[2]_{21} & S[2]_{22} & S[2]_{23} \\
S[2]_{31} & S[2]_{32} & S[2]_{33}
\end{pmatrix}
\begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i
\end{pmatrix}
\begin{pmatrix}
S[2]_{11} & S[2]_{12} & S[2]_{13} \\
S[2]_{21} & S[2]_{22} & S[2]_{23} \\
S[2]_{31} & S[2]_{32} & S[2]_{33}
\end{pmatrix}
\begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i
\end{pmatrix}
\begin{pmatrix}
S[2]_{11} & S[2]_{12} & S[2]_{13} \\
S[2]_{21} & S[2]_{22} & S[2]_{23} \\
S[2]_{31} & S[2]_{32} & S[2]_{33}
\end{pmatrix}
\begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i
\end{pmatrix}
\]

we have


(2.31)

In order to obtain the exact form of \( S[2] \), we consider \( S[2] \) to be as following [17].


(2.32)

where

\[ H_2[1] = \begin{pmatrix} \psi_2[1] & \varphi_2[1]^* & 0 \\ \phi_2[1] & 0 & \varphi_2^* \\ \varphi_2[1] & -\psi_2[1]^* & -\varphi_2[1]^* \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2^* & 0 \\ 0 & 0 & \lambda_2^* \end{pmatrix}. \]  

(2.33)
In the matrix $H_2[1]$, $\Psi_2[1] = (\psi_2[1], \phi_2[1], \varphi_2[1])^T$ is an eigenfunction of the Lax Pair Eq. (2.26) associated with the eigenvalue $\lambda = \lambda_2$. From the orthogonality condition it follows that $(\varphi_2[1]^*, 0, -\psi_2[1]^*)^T$ and $(0, \varphi_2[1]^*, -\phi_2[1]^*)^T$ are also the eigenfunctions of the Lax pair (2.26) at $\lambda = \lambda_2$.

The second iterated DT is given in Eq. (2.25). If $H_2[1]$ is the solution of $\Psi[2]$ at $\Lambda = \Lambda_2$ then it should satisfy (2.32) \[18\].

$$
$$

$$
$$

and the matrix form of Eq. (2.35) is,

$$
\begin{pmatrix}
S[2]_{11} & S[2]_{12} & S[2]_{13} \\
S[2]_{21} & S[2]_{22} & S[2]_{23} \\
S[2]_{31} & S[2]_{32} & S[2]_{33}
\end{pmatrix}
\begin{pmatrix}
\psi_2[1] \\
\phi_2[1] \\
\varphi_2[1]
\end{pmatrix}
\begin{pmatrix}
\varphi_2[1]^* \\
\psi_2[1]^* \\
0
\end{pmatrix}
= \begin{pmatrix}
\psi_2[1] \\
\phi_2[1] \\
\varphi_2[1]
\end{pmatrix}
\begin{pmatrix}
\varphi_2[1]^* \\
\psi_2[1]^* \\
0
\end{pmatrix}
\begin{pmatrix}
\lambda_2 \\
0 \\
0
\end{pmatrix}.
\quad (2.36)
$$

Then from Eq. (2.36) we generate the following system of equations:


Using the first six equations of the above algebraic equations, coefficients $S[2]_{13}$ and $S[2]_{23}$ are acquired by Cramer’s rule.
The determinant form of Eq. (2.37) will be,

\[
S_{[2]_{13}} = \begin{vmatrix}
\psi_2[1] & \phi_2[1] & \lambda_2\psi_2[1] \\
\phi_2[1]^* & 0 & \lambda_2^*\phi_2[1]^* \\
0 & \psi_2[1]^* & 0 \\
\end{vmatrix} = \begin{vmatrix}
\psi_2[1] & \varphi_2[1]^* & 0 \\
\phi_2[1] & 0 & \varphi_2[1]^* \\
\lambda_2[\psi_2[1]^*] & 0 & \lambda_2^*[\phi_2[1]^*] \\
\end{vmatrix},
\]

(2.37a)

\[
S_{[2]_{23}} = \begin{vmatrix}
\psi_2[1] & \phi_2[1] & \lambda_2\phi_2[1] \\
\varphi_2[1]^* & 0 & -\phi_2[1]^* \\
0 & \varphi_2[1]^* & 0 \\
\end{vmatrix} = \begin{vmatrix}
\psi_2[1] & \varphi_2[1]^* & 0 \\
\phi_2[1] & 0 & \varphi_2[1]^* \\
\lambda_2[\varphi_2[1]^*] & 0 & \lambda_2^*[\phi_2[1]^*] \\
\end{vmatrix}.
\]

(2.37b)

The determinant form of Eq. (2.37) will be,

\[
S_{[2]_{13}} = \frac{(\lambda_2 - \lambda_2^*)\psi_2[1]\varphi_2[1]^*}{|\psi_2[1]|^2 + |\phi_2[1]|^2 + |\varphi_2[1]|^2},
\]

(2.38a)

\[
S_{[2]_{23}} = \frac{(\lambda_2 - \lambda_2^*)\phi_2[1]\varphi_2[1]^*}{|\psi_2[1]|^2 + |\phi_2[1]|^2 + |\varphi_2[1]|^2}.
\]

(2.38b)

In order to determine \(S_{[2]_{13}}\) and \(S_{[2]_{23}}\) from Eq. (2.38) one needs to know the explicit expression of \(\psi_2[1], \phi_2[1]\) and \(\varphi_2[1]\) which are solutions of the eigenvalue problem (2.2) [18]. That is

\[
\begin{bmatrix}
\psi_2[1] \\
\phi_2[1] \\
\varphi_2[1]
\end{bmatrix}_x = (\lambda_2 J + P[1])
\begin{bmatrix}
\psi_2[1] \\
\phi_2[1] \\
\varphi_2[1]
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
\psi_2[1] \\
\phi_2[1] \\
\varphi_2[1]
\end{bmatrix}_t = (\lambda_2^2 V_0[1] + \lambda_1 V_1[1] + V_2[1])
\begin{bmatrix}
\psi_2[1] \\
\phi_2[1] \\
\varphi_2[1]
\end{bmatrix},
\]

\[
= \begin{pmatrix}
\end{pmatrix}
\begin{bmatrix}
\psi_2[1] \\
\phi_2[1] \\
\varphi_2[1]
\end{bmatrix}.
\]
\[
\psi_2[1]_x = i\lambda_2\psi_2[1] + p[1]\varphi_2[1],
\]
\[
\phi_2[1]_x = i\lambda_2\phi_2[1] + q[1]\varphi_2[1],
\]
\[
\varphi_2[1]_x = r_1[1]\psi_2[1] + r_2[1]\phi_2[1] - i\lambda_2\varphi_2[1],
\]
\[
\]
\[
\phi_2[1]_t = -2i\lambda_2^2 - iq[1]r_2[1]\phi_2[1] - iq[1]r_1[1]\psi_2[1] + (iq[1]_x - 2q[1]\lambda_2)\varphi_2[1],
\]
\[
\varphi_2[1]_t = (ir_1[1]_x - 2r_1[1]\lambda_2)\psi_2[1] + (-ir_2[1]_x - 2r_2[1]\lambda_2)\phi_2[1]
\]
\[
+ (2i\lambda_2^2 + ip[1]r_1[1] + iq[1]r_2[1])\varphi_2[1],
\]
\[
(2.39)
\]


Solving this system of equations given in (2.39) with appropriate seed solutions \( p[1] \) and \( q[1] \) one can obtain the explicit expression of \( \psi_2[1], \phi_2[1] \) and \( \varphi_2[1] \). Substitute them in Eq. (2.38) we can get the matrix element \( S_{213}^{[2]} \) and \( S_{223}^{[2]} \). If we plug (2.38) into (2.31) we can get a second iterated solution for Eq. (2.1)

\[
\]
\[
\]
\[
(2.40)
\]

Using this formula one can generate a class of solutions including solitary wave solutions, breather and RW solutions for the Eq. (2.1) [18].

Also as introduced in [18], we can also write equation (2.40) in the following form,

\[
\]
\[
\]
\[
(2.41)
\]
where

\[
N_1[2] = \begin{pmatrix}
\lambda_1 \psi_1 & \lambda_2 \psi_2 & \lambda_1^* \varphi_1 & \lambda_2^* \varphi_2 & 0 & 0 \\
\psi_1 & \psi_2 & \varphi_1^* & \varphi_2^* & 0 & 0 \\
\lambda_1 \phi_1 & \lambda_2 \phi_2 & 0 & 0 & \lambda_1^* \varphi_1 & \lambda_2^* \varphi_2 \\
\phi_1 & \phi_2 & 0 & 0 & \varphi_1^* & \varphi_2^* \\
\lambda_1^2 \psi_1 & \lambda_2^2 \psi_2 & \lambda_1^* \varphi_1 & \lambda_2^* \varphi_2 & 0 & 0 \\
\varphi_1 & \varphi_2 & -\psi_1^* & -\psi_2^* & -\phi_1^* & -\phi_2^*
\end{pmatrix},
\]

\[
N_2[2] = \begin{pmatrix}
\lambda_1 \psi_1 & \lambda_2 \psi_2 & \lambda_1^* \varphi_1 & \lambda_2^* \varphi_2 & 0 & 0 \\
\psi_1 & \psi_2 & \varphi_1^* & \varphi_2^* & 0 & 0 \\
\lambda_1 \phi_1 & \lambda_2 \phi_2 & 0 & 0 & \lambda_1^* \varphi_1 & \lambda_2^* \varphi_2 \\
\phi_1 & \phi_2 & 0 & 0 & \varphi_1^* & \varphi_2^* \\
\lambda_1^2 \psi_1 & \lambda_2^2 \psi_2 & \lambda_1^* \varphi_1 & \lambda_2^* \varphi_2 & 0 & 0 \\
\varphi_1 & \varphi_2 & -\psi_1^* & -\psi_2^* & -\phi_1^* & -\phi_2^*
\end{pmatrix},
\] (2.42)

and

\[
D[2] = \begin{pmatrix}
\lambda_1 \psi_1 & \lambda_2 \psi_2 & \lambda_1^* \varphi_1 & \lambda_2^* \varphi_2 & 0 & 0 \\
\psi_1 & \psi_2 & \varphi_1^* & \varphi_2^* & 0 & 0 \\
\lambda_1 \phi_1 & \lambda_2 \phi_2 & 0 & 0 & \lambda_1^* \varphi_1 & \lambda_2^* \varphi_2 \\
\phi_1 & \phi_2 & 0 & 0 & \varphi_1^* & \varphi_2^* \\
\lambda_1^2 \psi_1 & \lambda_2^2 \psi_2 & -\lambda_1^* \psi_1 & -\lambda_2^2 \psi_2 & -\lambda_1^* \phi_1 & -\lambda_2^2 \phi_2 \\
\varphi_1 & \varphi_2 & -\psi_1^* & -\psi_2^* & -\phi_1^* & -\phi_2^*
\end{pmatrix}.
\]

### 2.2.3 Third Iteration

Now we will construct a third order RW solution. However, for the third order RW solution we will only mention the essential expressions because the methodology is the same as for the first or second iteration.

The third iterated DT matrix is the following,


This DT changes the second iterated solution \( p[2] \) and \( q[2] \) into a new third order solution \( p[3] \) and \( q[3] \), that is

\[
\]

\[
\] (2.45)
where \((\psi_3[2], \phi_3[2], \varphi_3[2])^T\) is the column solution of Eq. (2.2) at \(\lambda = \lambda_3\) with \(p[2], q[2]\) as the seed solution. Using formula (2.45) with a given seed solution we can obtain three soliton solutions and third order breather solutions.

Again as shown in [18] we can also write equation (2.45) in more compact determinant form, namely

\[
\]

\[
\]

where

\[
N_1[3] = \begin{pmatrix}
\lambda_1^2 \psi_1 & \lambda_1^2 \psi_2 & \lambda_1^2 \psi_3 & \lambda_1^2 \varphi_1 & \lambda_2^2 \varphi_2 & \lambda_3^2 \varphi_3 \\
\lambda_1 \psi_1 & \lambda_1 \psi_2 & \lambda_1 \psi_3 & \lambda_1 \varphi_1 & \lambda_2 \varphi_2 & \lambda_3 \varphi_3 \\
\psi_1 & \psi_2 & \psi_3 & \varphi_1 & \varphi_2 & \varphi_3 \\
\lambda_1^2 \phi_1 & \lambda_2 \phi_2 & \lambda_3^2 \phi_3 \\
\lambda_1 \phi_1 & \lambda_2 \phi_2 & \lambda_3 \phi_3 \\
\phi_1 & \phi_2 & \phi_3
\end{pmatrix},
\]

\[
N_2[3] = \begin{pmatrix}
\lambda_1^2 \psi_1 & \lambda_1^2 \psi_2 & \lambda_1^2 \psi_3 & \lambda_1^2 \varphi_1 & \lambda_2^2 \varphi_2 & \lambda_3^2 \varphi_3 \\
\lambda_1 \psi_1 & \lambda_1 \psi_2 & \lambda_1 \psi_3 & \lambda_1 \varphi_1 & \lambda_2 \varphi_2 & \lambda_3 \varphi_3 \\
\psi_1 & \psi_2 & \psi_3 & \varphi_1 & \varphi_2 & \varphi_3 \\
\lambda_1^2 \phi_1 & \lambda_2 \phi_2 & \lambda_3^2 \phi_3 \\
\lambda_1 \phi_1 & \lambda_2 \phi_2 & \lambda_3 \phi_3 \\
\phi_1 & \phi_2 & \phi_3
\end{pmatrix},
\]

and

\[
D[3] = \begin{pmatrix}
\lambda_1^2 \psi_1 & \lambda_1^2 \psi_2 & \lambda_1^2 \psi_3 & \lambda_1^2 \varphi_1 & \lambda_2^2 \varphi_2 & \lambda_3^2 \varphi_3 \\
\lambda_1 \psi_1 & \lambda_1 \psi_2 & \lambda_1 \psi_3 & \lambda_1 \varphi_1 & \lambda_2 \varphi_2 & \lambda_3 \varphi_3 \\
\psi_1 & \psi_2 & \psi_3 & \varphi_1 & \varphi_2 & \varphi_3 \\
\lambda_1^2 \phi_1 & \lambda_2 \phi_2 & \lambda_3^2 \phi_3 \\
\lambda_1 \phi_1 & \lambda_2 \phi_2 & \lambda_3 \phi_3 \\
\phi_1 & \phi_2 & \phi_3
\end{pmatrix}.
\]
2.2.4 \textit{Nth Iteration}

The N-fold DT can be iterated successfully if we are given N distinct solutions of the Lax pair (2.2), that is \( \Psi_k = (\psi_k, \varphi_k, \varphi_k)^T \) at \( \lambda = \lambda_k \) for \( k = 1, 2, \ldots, N \). Then Nth iterated solution will be in the following form

\[
p[N] = p[N - 1] + 2i \frac{(\lambda_N - \lambda_N^*) \psi_N [N - 1] \varphi_N [N - 1]^*}{|\psi_N [N - 1]|^2 + |\varphi_N [N - 1]|^2 + |\varphi_N [N - 1]|^2},
\]
\[
q[N] = q[N - 1] + 2i \frac{(\lambda_N - \lambda_N^*) \phi_N [N - 1] \varphi_N [N - 1]^*}{|\psi_N [N - 1]|^2 + |\phi_N [N - 1]|^2 + |\varphi_N [N - 1]|^2},
\]

where from previous sections we can conclude that \( \Psi[N - 1] = T[N - 1] \Psi[N - 2] = T[N - 1] T[N - 2] \cdots T[1] T[0] \Psi \) and \( T[N] = \Lambda - H_N[N - 1] \Lambda N H_N[N - 1]^{-1} \) such that

\[
\Lambda_N = \begin{pmatrix}
\lambda_N & 0 & 0 \\
0 & \lambda_N & 0 \\
0 & 0 & \lambda_N^*
\end{pmatrix},
\]
\[
H_N[N - 1] = \begin{pmatrix}
\psi_N[N - 1] & \varphi_N^*[N - 1] & 0 \\
\phi_N[N - 1] & 0 & \varphi_N^*[N - 1] \\
\psi_N[N - 1] & -\psi_N^*[N - 1] & -\phi_N[N - 1]
\end{pmatrix}.
\]

The determinant forms of the Nth iterated solution is as following [18]

\[
p[N] = p + 2i \frac{\triangle_2}{\triangle_1},
\]
\[
q[N] = q + 2i \frac{\triangle_3}{\triangle_1},
\]

such that

\[
\triangle_1 = \begin{pmatrix}
\lambda_1^{N-1} \psi_1 & \cdots & \lambda_1^{N-1} \psi_N & \lambda_1^{(N-1)} \varphi_1^* & \cdots & \lambda_1^{(N-1)} \varphi_N^* & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda_1^{N-1} \phi_1 & \cdots & \lambda_1^{N-1} \phi_N & 0 & \cdots & 0 & \lambda_1^{(N-1)} \varphi_1^* & \cdots & \lambda_1^{(N-1)} \varphi_N^* \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_1 & \cdots & \phi_N & 0 & \cdots & 0 & \varphi_1^* & \cdots & \varphi_N^* \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda_1^{N-1} \varphi_1 & \cdots & \lambda_1^{N-1} \varphi_N & -\lambda_1^{(N-1)} \psi_1^* & \cdots & -\lambda_1^{(N-1)} \psi_N^* & -\lambda_1^{(N-1)} \phi_1^* & \cdots & -\lambda_1^{(N-1)} \phi_N^* \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\varphi_1 & \cdots & \varphi_N & -\psi_1^* & \cdots & -\psi_N^* & -\phi_1^* & \cdots & -\phi_N^*
\end{pmatrix}
\]
Eqs. (2.48) and (2.50) give the $N$th iterated DT solution formula. Using this formulas one can construct $N$-soliton solution and $N$-breather solution.

2.3 Generalized Darboux transformation (GDT)

Through our work we found that the $N$th iterated DT contains $N$-eigenfunctions, $\Psi_i, i = 1, 2, \ldots, N$ and $N$-distinct eigenvalues, $\lambda_i, i = 1, 2, \ldots, N$. However, the higher order RW solutions contain only one critical eigenvalue $\lambda_0$. In order to overcome this problem, as mentioned in [18], we can take limit $\lambda_i \rightarrow \lambda_1$ in the corresponding eigenvalues found in the DT. Using the methodology introduced in [18] and [27] we can obtain higher order RW solutions with only one eigenvalue, namely $\lambda_0$.

2.3.1 First Iteration

Since from Darboux theory we have that $T[1]\Psi_1 = 0$ that is why we cannot apply the limit procedure on $\Psi_1$. Now, let $\Psi_2 = \Psi_1(\lambda_1 + \delta)$ be a special solution for Lax pair (2.2). Then $\frac{\Psi_1(\lambda_1 + \delta)}{\delta}$ is also a solution of (2.2), where $\delta$ is a small parameter. Using Taylor series at $\lambda_1$ we can expand the eigenfunction $\Psi_2$, that is

$$\Psi_2 = \Psi_1(\lambda_1 + \delta) = \Psi_1^{[0]} + \Psi_1^{[1]} \delta + \Psi_1^{[2]} \delta^2 + \cdots + \Psi_1^{[N]} \delta^N + \cdots,$$

(2.51)

where

- $\Psi_1^{[k]} = \frac{1}{k!} \frac{\partial^k}{\partial \lambda^k} \Psi_1(\lambda)|_{\lambda = \lambda_1}, k = 1, 2, \cdots$. 

30
Since $\Psi_1^{[0]}$ is a solution of (2.2) at $\lambda = \lambda_1$, hence with seed solutions $p$ and $q$ the first iterated GDT solution of (1) becomes,

$$p[1] = p + 2i \frac{(\lambda_1 - \lambda_1^*) \psi_1^{[0]} \varphi_1^{[0]}}{|\psi_1^{[0]}|^2 + |\phi_1^{[0]}|^2 + |\varphi_1^{[0]}|^2},$$

$$q[1] = q + 2i \frac{(\lambda_1 - \lambda_1^*) \phi_1^{[0]} \varphi_1^{[0]}}{|\psi_1^{[0]}|^2 + |\phi_1^{[0]}|^2 + |\varphi_1^{[0]}|^2}.$$ 

(2.52)

Note that there is no difference between the conventional DT and GDT for the first iterated $\Psi$.

### 2.3.2 Second Iteration

Next we obtain the second iterated GDT through the limit process [18]

$$\lim_{\delta \to 0} \frac{T[1]|_{\lambda=\lambda_1+\delta} \Psi_2}{\delta} = \lim_{\delta \to 0} \frac{(\delta + T[1]|_{\lambda=\lambda_1}) \Psi_1(\lambda_1 + \delta)}{\delta}$$

$$= \lim_{\delta \to 0} \frac{(\delta + T[1]|_{\lambda=\lambda_1}) (\Psi_1^{[0]} + \Psi_1^{[1]} \delta + \cdots)}{\delta}$$

$$= \lim_{\delta \to 0} T[1]|_{\lambda=\lambda_1} \Psi_1^{[0]} + T[1]|_{\lambda=\lambda_1} (\Psi_1^{[1]} \delta + \cdots) + \delta (\Psi_1^{[0]} + \Psi_1^{[1]} \delta + \cdots)$$

$$= \lim_{\delta \to 0} (T[1]|_{\lambda=\lambda_1} \Psi_1^{[0]} + \delta (\Psi_1^{[0]} + T[1]|_{\lambda=\lambda_1} \Psi_1^{[1]}) + \delta^2 (\Psi_1^{[1]} + \cdots + T[1]|_{\lambda=\lambda_1} \Psi_1^{[2]} + \cdots))/\delta$$

$$= \lim_{\delta \to 0} (\Psi_1^{[0]} + T[1]|_{\lambda=\lambda_1} \Psi_1^{[1]} + \delta (\Psi_1^{[1]} + \cdots + T[1]|_{\lambda=\lambda_1} \Psi_1^{[2]} + \cdots)$$

$$= \Psi_1^{[0]} + T[1]|_{\lambda=\lambda_1} \Psi_1^{[1]}.$$ 

(2.53)

So we found that

$$\lim_{\delta \to 0} \frac{T[1]|_{\lambda=\lambda_1+\delta} \Psi_2}{\delta} = \Psi_1^{[0]} + T[1]|_{\lambda=\lambda_1} \Psi_1^{[1]} = \Psi_1[1].$$ 

(2.54)

In solving Eq. (2.53) we used that $T[1]|_{\lambda=\lambda_1} \Psi_1^{[0]} = 0$. Using GDT we found $\Psi_1[1]$ is the new iterated eigenfunction of $\lambda_1$ only. Since $T[1]$ is known and we also know $\Psi_1^{[0]}$ and $\Psi_1^{[1]}$ from expression (2.51), hence we can find the exact form of $\Psi_1[1] = (\psi_1[1], \phi_1[1], \varphi_1[1])$.

Substitute $\Psi_1[1]$ into Eq. (2.40) we will obtain the second iterated GDT solution of (1),


$$q[2] = q[1] + 2i \frac{(\lambda_1 - \lambda_1^*) \phi_1[1] \varphi_1[1]^*}{|\psi_1[1]|^2 + |\phi_1[1]|^2 + |\varphi_1[1]|^2}.$$ 

(2.55)
By comparing Eqs. (2.40) and (2.55), we see that eigenvalues and their corresponding eigenfunctions in (2.40) are replaced with $\lambda_1$ and its associated eigenfunction. In order to express (2.55) in a determinant form we need to perform the same limit process on (2.41). As shown in [18] the determinant form of (2.55) will in the following form,

$$p[2] = p + 2i \left| \frac{M_1[2]}{H[2]} \right|_{\lambda_2 \to \lambda_1},$$
$$q[2] = q + 2i \left| \frac{N_2[2]}{H[2]} \right|_{\lambda_2 \to \lambda_1},$$

(2.56)

where

$$H[2]_{\lambda_2 \to \lambda_1} =$$

$$
\begin{pmatrix}
\lambda_1 \psi_1 & \psi_1[1, 1] & \lambda_1^* \varphi_1^* & 0 & \varphi_1^*[1, 1] & 0 \\
\psi_1 & \psi_1[0, 1] & \varphi_1^* & 0 & \varphi_1^*[0, 1] & 0 \\
\lambda_1 \phi_1 & \phi_1[1, 1] & 0 & -\lambda_1^* \varphi_1^* & 0 & \varphi_1^*[1, 1] \\
\phi_1 & \phi_1[0, 1] & 0 & \varphi_1^* & 0 & \varphi_1^*[0, 1] \\
\lambda_1 \varphi_1 & \varphi_1[1, 1] & \lambda_1^* \psi_1^* & -\lambda_1^* \phi_1^* & -\psi_1^*[1, 1] & -\phi_1^*[1, 1] \\
\varphi_1 & \varphi_1[0, 1] & -\psi_1^* & -\phi_1^* & -\psi_1^*[0, 1] & -\phi_1^*[0, 1]
\end{pmatrix},
$$

$$M_1[2]_{\lambda_2 \to \lambda_1} =$$

$$
\begin{pmatrix}
\lambda_1 \psi_1 & \psi_1[1, 1] & \lambda_1^* \varphi_1^* & 0 & \varphi_1^*[1, 1] & 0 \\
\psi_1 & \psi_1[0, 1] & \varphi_1^* & 0 & \varphi_1^*[0, 1] & 0 \\
\lambda_1 \phi_1 & \phi_1[1, 1] & 0 & -\lambda_1^* \varphi_1^* & 0 & \varphi_1^*[1, 1] \\
\phi_1 & \phi_1[0, 1] & 0 & \varphi_1^* & 0 & \varphi_1^*[0, 1] \\
\lambda_1^2 \psi_1 & \psi_1[2, 1] & \lambda_1^2 \varphi_1^* & 0 & -\varphi_1^*[2, 1] & 0 \\
\varphi_1 & \varphi_1[0, 1] & -\psi_1^* & -\phi_1^* & -\psi_1^*[0, 1] & -\phi_1^*[0, 1]
\end{pmatrix},
$$

(2.57)

$$M_2[2]_{\lambda_2 \to \lambda_1} =$$

$$
\begin{pmatrix}
\lambda_1 \psi_1 & \psi_1[1, 1] & \lambda_1^* \varphi_1^* & 0 & \varphi_1^*[1, 1] & 0 \\
\psi_1 & \psi_1[0, 1] & \varphi_1^* & 0 & \varphi_1^*[0, 1] & 0 \\
\lambda_1 \phi_1 & \phi_1[1, 1] & 0 & -\lambda_1^* \varphi_1^* & 0 & \varphi_1^*[1, 1] \\
\phi_1 & \phi_1[0, 1] & 0 & \varphi_1^* & 0 & \varphi_1^*[0, 1] \\
\lambda_1^2 \phi_1 & \phi_1[2, 1] & \lambda_1^2 \varphi_1^* & 0 & -\varphi_1^*[2, 1] & 0 \\
\varphi_1 & \varphi_1[0, 1] & -\psi_1^* & -\phi_1^* & -\psi_1^*[0, 1] & -\phi_1^*[0, 1]
\end{pmatrix},
$$

with

$$\psi_1[j, n] = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1^n} [(\lambda_1 + \delta)^j \psi_1(\lambda_1 + \delta)]_{\delta = 0},$$

$$\phi_1[j, n] = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1^n} [(\lambda_1 + \delta)^j \phi_1(\lambda_1 + \delta)]_{\delta = 0},$$

$$\varphi_1[j, n] = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1^n} [(\lambda_1 + \delta)^j \varphi_1(\lambda_1 + \delta)]_{\delta = 0}, \quad j, n = 0, 1, 2.$$
2.3.3 Third Iteration

Now we will generate third iteration using the same limit process.

\[
\lim_{\delta \to 0} \frac{(T[2]_{\lambda_1+\delta})(\delta + T[1]_{\lambda_1}) \Psi_2}{\delta} = \lim_{\delta \to 0} \frac{(T[2]T[1])_{\lambda = \lambda_1+\delta} \Psi_2}{\delta^2} = \lim_{\delta \to 0} \frac{(\delta + T[2])(\delta + T[1]) \Psi_2}{\delta^2} = \lim_{\delta \to 0} \frac{(\delta + T[2])(\delta + T[1]) (\Psi_1^{[0]} + \Psi_1^{[1]} \delta + \Psi_1^{[2]} \delta^2 + \cdots)}{\delta^2}.
\]

(2.59)

Hence, this implies that

\[
\lim_{\delta \to 0} \frac{(T[2]_{\lambda_1+\delta})(\delta + T[1]_{\lambda_1}) \Psi_2}{\delta} = \Psi_1^{[0]} + (T[2] + T[1])\Psi_1^{[1]} + T[2]T[1]\Psi_1^{[2]} = \Psi_2.
\]

(2.60)

Where we used the results that

\[
T[2]T[1]\Psi_1^{[0]} = 0 \quad \text{and} \quad (T[2] + T[1])\Psi_1^{[0]} + T[2]T[1]\Psi_1^{[1]} = 0.
\]

Using GDT we found \(\Psi_1[2]\), which is the second iterated eigenfunction of \(\lambda_1\) only. Since \(T[1]\) and \(T[2]\) are known using expression (2.60) we can find exact form of \(\Psi_1[2]\). The final form of the third iterated GDT will be as following

\[
\]

\[
\]

where

\[
\]

\[
\]

\[
H[1] = \begin{pmatrix}
\psi[1] & \varphi^*[1] & 0 \\
\phi[1] & 0 & \varphi^*[1] \\
\varphi[1] & -\psi^*[1] & -\phi^*[1]
\end{pmatrix}.
\]

(2.61)
We can also apply this limit process on expression (2.45). As shown in [18] we will get the third iterated solution of the determinant form as following,

\[ p[3] = p + 2i \frac{|M_1[3]|}{|H_1[3]|} \big|_{\lambda \to \lambda_1}, \quad q[3] = q + 2i \frac{|M_2[3]|}{|H_1[3]|} \big|_{\lambda \to \lambda_1} \tag{2.63} \]

where

\[
|M_1[3]| = \begin{pmatrix} A & B & C \\ D & E & F \\ G_1 & H & J \end{pmatrix}, \quad |M_2[3]| = \begin{pmatrix} A & B & C \\ D & E & F \\ G_2 & H_2 & J_2 \end{pmatrix}, \quad |H[3]| = \begin{pmatrix} A & B & C \\ D & E & F \\ G_2 & H_2 & J_2 \end{pmatrix}
\]

and

\[
A = \begin{pmatrix} \lambda_1^2 \psi_1 & \psi_1[2,1] & \psi_1[2,2] \\ \lambda_1 \psi_1[1,1] & \psi_1[1,2] \\ \psi_1[0,1] & \psi_1[0,2] \end{pmatrix}, \quad B = \begin{pmatrix} \lambda_1^3 \varphi^*_1 & 0 & \varphi_1^*[2,1] \\ \lambda_1 \varphi_1[1,1] & 0 & \varphi_1^*[1,1] \\ \varphi_1[0,1] & 0 & \varphi_1^*[0,1] \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \varphi_1^*[2,1] & 0 \\ 0 & \varphi_1^*[1,2] & 0 \\ 0 & \varphi_1^*[0,2] & 0 \end{pmatrix},
\]

\[
D = \begin{pmatrix} \lambda_1^3 \phi_1 & \phi_1[2,1] & \phi_1[2,2] \\ \lambda_1 \phi_1[1,1] & \phi_1[1,2] \\ \phi_1[0,1] & \phi_1[0,2] \end{pmatrix}, \quad E = \begin{pmatrix} 0 & -\lambda_1^2 \varphi_1^*[2,1] & 0 \\ 0 & -\lambda_1^2 \varphi_1^*[1,2] & 0 \\ 0 & -\varphi_1^*[0,2] & 0 \end{pmatrix}, \quad F = \begin{pmatrix} \varphi_1^*[2,1] & 0 & \varphi_1^*[2,2] \\ \varphi_1^*[1,1] & 0 & \varphi_1^*[1,2] \\ \varphi_1^*[0,1] & 0 & \varphi_1^*[0,2] \end{pmatrix},
\]

\[
G_1 = \begin{pmatrix} \lambda_1^3 \psi_1 & \psi_1[3,1] & \psi_1[3,2] \\ \lambda_1 \varphi_1[1,1] & \varphi_1[1,2] \\ \varphi_1[0,1] & \varphi_1[0,2] \end{pmatrix}, \quad H = \begin{pmatrix} 0 & \lambda_1^3 \varphi^*_1 & 0 \\ -\lambda_1 \psi_1^* & -\lambda_1 \phi_1^* & -\varphi^*_1[1,1] \\ -\psi_1^* & -\phi_1^* & -\varphi^*_1[0,1] \end{pmatrix},
\]

\[
J = \begin{pmatrix} -\varphi^*_1[3,1] & 0 & \varphi^*_1[3,2] \\ -\phi^*_1[1,1] & -\psi^*_1[1,2] & -\phi^*_1[1,2] \\ -\phi^*_1[0,1] & -\psi^*_1[0,2] & -\phi^*_1[0,2] \end{pmatrix}, \quad G_2 = \begin{pmatrix} \lambda_1^3 \psi_1 & \psi_1[3,1] & \psi_1[3,2] \\ \lambda_1 \varphi_1[1,1] & \varphi_1[1,2] \\ \varphi_1[0,1] & \varphi_1[0,2] \end{pmatrix},
\]

\[
G_3 = \begin{pmatrix} \lambda_1^3 \psi_1 & \varphi_1[2,1] & \varphi_1[2,2] \\ \lambda_1 \varphi_1[1,1] & \varphi_1[1,2] \\ \varphi_1[0,1] & \varphi_1[0,2] \end{pmatrix}, \quad H_2 = \begin{pmatrix} -\lambda_1^2 \psi_1^* & \lambda_1^2 \phi_1^* & -\psi^*_1[2,1] \\ -\lambda_1 \psi_1^* & -\lambda_1 \phi_1^* & -\varphi^*_1[1,1] \\ -\psi_1^* & -\phi_1^* & -\varphi^*_1[0,1] \end{pmatrix},
\]

\[
J_2 = -\begin{pmatrix} \phi^*_1[2,1] & \psi^*_1[2,2] & \phi^*_1[2,1] \\ \phi^*_1[1,1] & \psi^*_1[1,2] & \phi^*_1[1,2] \\ \phi^*_1[0,1] & \psi^*_1[0,2] & \phi^*_1[0,2] \end{pmatrix},
\]

(2.65)

with

\[
\psi_1[j, n] = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1^n} [(\lambda_1 + \delta)^j \psi_1(\lambda_1 + \delta)]|_{\delta = 0},
\]

\[
\phi_1[j, n] = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1^n} [(\lambda_1 + \delta)^j \phi_1(\lambda_1 + \delta)]|_{\delta = 0},
\]

(2.66)

\[
\varphi_1[j, n] = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1^n} [(\lambda_1 + \delta)^j \varphi_1(\lambda_1 + \delta)]|_{\delta = 0}, \quad j, n = 0, 1, 2, \ldots .
\]

Hence, (2.63) provides third order RW solution of (2.1).
2.3.4 \textit{Nth Iteration}

We can construct a \textit{N}th step GDT by continuing to use the limit process. The final form of the \textit{N}th iterated GDT solution will be as follows,

\[
\begin{align*}
p[N] &= p[N-1] + 2i \frac{(\lambda_1 - \lambda_1^*) \psi_1[N-1] \varphi_1^*[N-1]}{\psi_1[N-1]^2 + |\phi_1[N-1]|^2 + |\varphi_1[N-1]|^2}, \\
q[N] &= q[N-1] + 2i \frac{(\lambda_1 - \lambda_1^*) \phi_1[N-1] \varphi_1^*[N-1]}{\psi_1[N-1]^2 + |\phi_1[N-1]|^2 + |\varphi_1[N-1]|^2},
\end{align*}
\]

(2.67)

where, \[\Psi_1[N-1] = \Psi_1^{[0]} + \left( \sum_{l=1}^{N-1} T_1[l](\lambda_1)\Psi_1^{[1]} + \left( \sum_{l=1}^{N-1} \sum_{k \neq l} T_1[k](\lambda_1)T_1[l](\lambda_1)\Psi_1^{[2]} + \ldots \\
+ \left( T_1[N-1](\lambda_1)T_1[N-2](\lambda_1) \ldots T_1[1](\lambda_1)\Psi_1^{[N-1]} \right),
\]

and

\[
T_1[k] = \Lambda[1] - H_1[k-1] \Lambda[1] H_1[k-1]^{-1}
\]

with \[H_1[k-1] = \begin{pmatrix}
\psi_1[k-1] & \varphi_1^*[k-1] & 0 \\
\phi_1[k-1] & 0 & \varphi_1^*[k-1] \\
\varphi_1[k-1] & -\psi_1[k-1] & -\phi_1^*[k-1]
\end{pmatrix} \]

Expression (2.67) is the \textit{N}th order RW solution of (2.1). Also, in order to obtain the \textit{N}th order iterated GDT, assuming \textit{N}-distinct solutions \(\Psi_i = (\psi_i, \phi_i, \varphi_i)^T\) are given for Lax pair (2.2) at \(\lambda = \lambda_1, \ldots, \lambda_i = \lambda_n\), we can perform the limit process on expression (2.50). We will obtain the following expression,

\[
\begin{align*}
p[N] &= p + 2i \frac{D_2}{D_1}, \\
q[N] &= q + 2i \frac{D_3}{D_1}
\end{align*}
\]

(2.68)

and the determinants \(D_1, D_2\) and \(D_3\) are as following

\[
D_1 = \begin{pmatrix}
(\lambda_1)^{N-1} \psi_1 & \cdots & \psi_1[N-1, N-1] & (\lambda_1)^{N-1} \varphi_1^* & \cdots & \varphi_1^*[N-1, N-1] & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\psi_1 & \cdots & \psi_1[0, N-1] & \varphi_1^* & \cdots & \varphi_1^*[0, N-1] & 0 & \cdots & 0 \\
(\lambda_1)^{N-1} \phi_1 & \cdots & \phi_1[N-1, N-1] & 0 & \cdots & (\lambda_1)^{N-1} \varphi_1^* & \cdots & \varphi_1^*[N-1, N-1] & \cdots & \cdots \\
\vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_1 & \cdots & \phi_1[0, N-1] & 0 & \cdots & \varphi_1^* & \cdots & \varphi_1^*[0, N-1] & \cdots & \cdots \\
(\lambda_1)^{N-1} \varphi_1 & \cdots & \varphi_1[N-1, N-1] & -(\lambda_1)^{N-1} \psi_1 & \cdots & -\psi_1^*[N-1, N-1] & -(\lambda_1)^{N-1} \phi_1^* & \cdots & -\phi_1^*[N-1, N-1] \\
\vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\varphi_1 & \cdots & \varphi_1[0, N-1] & -\psi_1^* & \cdots & -\psi_1^*[0, N-1] & -\phi_1^* & \cdots & -\phi_1^*[0, N-1]
\end{pmatrix}.
\]
2.4 First order RW solution

Using (2.68) through GDT we can obtain the formula for generating the plane wave solution as seed solution.

\[
D_2 = \begin{pmatrix}
(\lambda_1)^{N-1}\psi_1 & \cdots & \psi_1[N-1, N-1] & (\lambda_1)^{N-1}\varphi_1^* & \cdots & \varphi_1^*[N-1, N-1] & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\
\psi_1 & \cdots & \psi_1[0, N-1] & \varphi_1^* & \cdots & \varphi_1^*[0, N-1] & 0 & \cdots & 0 \\
(\lambda_1)^{N-1}\phi_1 & \cdots & \phi_1[N-1, N-1] & 0 & \cdots & (\lambda_1)^{N-1}\varphi_1^* & \cdots & \varphi_1^*[N-1, N-1] & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\
\phi_1 & \cdots & \phi_1[0, N-1] & 0 & \cdots & \varphi_1^* & \cdots & \varphi_1^*[0, N-1] & 0 & \cdots & 0 \\
(\lambda_1)^N\psi_1 & \cdots & \psi_1[N, N-1] & (\lambda_1)^N\varphi_1^* & \cdots & \varphi_1^*[N, N-1] & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\
\varphi_1 & \cdots & \varphi_1[0, N-1] & -\psi_1^* & \cdots & -\psi_1^*[0, N-1] & -\phi_1^* & \cdots & -\phi_1^*[0, N-1]
\end{pmatrix},
\]

where

\[
\psi_1[j, n] = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1^n} [(\lambda_1 + \delta)^j \psi_1(\lambda_1 + \delta)]|_{\delta=0},
\]

\[
\phi_1[j, n] = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1^n} [(\lambda_1 + \delta)^j \phi_1(\lambda_1 + \delta)]|_{\delta=0},
\]

\[
\varphi_1[j, n] = \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1^n} [(\lambda_1 + \delta)^j \varphi_1(\lambda_1 + \delta)]|_{\delta=0}, \quad j, n = 0, 1, 2, \ldots ,
\]

and

\[
(\lambda_i + \delta)^j \psi_i(\lambda_i + \delta) = \lambda_i^j \psi_i + \psi[j, 1] \delta + \psi[j, 2] \delta^2 + \cdots + \psi[j, m_i] \delta^m_i + \cdots ,
\]

\[
(\lambda_i + \delta)^j \phi_i(\lambda_i + \delta) = \lambda_i^j \phi_i + \phi[j, 1] \delta + \phi[j, 2] \delta^2 + \cdots + \phi[j, m_i] \delta^m_i + \cdots ,
\]

\[
(\lambda_i + \delta)^j \varphi_i(\lambda_i + \delta) = \lambda_i^j \varphi_i + \varphi[j, 1] \delta + \varphi[j, 2] \delta^2 + \cdots + \varphi[j, m_i] \delta^m_i + \cdots .
\]

Using (2.68) through GDT we can obtain the \(N\)th order RW solution of GCNLS (2.1) with the plane wave solution as seed solution.

2.4 Multi-RW solution of GCNLS system

In section 2.3 using GDT we obtained the formula for generating \(N\)th order RW solution for the GCNLS system. Therefore, in this section using the explicit formulas from previous section we will construct multi-RW solutions for GCNLS system (2.1).

2.4.1 First order RW solution

For our analysis in this section we choose plane wave solutions as the seed solution. Consider plane wave solutions in this form \(p[0] = a_1 e^{ic_1 t}\) and \(q[0] = a_2 e^{ic_2 t}\) where \(a_1, a_2\)
and $c_1$, $c_2$ are real constants. If we substitute $p[0]$ and $q[0]$ into equation (2.1) and restrict $c_1 = c_2 = c$ and $a = d = 1$ then we obtain the following consistent dispersion relation of the form $\frac{c}{2} = a_1^2 + a_2^2 + (b + b^*)a_1a_2$. Considering the following form of equations for

$$
\Psi = a_1 e^{\frac{ict}{2}} (k_1 e^{\mu_1 x + \mu_1 t + \Phi(f)} + k_2 e^{\mu_2 x + \mu_2 t + \Phi(f)}),
$$

$$
\phi = a_2 e^{\frac{ict}{2}} (k_3 e^{\mu_1 x + \mu_1 t + \Phi(f)} + k_4 e^{\mu_2 x + \mu_2 t + \Phi(f)}),
$$

$$
\varphi = e^{\frac{ict}{2}} (k_5 e^{\mu_1 x + \mu_1 t + \Phi(f)} + k_6 e^{\mu_2 x + \mu_2 t + \Phi(f)}).
$$

where

- $k_i, i = 1, 2, 3, 4, 5, 6$ are real constants.
- $\nu_i$ and $\mu_i$, $i = 1, 2$ are imaginary parameters.
- $\Phi(f)$ is an arbitrary parameter.

Then we can substitute the above given $\psi, \phi, \varphi$ and seed solution into equation (2.22). Solving the resultant set of equation we obtain the following special solution with $\lambda_1 = ih$,

$$
\Psi_1(\lambda_1) = \begin{pmatrix}
(k_1 \frac{a_1}{h + \sqrt{h^2 - c/2}} e^A + k_2 \frac{a_1}{h - \sqrt{h^2 - c/2}} e^{-A}) e^{\frac{ict}{2}} \\
(k_1 \frac{a_2}{h + \sqrt{h^2 - c/2}} e^A + k_2 \frac{a_2}{h - \sqrt{h^2 - c/2}} e^{-A}) e^{\frac{ict}{2}} \\
(k_1 e^A + k_2 e^{-A}) \frac{-ict}{2}
\end{pmatrix},
$$

(2.71)

where

$$
A = \sqrt{h^2 - \frac{c}{2}}(x - 2iht + \Phi(f)), \text{ where } \Phi(f) = \sum_{i=0}^{N} s_i f^{2i}, s_i \in \mathbb{C},
$$

$$
k_1 = \frac{2c[(h - \sqrt{h^2 - c/2})^2 - c/2]}{(c - 2h^2)(h - \sqrt{h^2 - c/2})},
$$

$$
k_2 = \frac{2c[(h + \sqrt{h^2 - c/2})^2 - c/2]}{(c - 2h^2)(h + \sqrt{h^2 - c/2})}.
$$

(2.72)

Substituting expressions (2.71) and (2.72) into (2.23), which is the first iterated DT formula, we can obtain the Akhmediev breather solution [18]. However, in order to obtain RW solution we fix the critical eigenvalue to be, $\lambda_1 = \lambda_0 = ih = i(\sqrt{c/2} + f^2)$. Next, we extend the Eq. (2.51) at $f = 0$ to obtain

$$
\Psi_2 = \Psi_1(\lambda_1 + f^2) = \Psi_1^{[0]} + \Psi_1^{[1]} f^2 + \Psi_1^{[2]} f^4 + \cdots + \Psi_1^{[N]} f^{2N} + \cdots
$$

(2.73)

In order to obtain $\Psi_1^{[0]}$, we need to take limit $f \rightarrow 0$ on the expression (2.71). However, when we take the limit $f \rightarrow 0$ of (2.71) we get indetermination. In order to overcome
this problem we use L’Hospital’s rule and differentiate the numerator and the denominator separately with respect to \( f \) 2 times. Then take the limit \( f \to 0 \) and obtain the expression for \( \Psi_1^{[0]} \). Next we need to obtain \( \Psi_1^{[1]} \). Again, we consider expression (2.71). First we take the derivative of (2.71) twice with respect to \( f \) and then we take limit \( f \to 0 \). However, when we take the limit \( f \to 0 \) we get indetermination and in order to overcome this problem we need to use L’Hospital’s rule. We differentiate the numerator and the denominator separately with respect to \( f \) three times and then take the limit \( f \to 0 \). After which we will obtain the expression for \( \Psi_1^{[1]} \). Similarly we can obtain the expression for \( \Psi_1^{[2]} \). For obtaining \( \Psi_1^{[2]} \), first we take the derivative of (2.71) with respect to \( f \) four times. Then use L’Hospital’s rule and differentiate the numerator and the denominator separately with respect to \( f \) five times. After which we can take limit \( f \to 0 \) and obtain the expression for \( \Psi_1^{[2]} \). We can use similar methodology N-times and obtain \( \Psi_1^{[N]} \).

\[
\Psi_1^{[0]} = \left( \frac{4\sqrt{2}e^{\frac{ict}{2}}[-\sqrt{2} - i\sqrt{2}ct + \sqrt{c}(1 + x)]a_1}{4\sqrt{2}e^{\frac{ict}{2}}[-\sqrt{2} - i\sqrt{2}ct + \sqrt{c}(1 + x)]a_2} \right),
\]

\[
\Psi_1^{[1]} = \left( \begin{array}{c}
\frac{\frac{4}{3}\sqrt{2}e^{\frac{ict}{2}}a_1[-4ic^{5/2}t^3 - 6\sqrt{2}(1 + x)(1 - c^2t^2) + 6c^{3/2}t(-2t + i(1 + x)^2)} + (2\sqrt{c} - \sqrt{2}c(1 + x))(12it + (1 + x)^2) - 6\sqrt{c}s_1}{\frac{4}{3}\sqrt{2}e^{\frac{ict}{2}}a_2[-4ic^{5/2}t^3 - 6\sqrt{2}(1 + x)(1 - c^2t^2) + 6c^{3/2}t(-2t + i(1 + x)^2)} + (2\sqrt{c} - \sqrt{2}c(1 + x))(12it + (1 + x)^2) - 6\sqrt{c}s_1} \\
\frac{4\sqrt{2}e^{-\frac{ict}{2}}[6i\sqrt{2}t - 2i\sqrt{2}c^2t^3 + 6c^{3/2}t^2(1 + x) + 3i\sqrt{2}ct(1 + x)^2 - \sqrt{c}(1 + x)^3 - 3\sqrt{2}s_1]}{\frac{4}{3}\sqrt{2}e^{\frac{ict}{2}}a_1[-4ic^{5/2}t^3 - 6\sqrt{2}(1 + x)(1 - c^2t^2) + 6c^{3/2}t(-2t + i(1 + x)^2)} + (2\sqrt{c} - \sqrt{2}c(1 + x))(12it + (1 + x)^2) - 6\sqrt{c}s_1}
\end{array} \right)
\]

\[
\Psi_1^{[2]} = \left( \begin{array}{c}
\frac{8}{5}e^{\frac{ict}{2}}a_1(\Psi_1^{[2,1]}) \\
\frac{8}{5}e^{\frac{ict}{2}}a_2(\Psi_1^{[2,2]}) \\
\frac{8}{5}e^{\frac{ict}{2}}c(\Psi_1^{[2,3]})
\end{array} \right),
\]

(2.74)
with

\[ \Psi^{[2,1]} = -60 + 30\sqrt{2c} - 10c + \sqrt{2c^3/2} - 240it + 360i\sqrt{2c}t - 300ict + 40i\sqrt{2c^3/2}t \]
\[ -10ict^2 + 60ict^2 - 420\sqrt{2c^3/2}t^2 + 120ict^2 - 20\sqrt{2c^5/2}t^2 + 360ict^2t^3 - 80ict^2t^3 \]
\[ + 40ict^3 - 40ict^4 + 20\sqrt{2c^7/2}t^4 - 8ic^4t^5 - 120x + 90\sqrt{2c}x - 40cx + 5\sqrt{2c^3/2}x \]
\[ + 360i\sqrt{2c}tx - 60ictx + 120i\sqrt{2c^3/2}tx - 40ictx - 420\sqrt{2c^3/2}t^2x + 240ict^2x \]
\[ - 60\sqrt{2c^5/2}t^2x - 80i\sqrt{2c^5/2}tx + 80ict^3x + 20\sqrt{2c^7/2}tx - 60x^2 + 90\sqrt{2c}x^2 \]
\[ - 60c^2x + 10\sqrt{2c^3/2}x^2 - 300ictx^2 + 120i\sqrt{2c^3/2}tx^2 - 60ictx^2 + 120c^2tx^2 \]
\[ - 60\sqrt{2c^5/2}t^2x^2 + 40ict^3x^2 + 30\sqrt{2c}x^3 - 40c^2x^3 + 10\sqrt{2c^3/2}x^3 + 40i\sqrt{2c^3/2}tx^3 \]
\[ - 40ictx^3 - 20\sqrt{2c^5/2}tx^3 - 10c^4 + 5\sqrt{2c^3/2}x^4 + 10ict^4x^4 + \sqrt{2c^3/2}x^5 \]
\[ + 60(2 - 2c^2t^2 - 2\sqrt{2c}(1 + x)/2\sqrt{2c^3/2}(1 + x) + c(4it + (1 + x)^2)s_1 + 60\sqrt{2c}s_2, \]
\[ \Psi^{[2,2]} = \Psi^{[2,1]}, \]

Next we can substitute \( \Psi^{[0]} = (\psi^{[0]}, \phi^{[0]}, \varphi^{[0]})^T \) into equation (2.52) with \( f = 0 \), we will obtain the first order RW solution in the form

\[ \psi^{[1]} = \frac{a_1e^{ict}(4\sqrt{2c}^3/2 - 3c^2 + 8ict - 6c^3t - 4\sqrt{2c}^3/2x - 6c^2x - 3c^2x^2 + 2ma_1^2 + 2ma_2^2)}{c^2(2ct^2 + (1 + x)^2) + 2ma_1^2 + 2ma_2^2}, \]
\[ q^{[1]} = \frac{a_2e^{ict}(4\sqrt{2c}^3/2 - 3c^2 + 8ict - 6c^3t - 4\sqrt{2c}^3/2x - 6c^2x - 3c^2x^2 + 2ma_1^2 + 2ma_2^2)}{c^2(2ct^2 + (1 + x)^2) + 2ma_1^2 + 2ma_2^2}, \]

(2.75)

where, \( m = (2 + 2c^2t^2 - 2\sqrt{2c}(1 + x) + c(1 + x)^2) \).

We can see that \( p^{[1]} \) and \( q^{[1]} \) differ only in amplitude.
Figure 2.1: (a) first order RW solution of the p component, (b) corresponding contour plot. With \( c = 1, a_1 = -1/2, a_2 = 1/2 \).

Figure 2.2: (a) first order RW solution of the q component, (b) corresponding contour plot. With \( c = 1, a_1 = -1/2, a_2 = 1/2 \).

**2.4.2 Second order RW solution**

To obtain the second order RW solution, we use the same limit process as in Eq. (2.53), that is

\[
\Psi_2[1] = \lim_{f \to 0} \frac{[if^2 + T_1[1]]\psi_1(f)}{f^2} = T_1[1]\Psi_1[1] + i\Psi_1[0] = \Psi_1[1]
\]  

(2.76)
with \( T_1[1] = \lambda_1 I - H_1[0] \Lambda[1] H_1[0]^{-1} \).

The expression for \( \Psi_1[0] = (\psi_1[0], \phi_1[0], \varphi_1[0])^T \), \( \Psi_1[1] = (\psi_1[1], \phi_1[1], \varphi_1[1])^T \) is given in Eq. (2.74) and we also know the expression for \( T_1[1] \). Hence, we can substitute \( \Psi_1[0], \Psi_1[1] \) and \( T_1[1] \) in Eq. (2.76) and find exact expression for \( \Psi_1[1] = (\psi_1[1], \phi_1[1], \varphi_1[1])^T \), that is

\[
\psi_1[1] = \frac{4}{3} e^{\frac{ict}{2}} a_1 \left( \frac{A}{B} \right), \\
\phi_1[1] = \frac{4}{3} e^{\frac{ict}{2}} a_2 \left( \frac{A}{B} \right), \\
\varphi_1[1] = \frac{4}{3} e^{-\frac{ict}{2}} c \left( \frac{C}{B} \right),
\]

(2.77)

where,

\[
A = -6i + 6ct + 3i\sqrt{2c(1 + x)} + (4c^2(-6i\sqrt{2c^{3/2}t^3} + 4\sqrt{2c^{5/2}t^4} + 8ic^2t^3(1 + x)
+ 2ct(1 + x)(3t + 2i(1 + x)^2) + 3(1 + x)(-2it + (1 + x)^2) + \sqrt{2c}(6t^2 - 3it(1 + x)^2
- (1 + x)^4) + 3(1 + i\sqrt{2ct + x}s_1)(\lambda_1 - \lambda_1^*)), \\
B = e^2(2ct^2 + (1 + x)^2) + 2ma_1^2 + 2ma_2^2, \\
C = 3\sqrt{2c} + 3i(1 + x) - (4i\sqrt{2} + \sqrt{2ct} - \sqrt{c}(1 + x))(a_1^2 + a_2^2)(4\sqrt{2c^2}t^3
+ 12ic^{3/2}t^2(1 + x) + 6\sqrt{2ct}(-it - (1 + x)^2) + 2\sqrt{c}(1 + x)(6t - i(1 + x)^2)
+ 3\sqrt{2}(2t + i(1 + x)^2) + 3i\sqrt{2}s_1(\lambda_1 - \lambda_1^*)),
\]

and \( m \) is defined as in section 2.4.1 and \( \lambda_1 = i(\sqrt{c/2} + f) \).

We already found exact expressions for \( p[1] \) and \( q[1] \) in Eq. (2.75) and we have expressions for \( \Psi_1[1] \) given in Eq. (2.77). Therefore, we can substitute Eqs. (2.75) and (2.77) into equation (2.55) and obtain the second order RW solution.

\[
p[2] = a_1 e^{ict} \frac{N}{D}, \\
q[2] = a_2 e^{ict} \frac{N}{D},
\]

(2.78)

However, the explicit expression for \( N \) and \( D \) are very long, therefore we will not present the obtained form in this paper. We will analyse the second order RW solution graphically.
Figure 2.3: (a) Second order RW solution of the p component for the values $c = 1, a_1 = -1/2, a_2 = 1/2, s_1 = 0.5 + 0.25i, s_2 = 0.75 + 0.25i$, (c) corresponding contour plot. (b) Second order RW solution of the p component for the values $c = 15, a_1 = -1/2, a_2 = 1/2, s_1 = 0, s_2 = 0$, (d) corresponding contour plot. Similar profile occurs for q also, not shown here.
Figure 2.4: (a) Second order RW solution of the p component for the values $c = 1, a_1 = -3, a_2 = 3, s_1 = 0.5 + 0.25i, s_2 = 0.75 + 0.25i$, (c) corresponding contour plot. (b) Second order RW solution of the p component for the values $c = 1, a_1 = -15, a_2 = 15, s_1 = 0.5 + 0.25i, s_2 = 0.75 + 0.25i$, (d) corresponding contour plot. Similar profile occurs for $q$ also, not shown here.

The results for the second order RW solution is shown in Figs. 2.3 and 2.4. The second order RW solution is derived with five free parameters, namely $c, a_1, a_2, s_1$ and $s_2$. We analyze based on these five parameters. When the values of $c, a_1, a_2$ are small and the values of $s_1$ and $s_2$ differ only by the real part then we have very similar to the classical second order RW solution (shown in Fig. 2.3(a) and (c)). When we increase the value of $c$ and $s_1 = s_2 = 0$ the RW gets deformed (Fig. 2.3 (b) and (d)). When we increase the value of the free parameters $a_1$ and $a_2$, the second order RW splits into two first order RWs, (shown in Fig. 2.4 (a) and (c)). Finally, when we increase the values of $a_1$ and $a_2$ more, the distance between the peaks increases (shown in Fig. 2.4 (b) and (d)).
2.4.3 Third order RW solution

Next, we want to construct the third order RW solution of the GCNLS system (2.1). Using the limit process from previous section we obtain that


(2.79)

We have the exact expressions for \( \Psi_1[0], \Psi_1[1] \) and \( \Psi_1[2] \) in Eq. (2.74). Moreover, we also know the exact expression for \( T_1[1] = \Lambda[1] - H_1[0]\Lambda[1]H_1[0]^{-1} \) and \( T_1[2] = \Lambda[1] - H_1[1]\Lambda[1]H_1[1]^{-1} \) with

\[ H_1[0] = \begin{pmatrix} \psi_1 & \varphi_1^* & 0 \\ \phi_1 & 0 & \varphi_1^* \\ \varphi_1 & -\psi_1^* & -\phi_1^* \end{pmatrix} \quad \text{and} \quad H_1[1] = \begin{pmatrix} \psi_1[1] & \varphi_1^*[1] & 0 \\ \phi_1[1] & 0 & \varphi_1^*[1] \\ \varphi_1[1] & -\psi_1^*[1] & -\phi_1^*[1] \end{pmatrix}. \]  

(2.80)

We can substitute the above equations into Eq. (2.79) and obtain the explicit expression for \( \Psi_1[2] \). Afterwards, we can substitute \( \Psi_1[2] = (\psi_1[2], \phi_1[2], \varphi_1[2])^T \) into equation (2.61) and obtain the third order RW solution for the GCNLS system (2.1). Again, since the explicit form of the third order RW solution is very long that is why we will not present the obtained results here.

2.5 Summary and Conclusion

We discussed the method for constructing \( N \)th order RW solution for generalized coupled nonlinear Schrödinger equations given by Eq. (2.1). We used GDT to construct a \( N \)th order RW solution. We also provided the explicit form of the first order RW solution. We provided explicit expressions for all the terms that are needed to calculate the second order RW solution. Since the final answer for the second order RW is lengthy we analyzed it graphically. Moreover, in this chapter we discussed in detail how to obtain the determinant representation of the \( N \)th order RW solution, which can be used to generate higher order RWs through symbolic manipulation.
Chapter 3

On the characterization of breather and rogue wave solutions and modulation instability of a generalized coupled nonlinear Schrödinger equations

3.1 Introduction

Rogue waves (RWs) in the NLS equations has been studied for many years; however, RWs in coupled NLS equations with higher order mixing effect has not been studied extensively. In this chapter we will study RWs in coupled NLS equations with the four-wave mixing effect.

\[
\begin{align*}
    ip_t + p_{xx} + (a|p|^2 + c|q|^2 + bpq^* + b^*qp^*)p &= 0, \\
    iq_t + q_{xx} + (a|p|^2 + c|q|^2 + bpq^* + b^*qp^*)q &= 0,
\end{align*}
\]

(3.1)

where,

- \(p(x, t)\) and \(q(x, t)\) are slowly varying pulse envelops.
- \(a\) and \(c\) are real constants, they describe self phase modulation and cross phase modulation respectively.
- \(b\) is a complex constant and \(^*\) denotes complex conjugation, describe the four wave mixing effects.

Four-wave mixing is a basic nonlinear phenomenon having fundamental relevance and practical applications particularly in nonlinear optic [19], optical processing [20], phase conjugate optics [21], real time holography [22] and measurement of atomic energy structures and decay rates [23, 24]. Even though for NLS equation the connection between the RW solution and modulation instability has been studied in detail; however, the connection between RWs and modulation instability of coupled NLS equation with four-wave mixing effect has not been studied that much [19]. As mentioned earlier, we will obtain breather and RW solutions for the GCNLS system using Darboux transformation (DT). In order to obtain breather and RW solutions we choose plane wave solution as the seed solution. Also, in order to construct Darboux transformation we need to obtain the Lax pair matrices for coupled NLS equation with four-wave mixing effect. The Lax pair for this equation is a 3×3 matrix. Since the eigenvalue equation of the transformed \(U\) is a cubic polynomial, therefore we will have three different cases of it. We will investigate each case separately because our general solution depends on these three cases. Case 1, all three roots of the cubic polynomial are all different; in this case we will obtain general breather solutions of GCNLS system. Case 2, we will consider one single root and double roots; in this case we will obtain RW solution of GCNLS system. Case 3, there are triple root; we only obtain a trivial solution. Moreover, we will see that for the first two cases we will obtain the solutions by considering a four-wave mixing parameter to be a real constant. After obtaining solutions then we examine the modulation instability of GCNLS system and show that it
can only occur when the four-wave mixing parameter is a real constant.
The plan for this chapter is as following. In section 2, we will construct DT of GCNLS
system (3.1). In section 3, we will derive breather and the RW solution of GCNLS system.
In section 4, we will investigate modulation instability (MI) of GCNLS system. Section 5,
summary and conclusion.

3.2 Darboux transformation (DT) of GCNLS system

First we can rewrite GCNLS system (3.1) into a matrix form [19]

\[ P_t - \frac{1}{2} \tilde{P}_x^2 + P^2 \tilde{P} = 0, \tag{3.2} \]

where,

\[
\begin{pmatrix}
0 & 0 & p \\
0 & 0 & q \\
r_1 & r_2 & 0
\end{pmatrix}
\text{ and } 
\begin{pmatrix}
i & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i
\end{pmatrix},
\tag{3.3}
\]

\( \tilde{P} = [J, P], \quad r_1 = -(a p^* + b q^*) \quad \text{and} \quad r_2 = -(b^* p^* + c q^*). \)

We can easily check that if we substitute (3.3) into (3.2) we will obtain the GCNLS system (3.1). As we introduced in
chapter 2, the associated linear equations of GCNLS system are as following

\[
\begin{align*}
\Psi_x &= (P + \lambda J) \Psi = U \Psi \\
\Psi_t &= (\frac{1}{2} (P \tilde{P} + \tilde{P}_x) - 2 \lambda P - 2 \lambda^2 J) \Psi = V \Psi
\end{align*}
\tag{3.4}
\]

where, \( \lambda \) is a complex spectral parameter and \( \Psi(x, t, \lambda) \) is a three component vector.
We will construct DT of (3.1) by using the following 5 steps [19].

**Step 1:** We choose the plane wave solutions as seed solutions for \( P \), with \( p = p^{[0]} \) and \( q = q^{[0]} \).

**Step 2:** We integrate the Lax pair (3.4) in order to obtain \( \Psi \).

**Step 3:** We consider \( \lambda = \lambda_1 \) and evaluate the linear function \( \Psi \), i.e. \( \Psi_1 = \Psi(x, t, \lambda = \lambda_1) \).

We also introduce two orthogonal auxiliary vectors \( R \) and \( W \), such that

\[
\langle \Psi | R \rangle = 0, \quad \langle \Psi | W \rangle = 0. \tag{3.5}
\]

**Step 4:** We obtain Darboux matrix \( D \) using vectors \( R \) and \( W \),

\[
D = [\Psi_1, R, W]. \tag{3.6}
\]

**Step 5:** Lastly, we obtain first iterated DT solution of GCNLS system,

\[
p[1] = p + 2i \frac{|N_1|}{|D|}, \quad q[1] = q + 2i \frac{|N_2|}{|D|}. \tag{3.7}
\]
where \( N_i, i = 1, 2 \) we obtain by replacing the third row of the matrix \( D \) with the \( i^{th} \) row of matrix \( D \) multiplied by \( \lambda_1, \lambda_1^* \) and \( \lambda_1^* \) respectively.

### 3.3 Breather and rogue wave solution of (1)

In this section we will construct breather and RW solutions of (3.1) by following the steps introduced in section 3.2. As we already said, we will consider plane wave solutions as our seed solutions, that is

\[
p[0] = a_1 e^{i(k_1 x + \omega_1 t)}, \quad q[0] = a_2 e^{i(k_2 x + \omega_2 t)},
\]

(3.8)

where \( a_1, a_2, k_1, k_2, \omega_1 \) and \( \omega_2 \) are all real constants. If we substitute seed solutions (3.8) into Eq. (3.1) we get the following expression

\[
-a_1 \omega_1 e^{i(k_1 x + \omega_1 t)} - a_1 k_1^2 e^{i(k_1 x + \omega_1 t)} + 2(aa_1^2 + ca_2^2 + ba_1 a_2 e^{i((k_1-k_2)x+(\omega_1-\omega_2)t)})
\]

\[
+ b^* a_1 a_2 e^{i((k_2-k_1)x+(\omega_2-\omega_1)t)} a_1 e^{i(k_1 x + \omega_1 t)} = 0
\]

\[
-a_2 \omega_2 e^{i(k_2 x + \omega_2 t)} - a_2 k_2^2 e^{i(k_2 x + \omega_2 t)} + 2(aa_1^2 + ca_2^2 + ba_1 a_2 e^{i((k_1-k_2)x+(\omega_1-\omega_2)t)})
\]

\[
+ b^* a_1 a_2 e^{i((k_2-k_1)x+(\omega_2-\omega_1)t)} a_2 e^{i(k_2 x + \omega_2 t)} = 0.
\]

(3.9)

In order to obtain a dispersion relation the presence of parameter \( b \) and \( b^* \) makes it difficult. Therefore, we will restrict parameters \( k_1 = k_2 \) and \( \omega_1 = \omega_2 \), which will help us to get a consistent dispersion in the following form

\[
-\omega_1 - k_1^2 + 2(aa_1^2 + ca_2^2 + b_R a_1 a_2) = 0,
\]

\[
\Rightarrow \omega_1 = -k_1^2 + 2h, \quad h = (aa_1^2 + ca_2^2 + 2b_R a_1 a_2).
\]

(3.10)

\( 2b_R = b + b^* \). Moreover, with this restrictions our seed solution will become

\[
p[0] = a_1 e^{i(k_1 x + \omega_1 t)}, \quad q[0] = a_2 e^{i(k_1 x + \omega_1 t)}.
\]

(3.11)

The second step of DT construction is integrating Lax pair(3.4). Consider

\( \Psi = (\psi(x,t), \phi(x,t), \varphi(x,t)) \) and follow the same methodology as in section 2.2.1 we get the following six coupled equations

\[
\psi_{1x} = i\lambda_1 \psi_1 + p \varphi_1,
\]

\[
\phi_{1x} = i\lambda_1 \phi_1 + q \varphi_1,
\]

\[
\varphi_{1x} = r_1 \psi_1 + r_2 \phi_1 - i\lambda_1 \varphi_1,
\]

\[
\psi_{1t} = (-2i\lambda_1^2 - i pr_1) \psi_1 - ipr_2 \phi_1 + (ipx - 2p \lambda_1) \varphi_1,
\]

\[
\phi_{1t} = (-2i\lambda_1^2 - i qr_2) \phi_1 - iq r_1 \psi_1 + (iqx - 2q \lambda_1) \varphi_1,
\]

\[
\varphi_{1t} = (-ir_{1x} - 2r_1 \lambda_1) \psi_1 + (-ir_{2x} - 2r_2 \lambda_1) \phi_1 + (2i\lambda_1^2 + ipr_1 + iqr_2) \varphi_1.
\]

(3.12)

Substitute (3.11) into (3.12) and integrating the set of differential equations (3.12) we can get the eigenfunction \( \Psi = (\psi(x,t), \phi(x,t), \varphi(x,t)) \). However, while we integrate (3.12)
we come across to a cubic polynomial. This cubic polynomial is the eigenvalue equation of the transformed $U$, which we obtain using the following expression $\det(\tau I - U) = 0$. Substitute expressions for $P$ and $J$ given in (3.3) into Eq. (3.4) we get matrix $U$ as following

$$U = \begin{pmatrix} i\lambda & 0 & p \\ 0 & i\lambda & q \\ -(ap^* + bq^*) & -(b^*p^* + cq^*) & -i\lambda \end{pmatrix}, \quad (3.13)$$

where $p$ and $q$ are plane wave equations given in Eq. (3.11), and $p^*$ and $q^*$ represent the complex conjugate. Next, we substitute $p,q,p^*$ and $q^*$ into (3.13) and we subtract the obtained matrix $U$ from $\tau I = \text{diag}(\tau,\tau,\tau)$. That is

$$\begin{pmatrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{pmatrix} - \begin{pmatrix} i\lambda & 0 & e^{i(k_1 x + \omega_1 t)}a_1 \\ 0 & i\lambda & e^{i(k_1 x + \omega_1 t)}a_2 \\ -ae^{-i(k_1 x + \omega_1 t)}a_1 - be^{i(k_1 x + \omega_1 t)}a_2 - e^{-i(k_1 x + \omega_1 t)}a_1 b^* - i\lambda \end{pmatrix}. \quad (3.14)$$

Then simplify (3.14) and calculate the determinant of it we obtain the following cubic polynomial

$$\tau^3 - i\lambda \tau^2 - \tau(a_1(-aa_1 - ba_2) - \lambda^2 + a_2(-b^*a_1 - ca_2)) + i\lambda(a_1(-aa_1 - ba_2) - \lambda^2 + a_2(-b^*a_1 - ca_2)) = 0. \quad (3.15)$$

There are three different cases for the roots $\tau_i$. Cubic polynomial (3.15) can have (i) three different roots, (ii) one single and one double root and (iii) three equal roots. Since the general solution of $\Psi$ depends on the roots of (3.15), therefore, we will integrate Eq. (3.12) by consider each case separately.

### 3.3.1 Case 1

First, we will consider the case when all three roots of cubic polynomial are different. Consider the following form of expressions for $\psi$, $\phi$ and $\varphi$, namely

$$\psi(x,t) = \Gamma_1 e^{\nu_1 x + \mu_1 t} + \Gamma_2 e^{\nu_2 x + \mu_2 t} + \Gamma_3 e^{\nu_3 x + \mu_3 t},$$

$$\phi(x,t) = \Gamma_4 e^{\nu_1 x + \mu_1 t} + \Gamma_4 e^{\nu_2 x + \mu_2 t} + \Gamma_4 e^{\nu_3 x + \mu_3 t},$$

$$\varphi(x,t) = e^{-i(k_1 x + \omega_1 t)}f[x,t](\Gamma_7 e^{\nu_1 x + \mu_1 t} + \Gamma_8 e^{\nu_2 x + \mu_2 t} + \Gamma_9 e^{\nu_3 x + \mu_3 t}). \quad (3.16)$$
Substitute (3.14) into equation (3.12) along with seed solution, we will obtain the following expressions

\[
\Gamma_4 = \frac{a_2 \Gamma_1}{a_1}, \quad \Gamma_5 = \frac{a_2 \Gamma_2}{a_1}, \quad \Gamma_6 = -\frac{(a a_1 + b a_2) \Gamma_3}{c a_2 + a_1 b^*}, \quad \Gamma_8 = \frac{\Gamma_2 \Gamma_7 (\lambda_1 + i \nu_2)}{\Gamma_1 (\lambda_1 + i \nu_1)}, \quad \Gamma_9 = 0
\]

\[
\mu_1 = -k_1 \nu_1 - 2 \lambda_1 \nu_1 + 2 i h + i k_1 \lambda_1, \quad \mu_2 = -k_1 \nu_2 - 2 \lambda_1 \nu_1 + 2 i h + i k_1 \lambda_1,
\]

\[
\nu_1 = \frac{1}{2} (i k_1 - \sqrt{-4 h - (k_1 - 2 \lambda_1)^2}), \quad \nu_2 = \frac{1}{2} (i k_1 + \sqrt{-4 h - (k_1 - 2 \lambda_1)^2})
\]

\[
\mu_3 = -2 i \lambda_1^2, \quad \nu_3 = i \lambda_1, \quad f(x, t) = \frac{\Gamma_1 (-i \lambda_1 + \nu_1)}{a_1 \Gamma_7}
\]

Then \(\Psi\) can be rearranged as:

\[
\psi = a_1 (c_1 e^A + c_2 e^B + c_3 e^{(i \lambda x - 2i \lambda^2 t)}),
\]

\[
\phi = a_2 (c_1 e^A + c_2 e^B - \frac{a_1 (a a_1 + b a_2)}{a_2 (c a_2 + b^* a_1)} c_3 e^{(i \lambda x - 2i \lambda^2 t)}),
\]

\[
\varphi = e^{-i(k_1 x + \omega_1 t)} (c_1 L_+ e^A + c_2 L_- e^B),
\]

with,

\[
A = \frac{1}{2} (i k_1 - \sqrt{s}) x + \left(\frac{1}{2} k_1 + \lambda\right) \sqrt{s} - i \left(\frac{1}{2} k_1^2 + i h\right) t,
\]

\[
B = \frac{1}{2} (i k_1 + \sqrt{s}) x + \left(-\frac{1}{2} k_1 + \lambda\right) \sqrt{s} - i \left(\frac{1}{2} k_1^2 + i h\right) t,
\]

\[
s = -k_1^2 - 4(h - k_1 \lambda + \lambda^2),
\]

\[
L_{\pm} = \frac{1}{2} (i k_1 \pm \sqrt{s}) - i \lambda \quad \text{and} \quad \lambda = f + i g.
\]

Where, \(a_1, a_2, k_1, f\) and \(g\) are real parameters and \(c_1, c_2\) and \(c_3\) are integration constants.

Now we can move to our next step for DT construction and choose auxiliary vectors (Step 3) [19], namely

\[
R = (\varphi_1^*, 0, -\psi_1^*)^T \quad \text{and} \quad W = (0, \varphi_1^*, -\phi_1^*)^T
\]

(3.19)

Once we have auxiliary vectors \(R\) and \(W\) then we can move to (Step 4) and define a Darboux matrix \(D\) in the following form

\[
D = \begin{pmatrix}
\psi_1 & \varphi_1^* & 0 \\
\phi_1 & 0 & \varphi_1^* \\
\varphi_1 & -\psi_1^* & -\phi_1^*
\end{pmatrix}
\]

(3.20)

Since we have matrix \(D\), then we can define matrices \(N_i, \quad i = 1, 2\). That is,

\[
N_1 = \begin{pmatrix}
\psi_1 & \varphi_1^* & 0 \\
\phi_1 & 0 & \varphi_1^* \\
\lambda_1 \psi_1 & \lambda_1^* \varphi_1^* & 0
\end{pmatrix}
\]

(3.21)

and

\[
N_2 = \begin{pmatrix}
\psi_1 & \varphi_1^* & 0 \\
\phi_1 & 0 & \varphi_1^* \\
\lambda_1 \phi_1 & 0 & \lambda_1^* \varphi_1^*
\end{pmatrix}
\]
Lastly, we can substitute expressions (3.20) and (3.21) into (3.7) and evaluate the determinants we will obtain the exact form of first iterated DT solution of GCNLS system.

\[
\begin{align*}
p[1] &= p + 2i \frac{(\lambda_1 - \lambda_1^*) \psi_1 \varphi_1^*}{|\psi|^2 + |\phi|^2 + |\varphi|^2}, \\
q[1] &= q + 2i \frac{(\lambda_1 - \lambda_1^*) \varphi_1 \psi_1^*}{|\psi|^2 + |\phi|^2 + |\varphi|^2}.
\end{align*}
\]

(3.22)

Since we obtained expressions for \(\psi, \phi\) and \(\varphi\) in (3.17) thus we can substitute those expressions into (3.22). After simplifying resultant expressions we arrive at the following general breather solution of GCNLS system,

\[
p = a_1 e^{i(k_1 x + \omega_1 t)} (1 + \frac{2i(\lambda - \lambda^*)(L^+ c_1^2 e^{A^+ A^*} + L^* c_1 c_2 e^{A+ B^*} + L^*_c c_1 c_2 e^{A^*+ B} + L^*_c c_2^2 e^{B+B^*})}{G_1 c_1^2 e^{A^+ A^*} + G_2 c_1 c_2 e^{A+ B^*} + G_3 c_1 c_2 e^{A^*+ B} + G_4 c_2^2 e^{B+B^*}}),
\]

\[
q = a_2 e^{i(k_1 x + \omega_1 t)} (1 + \frac{2i(\lambda - \lambda^*)(L^+ c_1^2 e^{A^+ A^*} + L^* c_1 c_2 e^{A+ B^*} + L^*_c c_1 c_2 e^{A^*+ B} + L^*_c c_2^2 e^{B+B^*})}{G_1 c_1^2 e^{A^+ A^*} + G_2 c_1 c_2 e^{A+ B^*} + G_3 c_1 c_2 e^{A^*+ B} + G_4 c_2^2 e^{B+B^*}}),
\]

(3.23)

with

\[
\begin{align*}
G_1 &= a_1^2 + a_2^2 + L^*_+, \\
G_2 &= a_1^2 + a_2^2 + L^*_+, \\
G_3 &= a_1^2 + a_2^2 + L^*_-, \\
G_4 &= a_1^2 + a_2^2 + L^*_-.
\end{align*}
\]

(3.24)

The general breather solution (3.23) satisfies equation (3.1) if the following conditions are satisfied

\[
(a - 1) + b \frac{a_2}{a_1} = 0, \quad b^* + (c - 1) \frac{a_2}{a_1}, \quad c_3 = 0.
\]

(3.25)

Since we restricted \(k_1 = k_2\) and \(\omega_1 = \omega_2\) therefore we see that \(p\) and \(q\) are linearly dependent. Since \(p[0]\) and \(q[0]\) differ only in amplitude, thus the solutions which we obtained from this seed solution also differ only in amplitude.
3.3.2 Case 2

In this sub-section we consider the case where the cubic polynomial has one single and one double roots. In this case, in order to have two equal roots we set the discriminant to be zero. Solving the discriminant equation

\[-4(aa_1^2 + ca_2^2 + a_1a_2(b + b^*))^2(\lambda^2 + aa_1^2 + ca_2^2 + a_1a_2(b + b^*)) = 0,\]

we obtain restriction between \(\lambda\) and parameters \(a, c,\) and \(b,\) namely

\[\lambda = \pm i \sqrt{(aa_1^2 + ca_2^2 + 2bRa_1a_2)} = \pm i \sqrt{h}.\]
Using this restriction we can solve equation (3.12) and obtain the exact form of $\Psi_1$. However, after obtaining $\Psi_1$ and following Step 3-5 we obtain RW solution of the GCNLS system. Since we already showed the detailed work for obtaining RW solutions in Chapter 2, thus we will not discuss the details for obtaining RW solutions in this subsection.

Figure 3.2: (a) First order RW profile of $p$ component. (b) First order RW profile of $q$ component for the parameter values $a_1 = 2, a_2 = 1, b_R = 1, c_1 = 0.5, c_2 = 1$ and $a- = 1/2, c = 1/2, k_1 = 1/2$, (c) and (d) corresponding contour plots for $p$ and $q$ respectively.

3.3.3 Case 3

Lastly, we consider the case when all three roots of the cubic polynomial are equal. We can rewrite the cubic polynomial as following

$$x^3 + \alpha x^2 + \beta x + \gamma = 0,$$

(3.26)
where
\[ \alpha = -i\lambda, \quad \beta = -(a_1(-a_1a_2 - b a_2 - \lambda^2 + a_2(-b^*a_1 - c a_2)), \]
\[ \gamma = i\lambda(a_1(-a_1a_2 - b a_2 - \lambda^2 + a_2(-b^*a_1 - c a_2)). \] (3.27)

Moreover, in order for the cubic polynomial (3.26) to have all equal roots \( \alpha, \beta \) and \( \gamma \) should satisfy the following two conditions
\[ \alpha^2 - 3\beta = 0, \quad \alpha^3 - 27\gamma = 0. \] (3.28)

If we substitute (3.27) into the first condition of (3.28) we find that
\[ \lambda = \pm \frac{1}{2}i\sqrt{3}\sqrt{h}. \] (3.29)

Then substitute (3.29) into the second condition of (3.26). We end up with \( h=0 \), which will result in \( \lambda \) being 0 as well. If \( \lambda = 0 \), the DT formula (3.20) will only provide a trivial solution. Hence, we can conclude that the third case does not provide any new solution.

### 3.4 Modulation instability and the criterion for existence of RW

The goal of this section is to analyze MI of GCNLS system. To study MI, we consider steady state solution as a seed solution of Eq. (3.1). That is
\[ p = a_1e^{i\omega t}, \quad q = a_2e^{i\omega t}, \] (3.30)
with \( \omega = 2h = 2(aa_1^2 + ca_2^2 + a_1a_2(b + b^*)). \) Then we check if the steady state equation (3.28) is stable against a small perturbation. Thus we perturb the steady state solution in the following way
\[ p = (a_1 + U_1)e^{i\omega t}, \quad q = (a_2 + U_2)e^{i\omega t} \] (3.31)

where \( U_j(x,t), j = 1, 2, \) are weak perturbations [19]. Next, substitute (3.29) into (3.1). Linearizing the equations in \( U_1 \) and \( U_2 \), that is substituting
\[ U_1^2 = 0, \quad U_2^2 = 0, \quad U_1^*U_1^* = 0, \quad U_2^*U_2^* = 0, \quad U_1U_2 = 0, \quad U_1U_1^* = 0, \quad U_2U_2^* = 0, \] (3.32)

Note that the \( U_i, i = 1, 2 \) are very small.

We obtain the following equations
\[ iU_{1t} + U_{1xx} + 2[aa_1^2(U_1 + U_1^*) + ca_2a_2(U_2 + U_2^*) + b(a_1^2U_2^* + a_2a_2U_2^* + b^*(a_2U_2 + a_1a_2U_2^*)) = 0, \]
\[ iU_{2t} + U_{2xx} + 2[aa_1a_2(U_1 + U_1^*) + ca_2^2(U_2 + U_2^*) + b(a_1a_2U_2^* + a_2^2U_1^* + b^*(a_1a_2U_2 + a_2^2U_1^*)) = 0. \] (3.33)

To solve equation (3.33) we assume the following form of general solution [19]
\[ U_j = u_j cos(Kt - \Omega x) + iv_j sin(Kt - \Omega x), \quad j = 1, 2, \] (3.34)
where

- $K$ is the wave number.
- $\Omega$ is the frequency of perturbation.

Then we can substitute Eq. (3.34) into (3.33). Splitting the resultant equation into imaginary and real parts and letting the coefficients be equal zero, we obtain the following equation

$$
\begin{align*}
(-\Omega^2 + 4aa_1^2 + 4b_Ra_1^2)a_1 + (4ca_1a_2 + 4b_Ra_2^2)a_2 - Kv_1 &= 0, \\
-(Ku_1 + \Omega^2v_1) &= 0, \\
(4aa_1a_2 + 4b_Ra_2^2)a_1 + (-\Omega^2 + 4aa_2^2 + 4b_Ra_1a_2)a_2 - Kv_2 &= 0, \\
-(Ku_2 + \Omega^2v_2) &= 0.
\end{align*}
$$

(3.35)

We obtained the above equation by restricting $b$ to be a real constant, that is $b = b^*$. If the determinant of equation (3.33) does not vanish then we will have nontrivial solution. We can rewrite Eq. (3.33) in a matrix form, that is

$$
\begin{pmatrix}
-\Omega^2 + 4aa_1^2 + 4b_Ra_1a_2 & 4ca_1a_2 + 4b_Ra_2^2 & -K & 0 \\
-K & 0 & -\Omega^2 & 0 \\
4aa_1a_2 + 4b_Ra_2^2 & -\Omega^2 + 4aa_2^2 + 4b_Ra_1a_2 & 0 & -K \\
0 & -K & 0 & -\Omega^2
\end{pmatrix},
$$

(3.36)

and the determinant of matrix (3.34) is as following

$$
K^4 + (4aa_1^2 + 4ca_2^2 + 8b_Ra_1a_2 - 2\Omega^2)\Omega^2 K^2 - (4aa_1^2 + 4ca_2^2 + 8b_Ra_1a_2 - \Omega^2)\Omega^6 = 0.
$$

(3.37)

Solving the above determinant equation we obtain the following dispersion relation

$$
K = \pm \sqrt{(-4aa_1^2 - 4ca_2^2 - 8b_Ra_1a_2 + \Omega^2)\Omega^2},
$$

(3.38)

We can clearly see that the stability of the steady state depends on whether light experiences self-focusing or self-defocusing effects inside the fiber [19]. When light experiences self-defocusing effect, that is when $a, c, b_R < 0$, then $K$ is real for all $\Omega$. Hence, steady state is stable against small perturbations. However, in the case of self-modulation, that is $a, c, b_R > 0$, $K$ becomes imaginary for $\Omega < \Omega_c = \sqrt{4aa_1^2 + 4ca_2^2 + 8b_Ra_1a_2}$. Thus, the perturbations $U_j(x, t), j = 1, 2$, grow exponentially with $t$. Moreover, this instability is referred to as the MI [19]. Consider the gain spectrum of MI.

$$
g(\Omega) = 2Im(K) = |\Omega|(\Omega^2 - (4aa_1^2 - 4ca_2^2 - 8b_Ra_1a_2 + \Omega^2))^{1/2}
$$

(3.39)

Fig. 3.3 shows the gain spectra at three power levels.
From our experiment we saw that MI occurs in Eq. (3.1) only when we restricted four wave mixing parameter $b$ to be real. Hence, we can conclude that the self-focusing GCNLS system (3.1) admits MI only when the four wave mixing parameter becomes real.

### 3.5 Summary and Conclusion

We studied breather and RW solutions of general coupled nonlinear Schrödinger equations given by Eq. (3.1) and we obtained these solutions by using the DT method. While solving the system of ODEs we came across a third order characteristic polynomial equation.

- When all three roots of the cubic polynomial are different we obtained general breather solution.
- When two of the roots are equal and one is different we obtained RW solution.
- When all three roots are equal, we end up at a trivial solution.

We also analyzed the MI of Eq. (3.1) and showed that it only occurs when $b^* = b$. 

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**Figure 3.3:** (a) Gain spectra of modulation instability for the parameter values $a = 1, c = 1, b_R = 0.5$ and for the different power levels (a) $a_1 = 0.5, a_2 = 0.6$, (b) $a_1 = 1, a_2 = 1.5$. (c) $a_1 = 1, a_2 = 2.5$
General Conclusion

In this thesis we briefly discussed the history of Lax pair. Also, we have showed how to derive the Lax equation for Nonlinear Schrödinger equation, Hirota Equation and general coupled nonlinear Schrödinger equation.

We have discussed the method for constructing a $N$th order RW solution for general coupled nonlinear Schrödinger equation. Since it is very hard to obtain the $N$th order RW solution using modified Darboux transformation (DT), thus we used generalized Darboux transformation (GDT) to construct $N$th order RW solution for GCNLS system. Using this transformation we were able to obtain explicit expression of first, second and third order RW solution. Moreover, we were able to discuss in detail how to obtain the recursive formula of $N$th order RW solution. Even though the purpose of this thesis was to provide the recursive formula of $N$th order RW solution but we also provided the determinant expression for $N$th order RW solution.

Moreover, we derived the explicit form for first, second and third order RW solutions. Since some of the expressions were very long therefore for third iterated GDT we only provided the formula of it. We have derived the second order RW solution with five parameters. We also analyzed the second order RW solutions by varying these free parameters. Through our construction we were able to give recursive expression for $N$ order, which can be used to generate higher order RW solutions.

In addition to the above work, we also studied the breather solution of GCNLS system. In order to obtain the breather solution we used the DT method. First we chose plane wave solution as the seed solutions. Since the Lax pair for GCNLS system is a $3 \times 3$ matrix, therefore, the characteristic polynomial equation that we obtained was a cubic polynomial. The nature of the solutions depends on the roots of the characteristic polynomial. Therefore we considered each case separately.

When all the roots of the polynomial are different, we have obtained the general breather solution. In the case, when all three roots are equal we ended up with trivial solution. We have demonstrated that one can only obtain the general breather and RW solution for GC-NLS system.

Moreover, we investigated MI of GCNLS system and we saw that it can only occur when our four wave mixing parameter is real.
References


