INITIAL-BOUNDARY VALUE PROBLEMS FOR THE WAVE EQUATION

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by

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Abstract

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In this thesis we give an overview of the wave equation in two and three dimensional space variables. We study initial value problems, known as Cauchy problems, for the wave equation in two and three dimensional space variables. Huygen’s principle and d’Alembert’s formula are discussed. Several examples are studied in detail. Also, we consider initial-boundary value problems for the wave equation. We state and prove the existence and uniqueness results for both the Cauchy and initial-boundary value problems associated with the wave equation.
Introduction

For several of the wave phenomena that we might see or experience around us, we can derive the partial differential equations that govern their behavior. Even though the primary physical factors that give rise to these wave phenomena differ, by examining these partial differential equations it is possible to find some common form for them. Vibration and wave propagation phenomena are governed by partial differential equations known as wave equations.

Our goal in this thesis is to study initial and initial-boundary value problems arising in the study of wave equations. This thesis is organized as follows.

In Chapter 1, we describe some simple solutions of the equation, including solutions known as plane waves and spherical waves. Also, we study and derive the solution for the one-dimensional wave equation. These problems have been extensively studied, for example, see [1], [2], [5], [8] and [10].

In Chapter 2, we study initial-value problems, and show that Poisson's formula gives the solution of initial-value problems in three space dimensions. The method of descent is then used to obtain the corresponding solution in two space dimensions. The uniqueness of solution of an initial-value problem is also proved.

In Chapter 3, we study initial-boundary value problems in one and two space dimensions. The method of solution involves separation of variables. Also, we study the wave equation on the half-line. The uniqueness of solution of an initial-boundary value problem is also proved.
In Chapter 4, we study initial value problems for the nonhomogeneous wave equation in one space dimension.
Chapter 1

1.1 Special Solutions of the Wave Equation. Plane and Spherical Waves

The most useful and simplest partial differential equation of hyperbolic type is the wave equation,

$$\frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{1.1}$$

where $x_1, \ldots, x_n$ and $t$ are called space and time variables, respectively.

Let $x = (x_1, \ldots, x_n)$ where $x \in \mathbb{R}^n$. Here, $\mathbb{R}^n$ is known as the $x$-space. In wave equation (1.1), $n$ indicates the number of space variables. Equation (1.1) is an $n$-dimensional wave equation where $u$ is a function defined in the space $\mathbb{R}^{n+1}$. So, the solution of the wave equation (1.1) is a function of $n+1$ variables. Sometimes we represent $x$-space by one axis for $n \geq 3$ as shown in Figure 1.1

![Figure 1.1](image)

In order to find some solutions of the wave equation, we need a unit vector $\xi_x = (\xi_1, \ldots, \xi_n)$, in $\mathbb{R}^n$
\[ \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 = 1. \] (1.2)

The equation
\[ \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n - t = c \]
represents a plane in \( \mathbb{R}^n \) for each constant \( c \) and each fixed \( t \). \( \xi \) is unit vector which is normal to this plane. As \( t \) increases, the plane travels in the direction of \( \xi \) with speed 1. Let \( G(y) \) be a \( C^2 \) function of a single variable \( y \). Then,
\[ u(x_1, \ldots, x_n, t) = G(\xi_1 x_1 + \ldots + \xi_n x_n - t) \] (1.3)
is a solution (known as plane wave) of the wave equation since if we put
\[ y = \xi_1 x_1, \ldots, \xi_n x_n - t, \]
then
\[ \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} - \frac{\partial^2 u}{\partial t^2} = G'(y)\xi_1^2 + \cdots + G'(y)\xi_n^2 + G'(y). \]

We illustrate plane wave solutions of wave equation for \( n = 1 \) and \( n = 2 \).

Let us look at the special case \( \xi = (1) \), and consider one-dimensional wave equation
\[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \] (1.4)
then from (1.2) we get \( \xi_1^2 = 1 \), which implies that \( \xi_1 = \pm 1 \). Then the traveling points in \( \mathbb{R}^1 \) are
\[ c = x_1 - t \] (1.5)
and
\[ c = x_1 + t. \] (1.6)
Equations (1.5) and (1.6) describe a point traveling in the positive \( x_1 \) direction and traveling in the negative direction with speed 1, respectively. Since \( c \) is constant, the
point \( x_i \) satisfies \( \frac{\partial x_i}{\partial t} = 1 \) and or \( x_i \) is traveling in the positive direction with speed 1.

The corresponding plane wave solutions are

\[ u(x_i, t) = F(x_i - t) \tag{1.7} \]

and

\[ u(x_i, t) = G(x_i + t), \tag{1.8} \]

where \( F \) and \( G \) are arbitrary \( C^2 \) functions of a single variable. Solutions of the form (1.7) represent waves traveling in the positive \( x_i \) direction with speed 1. The one-dimensional wave equation describes the motion of a stretched string which lies along the \( x_1 \)-axis when in equilibrium. Figure 1.2 shows the \((x_1, t)\)-plane on which the solution of (1.7) is defined. The region \( u(x_i, t) \neq 0 \) is indicated the shaded strip. Each line of slope are within the region corresponds to a value of \( c \) that gives a vertical displacement \( G(c) \) of the string. Similarly, we can describe the solution of the form (1.8) which represents waves traveling in the negative \( x_i \) direction.

Let us look at plane wave solution of the wave equation for \( n = 2 \). We consider the wave equation in three independent variables

\[ \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial t^2} = 0. \tag{1.9} \]

There are infinitely many unit vectors \( \xi \). Let us consider the case where \( \xi = \left( \frac{1}{3}, \frac{2\sqrt{2}}{3} \right) \).

Then a traveling line in \( \mathbb{R}^2 \) is given by

\[ \frac{1}{3} x_1 + \frac{2\sqrt{2}}{3} x_2 - t = c. \tag{1.10} \]
This equation describes a line in the space $\mathbb{R}^3$ traveling in the direction of its normal $(\frac{1}{3}, \frac{2\sqrt{2}}{3})$ with speed 1 (see Figure 1.3). The corresponding plane wave solutions of (1.9) are

$$u(x_1, x_2, t) = F(\frac{1}{3} x_1 + \frac{2\sqrt{2}}{3} x_2 - t),$$

(1.11)

and

$$u(x_1, x_2, t) = G(\frac{1}{3} x_1 + \frac{2\sqrt{2}}{3} x_2 + t),$$

(1.12)

where $F$ and $G$ are arbitrary $C^2$ functions of a single variable.

Every solution for one-dimensional wave equation can be expressed as the sum of two plane waves, as we show in section 1.2, but for $n \geq 2$ not all solutions are finite sums of plane waves.
Therefore, we introduce another type of solution of the wave equation known as a spherical wave. It is a solution of the wave equation whose value is constant on spheres in the $x$-space centered at the origin. To define the spherical waves, we introduce spherical coordinates in the $x$-space.

\[ \frac{3(t_0 + c)}{2\sqrt{2}} \]

\[ 3(t_0 + c) \]

Figure 1.3

The Laplacian operator in terms of spherical coordinates in $\mathbb{R}^n$, is given by the following formula

\[ \Delta u = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} (r^{n-1} \frac{\partial u}{\partial r}) + \frac{1}{r^2} \Lambda_s u \]

where $\Lambda_s$ is a second order partial differential operator involving differentiations with respect to the angular variables only, see [11]. In $\mathbb{R}^3$, 

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\[ \Lambda_3 u = \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2}. \]

Hence, the wave equation (1.1), using spherical coordinates in the \( \mathbb{R}^n \), takes the form

\[ \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \Lambda_n u - \frac{\partial^2 u}{\partial t^2} = 0. \]  

(1.13)

By definition, a spherical wave is a solution of (1.13) which depends only on \( r \) and \( t \), and does not depend on angular variables. Therefore a spherical wave is a function \( u = u(r, t) \) which satisfies the wave equation

\[ \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial u}{\partial r} \right) - \frac{\partial^2 u}{\partial t^2} = 0, \]  

(1.14)

and a function \( u = u(r, t) \) is a spherical wave which satisfies equation (1.14). Equation (1.14) is the equation of spherical waves.

The equation for spherical waves for \( n = 3 \) is

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) - \frac{\partial^2 u}{\partial t^2} = 0. \]  

(1.15)

Substituting \( h = ru \) for the dependent variable into (1.15), then the equation for \( h \) is found to be a one-dimensional wave equation

\[ \frac{\partial^2 h}{\partial r^2} - \frac{\partial^2 h}{\partial t^2} = 0. \]  

(1.16)

In this section, we show that every solution of one-dimensional wave equation can be expressed as the sum of solutions of the form \( F(r - t) \) and \( G(r + t) \), where \( F \) and \( G \) are \( \mathcal{C}^2 \) functions of a single variable. Using this fact, we can express the solution of (1.16) as the sum of solutions of the form
\[ u(r, t) = \frac{F(r - t)}{r} \]  
\[ u(r, t) = \frac{G(r + t)}{r} . \]  

Solutions (1.17) and (1.18) represent expanding and contracting spherical waves. The speed of expanding or contracting of these spherical waves is one. 

For \( n = 2 \), equation (1.14) is known as equation of cylindrical waves, which takes the form 

\[ \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) - \frac{\partial^2 u}{\partial t^2} = 0 . \]  

To solve equation (1.19), we use method of separation of variables, because it is not easy to solve as (1.14). Further study of finding solutions of the wave equation by using the method of separation of variables is indicated in Chapter 3.

### 1.2 D’Alembert’s Formula

Let us find the general solution of the wave equation for \( n = 1 \). We consider the wave equation in two independent variables 

\[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 . \]  

Suppose that the function \( u(x, t) \) is a solution of (1.4), and \( u(x, t) \) belongs to the class of \( C^2(\mathbb{R}^2) \), where \( \mathbb{R}^2 \) denotes the \( x, t \)-plane. Introduce new independent variables \( \xi \) and \( \tau \) by the equations,
\[ \xi = x + t \]

and

\[ \tau = x - t , \]

and set

\[ \nu(\xi, \tau) = u\left(\frac{\xi + \tau}{2}, \frac{\xi - \tau}{2}\right). \]

The function \( \nu(\xi, \tau) \) belongs to the class of \( C^2(\mathbb{R}^2) \), where \( \mathbb{R}^2 \) denotes the \( \xi, \tau \) plane.

By the chain rule,

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \tau} , \]

\[ \frac{\partial^3 u}{\partial x^2} = \frac{\partial^2 v}{\partial \xi^2} + 2 \frac{\partial^2 v}{\partial \xi \partial \tau} + \frac{\partial^2 v}{\partial \tau^2} , \]

\[ \frac{\partial u}{\partial t} = \frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \tau} , \]

and

\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial \xi^2} - 2 \frac{\partial^2 v}{\partial \xi \partial \tau} + \frac{\partial^2 v}{\partial \tau^2} . \]

Then

\[ \frac{\partial^2 u - \partial^2 u}{\partial x^2 - \partial t^2} = 0 = -4 \frac{\partial^2 v}{\partial \xi \partial \tau} \]

and we have derived the canonical form of the wave equation (1.4):

\[ \frac{\partial^2 v}{\partial \xi \partial \tau} = 0 . \]

Then there is a function \( g(\tau) \) in the class of \( C^1(\mathbb{R}) \) and a function \( G(\tau) \) in the class of \( C^2(\mathbb{R}) \) such that
\[
\frac{\partial \nu}{\partial \tau} = g(\tau)
\]

and

\[
\frac{\partial}{\partial \tau} (\nu - G) = 0.
\]

Further there is a function \( F(\xi) \) in the class \( C^2(\mathbb{R}) \) such that

\[
\nu(\xi, \tau) - G(\tau) = F(\xi).
\]

It follows that

\[
u(x,t) = F(x+t) + G(x-t).\] (1.20)

Equation (1.20) is known as d'Alembert's solution, which is the general solution of the wave equation for (1.4), where \( F \) and \( G \) are arbitrary functions having two continuous derivatives. The solution \( F(x+t) \) can be interpreted as a solution of the wave equation moving to the left, and \( G(x-t) \) can be interpreted as a solution of the wave equation moving to the right. The speed of these traveling waves is 1.

If we consider the unnormalized wave equation

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}
\] (1.21)

then the corresponding general solution will be

\[
u(x,t) = F(x+ct) + G(x-ct),
\]

as we now show. Let

\[
\xi = \alpha x + \beta t
\]

and

\[
\tau = \gamma x - \delta t
\]

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where \( \alpha, \beta, \gamma \) and \( \delta \) are constants. We need to reduce partial differential equation to a canonical form, then

\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \tau} \frac{\partial \tau}{\partial t} = \beta \frac{\partial u}{\partial \xi} - \delta \frac{\partial u}{\partial \tau},
\]

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left( \beta \frac{\partial u}{\partial \xi} - \delta \frac{\partial u}{\partial \tau} \right)
\]

and

\[
\frac{\partial^2 u}{\partial t^2} = \beta^2 \frac{\partial^2 u}{\partial \xi^2} - 2\beta \delta \frac{\partial^2 u}{\partial \xi \partial \tau} + \delta^2 \frac{\partial^2 u}{\partial \tau^2}.
\]

Similarly,

\[
\frac{\partial^2 u}{\partial x^2} = \alpha^2 \frac{\partial^2 u}{\partial \xi^2} + 2\alpha \gamma \frac{\partial^2 u}{\partial \xi \partial \tau} + \gamma^2 \frac{\partial^2 u}{\partial \tau^2}.
\]

Substituting these into wave equation (1.21), yields

\[
(\alpha^2 - c^{-2} \beta^2) \frac{\partial^2 u}{\partial \xi^2} - 2(\alpha \gamma - c^{-2} \delta \beta) \frac{\partial^2 u}{\partial \xi \partial \tau} + \gamma^2 \frac{\partial^2 u}{\partial \tau^2} = 0.
\]

Select \( \alpha, \beta, \delta \) and \( \gamma \) so that

\[
\begin{align*}
\alpha^2 - c^{-2} \beta^2 &= 0 \\
\gamma^2 - c^{-2} \delta^2 &= 0,
\end{align*}
\]

we get \( \alpha^2 = \beta^2 c^{-2} \), which implies that

\[\gamma = \delta c^{-1} \quad \text{and} \quad \alpha = \beta c^{-1}.\]

Therefore, our characteristic equation is
\[ \xi = \alpha(x + ct), \]
\[ \tau = \gamma(x - ct), \]
and our wave equation reduces to \[ 4\alpha\gamma \frac{\partial^2 u}{\partial \xi \partial \tau} = 0. \] There is no loss in generality in assuming \( \alpha = 1 \) and \( \gamma = 1 \). Then
\[ \frac{\partial^2 u}{\partial \xi \partial \tau} = 0. \]
This implies that \( \frac{\partial u}{\partial \xi} = f(\xi) \) for some arbitrary function of \( \xi \). Integration of this equation with respect to \( \xi \) yields
\[ u(\xi, \tau) = \int F(\xi) d\xi + G(\tau) \]
and
\[ u(\xi, \tau) = F(\xi) + G(\tau). \]
The general solution would be
\[ u(x, t) = F(x + ct) + G(x - ct) \]
where \( F(x + ct) \) can be interpreted as a solution of the wave equation moving to the left, and \( G(x - ct) \) can be interpreted as a solution of the wave equation moving to the right. The speed of these traveling waves is \( c \).

Now, we are going to use the general solution of wave equation to derive d'Alembert's formula for the initial value problem. To do so, we need to find a twice continuously differentiable function \( u = (x, t) \) such that
\[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad x \in \mathbb{R}, \quad 0 \leq t < \infty, \quad (1.22) \]
\[ u(x,0) = \phi_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \phi_1(x), \quad (1.23) \]

where \( \phi_0 \in C^2(-\infty, \infty) \) and \( \phi_1 \in C^1(-\infty, \infty) \). We can write \( u(x,t) = \phi(x-t) + \psi(x+t) \), as shown by (1.20). Then

\[ \phi_0(x) = \phi(x) + \psi(x) \]
\[ \phi_1(x) = -\phi'(x) + \psi'(x) \]

or

\[ \phi(x) + \psi(x) = \phi_0(x) \]
\[ \phi(x) - \psi(x) = -\int_0^x \phi_1(s) ds + c. \]

Solving the system of equations where \( c \) is an arbitrary constant gives

\[ \phi(x) = \frac{1}{2} \phi_0(x) - \frac{1}{2} \int_0^x \phi_1(s) ds + \frac{c}{2} \]
\[ \psi(x) = \frac{1}{2} \phi_0(x) + \frac{1}{2} \int_0^x \phi_1(s) ds - \frac{c}{2}. \]

Hence from (1.20) we see that the solution of (1.22) and (1.23) is

\[ u(x,t) = \frac{\phi_0(x-t) + \phi_0(x+t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \phi_1(s) ds. \quad (1.24) \]

Equation (1.24) is known as d'Alembert's formula. Equation (1.22) together with initial condition (1.23) is a Cauchy problem, and the solution of the Cauchy problem is given by (1.24) is unique. Equation (1.22) has infinitely many solutions and in order to pick out particular solution subject to the initial conditions. We can summarize the above result as the following theorem:
Theorem 1.1 The solution of initial value problem (1.22) and (1.23) for the one-dimensional wave equation is given by

\[ u(x,t) = \frac{\phi_0(x-t) + \phi_0(x+t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \phi_1(s) ds \]

where \( \phi_0 \in C^2(\mathbb{R}) \) and \( \phi_1 \in C^1(\mathbb{R}) \), and the solution is in \( C^2 \) for \( x \in \mathbb{R} \) and \( t \geq 0 \).

If (1.22) is replace by (1.21), it is easy to check that the corresponding d’Alembert’s formula is as above, but with \( t \) replaced by \( ct \).

1.3 The Cauchy Problem for the Wave Equation in Two Independent Variables

In this section we consider a few examples, and then we conclude with the existence and uniqueness theorem for the Cauchy problem.

Example 1. Let us solve the following Cauchy problem

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x \in \mathbb{R}, \quad 0 \leq t < \infty \]

\[ u(x,0) = \cos(x), \quad \frac{\partial u}{\partial t}(x,0) = \sin(2x). \]

Using d’Alembert’s formula

\[ u(x,t) = \frac{\phi_0(x-ct) + \phi_0(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \phi_1(s) ds \]

with \( \phi_0(x) = \cos(x) \), \( \phi_1(x) = \sin(2x) \) and \( c = 3 \), we get
\[ u(x,t) = \frac{1}{2} (\cos(x-3t) + \cos(x+3t)) + \frac{1}{6} \int_{x-3t}^{x+3t} \sin(2s) \, ds \]

\[ = \frac{1}{2} (\cos(x-3t) + \cos(x+3t)) - \frac{1}{12} [\cos(2(x+3t)) - \cos(2(x-3t))]. \]

This solution can be written as

\[ u(x,t) = \cos(x) \cos(3t) + \frac{1}{6} \sin(2x) \sin(6t). \]

\[ \square \]

**Example 2.** Consider the following Cauchy problem

\[ 4 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x \in \mathbb{R}, \quad 0 \leq t < \infty, \]

\[ u(x,0) = x^2, \quad \frac{\partial u}{\partial t}(x,0) = \sin(2x). \]

\[ u(x,t) = \frac{1}{2} ((x-2t)^2 + (x+2t)^2) + \frac{1}{4} \int_{x-2t}^{x+2t} \sin(2s) \, ds \]

\[ = \frac{1}{2} ((x-2t)^2 + (x+2t)^2) - \frac{1}{8} [\cos(2(x+2t)) - \cos(2(x-2t))]. \]

If we simplify the above expression, we will get that

\[ u(x,t) = x^2 + 4t^2 + \frac{1}{4} \sin(2x) \sin(4t). \]

\[ \square \]

**Example 3.** Consider the following Cauchy problem

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x \in \mathbb{R}, \quad 0 \leq t < \infty, \]

\[ u(x,0) = 2 + x, \quad \frac{\partial u}{\partial t}(x,0) = e^x. \]
Using d'Alembert’s formula

\[ u(x,t) = \phi_0(x-t) + \phi_0(x+t) + \frac{1}{2} \int_{x-t}^{x+t} \phi_1(s) ds \]

we get

\[ u(x,t) = \frac{1}{2} ((2 + x - t) + (2 + x + t)) + \frac{1}{2} \int_{x-t}^{x+t} \phi_1(s) ds \]

\[ = \frac{1}{2} (4 + 2x) + \frac{1}{2} e^{x} \bigg|_{x-t}^{x+t} \]

\[ = 2 + x + \frac{1}{2} e^{x} (e^{2t} - \frac{1}{e^{t}}). \]

Example 4. Consider the following Cauchy problem

\[ 4 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x \in \mathbb{R}, \quad 0 \leq t < \infty, \]

\[ u(x,0) = \cos(x), \quad \frac{\partial u}{\partial t}(x,0) = xe^{-x}. \]

Again, using

\[ u(x,t) = \phi_0(x-ct) + \phi_0(x+ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} \phi_1(s) ds \]

with \( \phi_0(x) = \cos(x) \), \( \phi_1(x) = xe^{-x} \) and \( c = 2 \), we get

\[ u(x,t) = \frac{1}{2} (\cos(x-2t) + \cos(x + 2t)) + \frac{1}{4} \int_{x-2t}^{x+2t} e^{-s} ds. \]

To apply integration by parts, we need to write the integral in the form \( \int UdV \). There are several ways to do this. If \( U = -s \) and \( dV = e^{-s} ds \), then

\[ v = \int dV = \int e^{-s} ds = -e^{-s}. \]
\[ dU = -ds. \]

Now, integration by parts produces the following.

\[ \int U dv = U v - \int v dU \]
\[ -\int se^{-s} ds = se^{-s} - \int e^{-s} ds = se^{-s} + e^{-s} + C. \]

Then

\[ u(x,t) = \frac{1}{2} (\cos(x - 2t) + \cos(x + 2t)) - \frac{1}{4} \left[ se^{-s} + e^{-s} \right] + \frac{1}{4} \left[ (x + 2t)e^{-x-2t} + e^{-x-2t} - (x - 2t)e^{-x+2t} - e^{-x+2t} \right] \]
\[ = \cos(x) \cos(2t) - \frac{1}{4} [(x + 1)(e^{-2t} - e^{2t}) + 2t(e^{-2t} + e^{2t})] \]
\[ = \cos(x) \cos(2t) - \frac{1}{2} \left[ (x + 1) \sinh(2t) - t \cosh(2t) \right]. \]

\[ \square \]

The d'Alembert formula shows that the solution of the Cauchy problem for the wave equation exists and is unique.

**Theorem 1.2** Let \( u_1 \) and \( u_2 \) be the solutions of \( \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \) on \( x \in \mathbb{R}, 0 < t \) satisfying

\[ u_1(x,0) = \phi_1(x), \quad \frac{\partial u_1}{\partial t}(x,0) = \psi_1(x) \]

and

\[ u_2(x,0) = \phi_2(x), \quad \frac{\partial u_2}{\partial t}(x,0) = \psi_2(x). \]
Let $\varepsilon > 0$ and $T > 0$. Then there exists a positive number $\delta$ such that, if

$$|\phi_1(x) - \phi_2(x)| < \delta$$

and

$$|\psi_1(x) - \psi_2(x)| < \delta$$

for all $x \in \mathbb{R}$, then

$$|u_1(x, t) - u_2(x, t)| < \varepsilon$$

for all $x \in \mathbb{R}$ and $0 \leq t \leq T$. See [8].

**Proof:** We know that

$$u_1(x, t) = \frac{\phi_1(x - t) + \phi_1(x + t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \psi_1(s) ds$$

and

$$u_2(x, t) = \frac{\phi_2(x - t) + \phi_2(x + t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \psi_2(s) ds.$$ 

For any given positive numbers $\varepsilon$ and $T$, we have

$$|u_1(x, t) - u_2(x, t)| \leq \frac{1}{2} |\phi_1(x - t) - \phi_1(x + t)| + \frac{1}{2} |\phi_2(x - t) - \phi_2(x + t)|$$

$$+ \frac{1}{2} \int_{x-t}^{x+t} |\psi_1(s) - \psi_2(s)| ds$$

$$< \frac{1}{2} \delta + \frac{1}{2} \delta + \frac{1}{2} (2T) \delta = (1 + T) \delta.$$ 

Then

$$|u_1(x, t) - u_2(x, t)| < \varepsilon$$

for all $x$ and $0 \leq t \leq T$ if we choose $\delta$ to be any positive number such that
Thus, we may choose \( \delta \) as any number satisfying \( 0 < \delta < \varepsilon / (1 + T) \).

Chapter 2

2.1 Solution of the Initial Value Problem

As we have seen in the previous chapter, the Cauchy problem is an initial value problem for the wave equation,

\[
\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} - \frac{\partial^2 u}{\partial t^2} = 0. \tag{2.1}
\]

The Cauchy problem asks for the solution \( u(x_1, \ldots, x_n, t) \) of (2.1) which satisfies the given initial conditions

\[
u(x_1, \ldots, x_n, 0) = \phi_0(x_1, \ldots, x_n), \tag{2.2}
\]

\[
\frac{\partial u}{\partial t}(x_1, \ldots, x_n, 0) = \phi_1(x_1, \ldots, x_n), \tag{2.3}
\]

where \( \phi_0 \) and \( \phi_1 \) are some given functions which are defined in the \( x \)-space at \( t = 0 \). The plane \( t = 0 \) in the \( (x, t) \)-space \( \mathbb{R}^{n+1} \) is known as the initial surface.

For \( n = 2 \), equation (2.1) is a two-dimensional wave equation with the initial conditions

\[
u = (x_1, x_2, 0) = \phi_0(x_1, x_2), \tag{2.4}
\]

\[
\frac{\partial u}{\partial t}(x_1, x_2, 0) = \phi_1(x_1, x_2). \tag{2.5}
\]
The initial value problem asks for the solution \( u = (x_1, x_2, t) \) of a two-dimensional wave equation satisfying the initial conditions (2.4) and (2.5), and defined in the three-dimensional \((x_1, x_2, t)\)-space. Equations (2.4) and (2.5) are specified on the initial surface \( t = 0 \). We must show that initial value problem is well-posed under certain conditions on the initial data \( \phi_1 \) and \( \phi_0 \).

A partial differential equation is well-posed, if

(i) a solution of the problem exists,

(ii) its solution is unique,

(iii) the solution depends continuously on the initial conditions.

When solving an initial value problem, it is enough to solve the problem only in the upper half-space \( t \geq 0 \), because the problem for the lower half-space can be reduced to the problem for the upper half-space by making the transformation \( t' = -t \). Using this transformation, wave equation (2.1) remains unchanged. Therefore, we are going to find the solution of an initial value problem for \( t \geq 0 \). The initial value problem with initial surface the plane \( t = t^0 \) can be immediately reduced to the problem given by equations (2.1), (2.2), (2.3) with initial surface \( t = 0 \) by making the transformation \( t' = t - t^0 \). Again, under this transformation, equation (2.1) remains unchanged.

We have seen that the initial value problem for one-dimensional wave in Chapter 1, equations (1.22)-(1.23) is a well-posed problem and its solution is given by the formula (1.24). Existence can be proved by verifying that equation (1.24) satisfies equations (1.22)-(1.23). Uniqueness of the solution to the problem follows from the fact that the general solution of (1.22) is (1.20) and from the fact that initial conditions (1.23) uniquely determine the functions \( F \) and \( G \). The continuous dependence of the solution
on the initial conditions was proved in Theorem 1.2. Since we do not have a simple formula for the general solution of the wave equation in more than one space variable, it means that we cannot follow this simple procedure to prove existence and uniqueness. Instead, we will prove the existence by finding the formula for the solution of $n \geq 2$. Then, we use the energy method to prove the uniqueness, and the last we will prove continuous dependence on the initial data using the formula for the solution of the problem.

2.2 Initial Value Problems for the Wave Equation in More Than Two Independent Variables. Huygen's Principle.

Let us solve the Cauchy problem for the wave equation in four independent variables

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0,$$

$$u(x, 0) = \phi_0(x),$$

$$\frac{\partial u}{\partial t}(x, 0) = \phi_1(x),$$

where $\phi_0 \in C^3(\mathbb{R}^3)$ and $\phi_1 \in C^2(\mathbb{R}^3)$. We need to show that if $\phi \in C^2(\mathbb{R}^3)$ then

$$u(x, t) = \frac{1}{4\pi} \int_{|x - \xi| = t} \phi(\xi) d\sigma$$

is a solution of wave equation (2.6). The integral in (2.8) is the surface integral of $\phi(\xi)$ over the sphere of radius $t$ centered at $x$. The right hand side of (2.8) is called the spherical mean of $\phi(\xi)$. Equation (2.8) can be written as
Then

$$\Delta_3 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$  \hspace{1cm} (2.10)$$

$$= \frac{t}{4\pi} \int_{|\xi - x|=t} \Delta_3 \phi(x + \xi \xi d\sigma$$

$$= \frac{1}{4\pi t} \int_{|\xi - x|=t} \Delta_3 \phi(\xi) d\sigma. $$

Taking the derivative of both sides of (2.9) we get

$$\frac{\partial u}{\partial t} = \frac{1}{4\pi} \int_{|\xi - x|=1} \phi(x + \xi \xi d\sigma + \frac{t}{4\pi} \int_{|\xi - x|=1} \nabla \phi(x + \xi \xi \xi d\sigma$$

$$= \frac{u(x,t)}{t} + 1 \int_{|\xi - x|=t} \nabla \phi(\xi) \cdot \eta \eta d\sigma$$

where \( \eta \) is the unit outward normal to the sphere \(|\xi - x| = t\). Using the divergence theorem, see [7], equation (2.11) takes the form

$$\frac{\partial u}{\partial t} = \frac{u(x,t)}{t} + \frac{1}{4\pi t} \int_{|\xi - x|=t} \Delta_3 \phi(\xi) d\xi.$$  \hspace{1cm} (2.12)$$

Let

$$I(x,t) = \int_{|\xi - x|=t} \Delta_3 \phi(\xi) d\xi.$$  \hspace{1cm} (2.13)$$

Calculate the second derivative of (2.9), we see that

$$\frac{\partial^2 u}{\partial t^2} = \frac{u}{t^2} + \frac{1}{t} \frac{\partial u}{\partial t} + \frac{1}{4\pi t} \frac{\partial I}{\partial t} - \frac{1}{4\pi t^2} I.$$  \hspace{1cm} (2.13)$$
Furthermore,

\[
\frac{\partial I}{\partial t} = \int_{|\xi - \lambda| = \epsilon} \Delta_{\lambda} \phi(\xi) d\sigma,
\]

and hence from equations (2.10), (2.13), and (2.14), we see that equation (2.8) is the solution of wave equation (2.6).

Using the above result now it is easy to obtain the solution of Cauchy problem given by equations (2.6) and (2.7). If \( u(x,t) \) is defined by (2.8) then from equations (2.9) and (2.11), we have

\[
u(x,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = \phi(x). \tag{2.15}
\]

On the other hand, if \( u(x,t) \) is a solution of the wave equation satisfying (2.15), then it is easily seen that \( v = \frac{\partial u}{\partial t} \) is a solution of the wave equation satisfying

\[
u(x,0) = \phi(x),
\]

and

\[
\frac{\partial v}{\partial t}(x,0) = 0.
\]

Using initial conditions for \( \phi_0(x) \), and \( \phi_1(x) \) we can now conclude that a solution of the Cauchy problem given by equations (2.6) and (2.7) is given by

\[
u(x,t) = \frac{1}{4\pi} \frac{\partial}{\partial t} \left[ \frac{1}{t} \int_{|\xi - \lambda| = \epsilon} \phi_0(\xi) d\sigma \right] + \frac{1}{4\pi} \int_{|\xi - \lambda| = \epsilon} \phi_1(\xi) d\sigma. \tag{2.16}
\]
Equation (2.16) is known as Poisson's formula. We cannot conclude that equation (2.16) is unique. We show that later, but for now assume it is unique. From Poisson's formula, we can see that the solution of (2.6) and (2.7) depends continuously on the data $\phi_1(x), \phi_0(x)$ and $\nabla \phi_0(x)$.

First we state a lemma which reduces the problem of finding the solution of (2.6) and (2.7) to the problem of finding the solution of (2.6) satisfy the special initial conditions,

$$u(x, 0) = 0, \quad x \in \mathbb{R}^3 \quad (2.17)$$

$$\frac{\partial u}{\partial t}(x, 0) = \phi(x), \quad x \in \mathbb{R}^3. \quad (2.18)$$

**Lemma 2.1.** Let $u_\phi$ be the solution of the initial value problem (2.6), (2.17), and (2.18).

If $u_\phi$ is in $C^3$ for $x \in \mathbb{R}^3$ and $t \geq 0$, then $v = \frac{\partial u_\phi}{\partial t}$ satisfies (2.6) with initial conditions, $u(x, 0) = \phi(x), \frac{\partial u}{\partial t}(x, 0) = 0$, and $v$ is in $C^2$ for $x \in \mathbb{R}^3$ and $t \geq 0$. See [11].

**Proof.** Using the fact that any derivative of a solution of a homogeneous partial differential equation with constant coefficients is also a solution of the equation, implies that $v$ satisfies equation (2.6), and since $u_\phi$ satisfies (2.18), we get

$$v(x, 0) = \frac{\partial}{\partial t} u_\phi(x, 0) = \phi(x).$$

If we take the derivative of $v$ respect to $t$, and use the facts that $u_\phi$ satisfies (2.6) and it vanishes for $t = 0$, then we see that

$$\frac{\partial v}{\partial t}(x, 0) = \frac{\partial^2 u_\phi}{\partial t^2}(x, t) = \left(\frac{\partial^2 u_\phi}{\partial x_1^2} + \frac{\partial^2 u_\phi}{\partial x_2^2} + \frac{\partial^2 u_\phi}{\partial x_3^2}\right)|_{t=0} = 0.$$
Lemma 2.2. The solution of the initial value problem (2.6) and (2.7), which is in $C^2$ for $x \in \mathbb{R}^3$ and $t \geq 0$, is given by

$$u = \frac{\partial u_{\phi_0}}{\partial t} + u_{\phi}$$

(2.19)

where $u_{\phi_0}$ is in $C^3$ and $u_{\phi}$ is in $C^2$ for $x \in \mathbb{R}^3$ and $t \geq 0$. See [11].

From lemma 2.2, we can say that if we know the solution on the special initial value problem, then it is easy to solve the initial value problem (2.6) and (2.7).

Lemma 2.3 For any integer $k \geq 2$ if $\phi \in C^k(\mathbb{R}^3)$, then the solution of the initial value problem (2.6), (2.17) and (2.18) is given by the formula

$$u_{\phi}(x,t) = \frac{1}{4\pi} \int_{|\xi-x|=t} \phi(\xi)d\sigma$$

(2.20)

and $u_{\phi}$ is in $C^2$ for $x \in \mathbb{R}^3$ and $t \geq 0$. See [11].

Combining Lemmas 2.2 and 2.3, we can write the formula for the solution of the initial value problem given by (2.6) and (2.7), as the following theorem:

Theorem 2.1 The solution of the initial value problem (2.6) and (2.7) for the three-dimensional wave equation is given by
\[ u(x,t) = \frac{1}{4\pi} \frac{\partial}{\partial t} \left[ \frac{1}{t} \int_{|\xi - x| = t} \phi_0(\xi) d\sigma \right] + \frac{1}{4\pi} \int_{|\xi - x| = t} \phi_1(\xi) d\sigma \]

and the solution is in $C^2$ for $x \in \mathbb{R}^3$ and $t \geq 0$, whenever $\phi_0 \in C^2(\mathbb{R}^3)$ and $\phi_1 \in C^2(\mathbb{R}^3)$.

See [2], and [3].

For a given point $(x, t)$, the domain of dependence of $u(x, t)$ is the surface of the ball of radius $t$ centered at $x$. The initial disturbances that vanish outside a ball centered at the origin in $\mathbb{R}^3$ reach a point $x$ outside of this ball in a finite time, are then "heard" for a finite length of time, and then sharply disappear. This phenomenon is known as Huygen's principle. See [8].

2.3 Initial Value Problems for the Two-Dimensional Wave Equation.

Uniqueness.

Now, consider the initial value problem for two-dimensional space

\[
\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0, \quad (2.21)
\]

\[
u(x,0) = \phi_0(x), \quad (2.22)
\]

\[
\frac{\partial u}{\partial t}(x,0) = \phi_1(x)
\]

where $\phi_0 \in C^3(\mathbb{R}^2)$ and $\phi_1 \in C^2(\mathbb{R}^2)$. The solution of this problem can be obtained form the solution of the three-dimensional problem. Using equation (2.16) and
Hadamard's method of descent, we can solve the Cauchy problem for the wave equation in three independent variables. If \( \phi_0(x) \) and \( \phi_1(x) \) are independent of \( x_3 \), then using (2.16), we can find the solution of (2.21) and (2.22). This follows from the fact that here the integrals in (2.16) are invariant with respect to translation along the \( x_3 \) axis. Let \( \nu \) be the unit outward normal to the sphere \( |\xi - x| = t \) and let \( k \) be the unit vector in the \( \zeta \) direction, \( \zeta = (\mu, \eta, \zeta) \). Consider a typical surface element having area \( d\sigma \). This is a projection onto, \( d\mu \) by \( d\eta \), a rectangle centered at \((x_1, x_2, 0)\) in the \((\mu, \eta)\) plane. The area of this rectangle is related to the area of the surface element by

\[
d\mu d\eta = \nu \cdot k d\sigma = \frac{1}{t} \sqrt{t^2 - (\mu - x_1)^2 - (\eta - x_2)^2} \, d\sigma.
\]

Here, we have used the fact that the sphere of radius \( t \) about \((x_1, x_2, 0)\) has equation

\[
t^2 = (\mu - x_1)^2 - (\eta - x_2)^2 + \zeta^2.
\]

Since a sphere has both an upper and lower hemisphere, we introduce a factor of \( 2 \) and write

\[
\int_{|\xi - x|=t} \phi(\xi) d\sigma = 2t \iint_{(\mu-x_1)^2+(\eta-x_2)^2 \leq t^2} \frac{\phi(\mu, \eta) d\mu d\eta \sqrt{t^2 - (\mu - x_1)^2 - (\eta - x_2)^2}}{\sqrt{(\mu-x_1)^2+(\eta-x_2)^2}}.
\]

Letting \( \xi = (\mu, \eta), \ x = (x_1, x_2) \) we now see from equation (2.16) that the solution of the wave equation in three independent variables (2.21) and (2.22) is given by

\[
u(x,t) = \frac{1}{2\pi} \frac{\partial}{\partial t} \int_{|\xi - x|=t} \frac{\phi_0(\xi) d\xi}{\sqrt{t^2 - |\xi - x|^2}} + \frac{1}{2\pi} \int_{|\xi - x|=t} \frac{\phi_1(\xi) d\xi}{\sqrt{t^2 - |\xi - x|^2}}.
\] (2.23)
Thus, by “descending” from three-dimensions to two-dimensions we obtain the solution of the two-dimensional initial value problem (2.21) and (2.22) given in the following theorem.

**Theorem 2.2** Suppose that \( \phi_0 \in C^3(\mathbb{R}^2) \) and \( \phi_1 \in C^2(\mathbb{R}^2) \). Then solution of initial value problem (2.21) and (2.22) for the two-dimensional wave equation is given by

\[
 u(x,t) = \frac{1}{2\pi} \frac{\partial}{\partial t} \int_{\mathbb{R}^2} \frac{\phi_0(\xi) d\xi}{\sqrt{t^2 - |\xi - x|^2}} + \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\phi_1(\xi) d\xi}{\sqrt{t^2 - |\xi - x|^2}}
\]

and the solution is in \( C^2 \) for \((x_1, x_2) \in \mathbb{R}^2 \) and \( t \geq 0 \).

Notice that there is a significant difference between the solution given by (2.16) in 3-space and (2.23) in the plane. The domain of influence of the initial data at a point in 3-space at time \( t \) is the surface of a sphere of radius \( t \) about the point, and Huygen’s principle holds. In the plane, however, the domain of influence at \((x_1, x_2)\) time \( t \) is the solid disk of radius \( t \) about the point, not just its boundary curve circle. A disturbance will eventually be felt at \((x_1, x_2)\) and will then be felt at all later times. Huygen’s principle does not hold in the plane.

**Proof of uniqueness of solution:**

Now, let us show the uniqueness of the solution of equations (2.21), and (2.22). Assume that there are two solutions of equations (2.21) and (2.22). Considering their difference shows that it suffices to prove that the solution of
\[
\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0, \tag{2.24}
\]

\[
u(x, 0) = 0, \tag{2.25}
\]

\[
\frac{\partial u}{\partial t}(x, 0) = 0
\]

is identically zero for \( t > 0 \). Before we begin the proof, let us discuss some facts about characteristic cone and characteristic directions for the wave equation. Let \( \nu = (\nu_1, \ldots, \nu_n, \nu_t) \) be a unit vector in \( \mathbb{R}^{n+1} \) describing a characteristic direction. The normal to a characteristic surface must always make an 45° angle with the \( t \)-axis. Let \((x^0, t^0) = (x_1^0, \ldots, x_n^0, t^0)\) be a fixed point in \( \mathbb{R}^{n+1} \). The equation

\[
(x_1 - x_1^0)^2 + \ldots + (x_n - x_n^0)^2 - (t - t^0)^2 = 0
\]

describes a double conical surface with apex at \((x^0, t^0)\). Now, let us go back to our problem, and consider the characteristic cone

\[
(t - t^0)^2 - (x_1 - x_1^0)^2 - (x_2 - x_2^0)^2 = 0
\]

which corresponds to the characteristic lines \( (x_1 - x_1^0) = \pm (t - t^0) \) where \((x_1^0, x_2^0, t^0) \in \mathbb{R}^3, t^0 > 0 \). Let \( D \) be the region bounded by this cone and the plane \( t = 0 \), where \( \Gamma \) is the cone (with unit outward normal \( \nu \)) and \( \sigma_0 \) is the disk cut out of the plane \( t = 0 \) (see Figure 2.1). We need to show that \( u(x_1^0, x_2^0, t^0) = 0 \). Using the identity

\[
2 \frac{\partial u}{\partial t} \left( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} \right) = \frac{\partial}{\partial t} \left( \left( \frac{\partial u}{\partial x_1} \right)^2 + \left( \frac{\partial u}{\partial x_2} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \right)
\]

\[-2 \frac{\partial}{\partial x_1} \left( \frac{\partial u}{\partial t} \frac{\partial u}{\partial x_1} \right) - 2 \frac{\partial}{\partial x_2} \left( \frac{\partial u}{\partial t} \frac{\partial u}{\partial x_2} \right),
\]
if \( u(x,t) \) is the solution of the wave equation (2.24), then

\[
\int \int_\Omega \left[ \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x_1} \right)^2 + \left( \frac{\partial u}{\partial x_2} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \right] - 2 \frac{\partial}{\partial x_1} \left( \frac{\partial u}{\partial t} \frac{\partial u}{\partial x_1} \right) - 2 \frac{\partial}{\partial x_2} \left( \frac{\partial u}{\partial t} \frac{\partial u}{\partial x_2} \right) \, dx \, dt = 0.
\]

However, from (2.25), we have

\[
\frac{\partial u}{\partial x_1}(x,0) = \frac{\partial u}{\partial x_2}(x,0) = \frac{\partial u}{\partial t}(x,0) = 0,
\]

and applying the divergence theorem, see [9], we get

\[
\int \int \left\{ \left( \frac{\partial u}{\partial x_1} \right)^2 + \left( \frac{\partial u}{\partial x_2} \right)^2 + \left( \frac{\partial u}{\partial t} \right)^2 \right\} e_i - 2 \left( \frac{\partial u}{\partial t} \frac{\partial u}{\partial x_1} \right) e_{x_1} - 2 \left( \frac{\partial u}{\partial t} \frac{\partial u}{\partial x_2} \right) e_{x_2} \right\} \cdot v \, d\sigma = 0 \quad (2.26)
\]

where \( e_{x_1}, e_{x_2} \) and \( e_t \) are the unit vectors along the positive \( x_1, x_2 \) and \( t \)-axes, respectively.

Using the equation \( \Gamma : (x_1 - x_1^0)^2 - (x_2 - x_2^0)^2 - (t - t^0)^2 = 0 \) and the fact that \( v \) is parallel to the vector

\[\text{Figure 2.1}\]

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\[(x_1 - x_1^0)e_{x_1} + (x_2 - x_2^0)e_{x_2} - (t - t^0)e_t,\]

we get

\[(e_{x_1} \cdot \nu)^2 - (e_{x_2} \cdot \nu)^2 - (e_t \cdot \nu)^2 = 0.\]

Using the fact that \(e_i \cdot \nu = \frac{\sqrt{2}}{2}\) and equation (2.26), we have

\[
\int_\Gamma \left\{ \left( \frac{\partial u}{\partial x_1} e_t \cdot \nu \right)^2 + \left( \frac{\partial u}{\partial x_2} e_t \cdot \nu \right)^2 \right\} d\sigma = 0.
\]

This implies

\[
\frac{\partial u}{\partial x_1} e_t \cdot \nu = \frac{\partial u}{\partial t} e_{x_1} \cdot \nu,
\]

and

\[
\frac{\partial u}{\partial x_2} e_t \cdot \nu = \frac{\partial u}{\partial t} e_{x_2} \cdot \nu.
\]

Thus,

\[
\frac{\partial u}{\partial x_1} e_{x_1} \cdot \nu = \frac{\partial u}{\partial x_2} e_{x_2} \cdot \nu = \frac{\partial u}{\partial t} e_t \cdot \nu = \lambda
\]

where \(\lambda = \lambda(x,t)\).

Then

\[
\frac{\partial u}{\partial \mathbf{I}} = \frac{\partial u}{\partial x_1} e_{x_1} \cdot \mathbf{I} + \frac{\partial u}{\partial x_2} e_{x_2} \cdot \mathbf{I} + \frac{\partial u}{\partial t} e_t \cdot \mathbf{I}
\]

\[
= \lambda \left( e_{x_1} \cdot \nu (e_{x_1} \cdot \mathbf{I}) + (e_{x_2} \cdot \nu (e_{x_2} \cdot \mathbf{I}) + (e_t \cdot \nu (e_t \cdot \mathbf{I})
\]

\[
= \lambda \mathbf{I} \cdot \nu = 0
\]
where \( I \) is a unit vector lying along the generator of the characteristic cone. We can conclude that \( u(x,t) \) is constant along the generator, and since \( u(x,0) = 0 \), we have that \( u(x,t) = 0 \) along this generator, implies that \( u(x_1^0, x_2^0, t) = 0 \) which shows the uniqueness of the solution of wave equation (2.24), (2.25).

\[ \] 2.4 Uniqueness of Solution to an Initial Value Problem.

In this section, we use the domain of dependence inequality to show the uniqueness of solution in an initial value problem,

\[
\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} - \frac{\partial^2 u}{\partial t^2} = 0, 
\]

\[ (2.27) \]

\[ u(x_1, \ldots, x_n, 0) = \phi_0(x_1, \ldots, x_n), \]

\[ \text{at} \]

\[ \frac{\partial u}{\partial t} (x_1, \ldots, x_n, 0) = \phi_1(x_1, \ldots, x_n), \]

\[ (2.29) \]

where \( t > 0 \), and \( x \in \mathbb{R}^n \). The surface

\[
(x_1 - x_1^0)^2 + \cdots + (x_n - x_n^0) - (t - t^0)^2 = 0
\]

is known as the characteristic cone with apex at \((x^0, t^0)\). The forward characteristic cone with apex at \((x^0, t^0) \in \mathbb{R}^{n+1}\) is the upper part for which \( t \geq t^0 \), and the surface and interior of this cone is
\[(x_1 - x_1^0)^2 + \cdots + (x_n - x_n^0)^2 - (t - t^0)^2 \leq 0.\]

The backward characteristic cone with apex at \((x^0, t^0) \in \mathbb{R}^{n+1}\) is the lower part for which \(t \leq t^0\). Let us consider the plane \(t = T\) in \(\mathbb{R}^{n+1}\), where \(T\) is any number \(\leq t^0\), and then the part of this plane cut off by the backward characteristic cone with apex at \((x^0, t^0)\) is the set of points \((x, t)\) satisfying both above conditions and is given by

\[(x_1 - x_1^0)^2 + \cdots + (x_n - x_n^0)^2 \leq (t^0 - T)^2.\]

This set is the closed ball \(B(x^0, t^0 - T)\) in the \(x\)-space, \(\mathbb{R}^n\) at \(t = T\). Figure 2.2 illustrates these cut-off balls. Now, we consider the following theorem:

**Theorem 2.3** If \(u\) is in \(C^2(\bar{\Omega})\) and satisfies the wave equation in \(\Omega\), where \(\Omega\) the conical domain in \(\mathbb{R}^{n+1}\) bounded by the backward characteristic cone with apex at \((x^0, t^0) \in \mathbb{R}^{n+1}\) with \(t^0 > 0\) and by the plane \(t = 0\), then for any \(T, 0 \leq T \leq t^0\), the following inequality holds:

\[\int_{\tilde{B}(x^0, r^0 - T)} (u^2 + \cdots + u_n^2 + u_t^2) \, dx \leq \int_{\tilde{B}(x^0, r^0)} (u^2 + \cdots + u_n^2 + u_t^2) \, dx.\quad (2.30)\]

**Proof.** We use a differential identity to prove the theorem. Claim that

\[2u_t (u_{x_1} + \cdots + u_{x_n} - u_{tt}) = (2u_t u_{x_1})_x + \cdots + (2u_t u_{x_n})_x - (u^2_{x_1} + \cdots + u^2_{x_n} + u^2_t).\quad (2.31)\]

Let \(C_T\) be the part of backward characteristic cone with the apex at \((x_1^0, \ldots x_n^0, t^0)\) which lies between the planes \(t = 0\) and \(t = T\). Now, integrate (2.31) over \(\Omega_T\), where \(\Omega_T\) is the part of \(\Omega\) below the plane \(t = T\). Since \(u\) satisfies the wave equation in \(\Omega\), then
\[
\int_{\Omega_T} [(2u, u_{x_i}) x_i + \cdots + 2u, u_{x_n}) x_n - (u^2_{x_i} + \cdots + u^2_{x_n} + u^2_t)] dx_1 \cdots dx_n dt = 0. \quad (2.32)
\]

Since the integrand in (2.32) is the divergence of the vector field in the \((x, t)\)–space, we can apply the divergence theorem to (2.32) which gives us

\[
\int \left[ (2u, u_{x_i}) v_i + \cdots + 2u, u_{x_n}) v_n - (u^2_{x_i} + \cdots + u^2_{x_n} + u^2_t) v_t \right] d\sigma = 0.
\quad (2.33)
\]

In equation (2.33), \(v\) is unit normal vector on \(\partial \Omega_T\) in the direction exterior to \(\Omega_T\) where \(\partial \Omega_T\) is the boundary of \(\Omega_T\), and \(d\sigma\) is the surface element on \(\partial \Omega_T\). Since the boundary of \(\partial \Omega_T\) consists of three parts, the top and the bottom discs and conical surface \(C_T\), we can write integral (2.33) as the sum of three integrals. The integral on the top disc is

\[
\int_{B(x^0_1, \ldots, x^0_n, t^0 - T)} (u^2_{x_i} + \cdots + u^2_{x_n} + u^2_t)|_{t=T} dx_1 \cdots dx_n \quad (2.34)
\]

where \(v = (0, \cdots, 0, 1)\) and \(x \in B(x^0_1, \ldots, x^0_n, t^0 - T), \ t = T\). The integral on the bottom disc is
\[
\frac{1}{\delta(x_{10}^{0}, \ldots, x_{n0}^{0}, t^{0})} \left( u_{x_{1}^{0}}^{2} + \cdots + u_{x_{n}^{0}}^{2} + u_{t^{0}}^{2} \right)_{|t=0} \, dx_{1} \cdots dx_{n}
\]  

(2.35)

where \( \nu = (0, \cdots, 0, -1) \) and \( x \in \bar{B}(x_{1}^{0}, \ldots, x_{n}^{0}, t^{0}), t = 0 \). On the conical surface \( C_{r} \) the exterior normal \( \nu = (\nu_{1}, \ldots, \nu_{n}, \nu_{t}) \) defines a characteristic direction and hence must satisfy the relations

\[
\nu_{1}^{2} + \cdots + \nu_{n}^{2} - \nu_{t}^{2} = 0, \quad \nu_{t} = \frac{1}{\sqrt{2}}.
\]

Therefore, the integral on \( C_{r} \) is

\[
\int_{C_{r}} \left[ 2u_{x_{1}} \nu_{1} + \cdots + 2u_{x_{n}} \nu_{n} - (u_{x_{1}}^{2} + \cdots + u_{x_{n}}^{2} + u_{t}^{2})\nu_{t} \right] d\sigma
\]

\[
= \sqrt{2} \left[ 2u_{x_{1}} \nu_{1} + \cdots + 2u_{x_{n}} \nu_{n} - (u_{x_{1}}^{2} + \cdots + u_{x_{n}}^{2} + u_{t}^{2})\nu_{t} \right] d\sigma
\]

\[
= \sqrt{2} \left[ 2u_{x_{1}} \nu_{1} + \cdots + 2u_{x_{n}} \nu_{n} - u_{x_{1}}^{2} \nu_{1}^{2} - \cdots - u_{x_{n}}^{2} \nu_{n}^{2} - u_{t}^{2} \nu_{t}^{2} \right] d\sigma.
\]

If we combine three terms in the last integrand, we will get the difference of the squares, so we can write the integral on \( C_{r} \) as

\[
- \sqrt{2} \int_{C_{r}} [(u_{x_{1}} \nu_{1} - u_{t} \nu_{t})^{2} + \cdots + (u_{x_{n}} \nu_{n} - u_{t} \nu_{t})^{2}] d\sigma.
\]

(2.36)

Equation (2.33) can be written as the sum of the three integrals (2.36), (2.35) and (2.34).

Equation (2.33) becomes

\[
0 = - \left[ (u_{x_{1}}^{2} + \cdots + u_{x_{n}}^{2} + u_{t}^{2})_{|t=0} \, dx_{1} \cdots dx_{n}
\right]

\[\left. \right|_{t=0} dx_{1} \cdots dx_{n}
\]  

\[
+ \int_{\delta(x_{1}^{0}, \ldots, x_{n}^{0}, t^{0})} \left[ (u_{x_{1}}^{2} + \cdots + u_{x_{n}}^{2} + u_{t}^{2}) \right]_{t=0} \, dx_{1} \cdots dx_{n}
\]

\[
- \sqrt{2} \int_{C_{r}} [(u_{x_{1}} \nu_{1} - u_{t} \nu_{t})^{2} + \cdots + (u_{x_{n}} \nu_{n} - u_{t} \nu_{t})^{2}] d\sigma.
\]

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So,

\[ \int_{\tilde{B}(x_0^0, t_0^0, D^0-T)} (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2) |_{t=T} \, dx_1 \cdots dx_n \]

\[ = \int_{\tilde{B}(x_0^0, t_0^0, D^0)} (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2) |_{t=0} \, dx_1 \cdots dx_n \]

\[ - \sqrt{2} \int_{C_T} [(u_{x_1} v_t - u_t v_1)^2 + \cdots + (u_{x_n} v_t - u_t v_n)^2] \, d\sigma. \]

Since the second quantity on the right side of the above equation is always negative, then we obtain the inequality

\[ \int_{\tilde{B}(x_0^0, t_0^0, D^0-T)} (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2) |_{t=T} \, dx_1 \cdots dx_n \]

\[ \leq \int_{\tilde{B}(x_0^0, t_0^0, D^0)} (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2) |_{t=0} \, dx_1 \cdots dx_n \]  \hspace{1cm} (2.37)

which is the inequality for general \( n \).

\[ \blacksquare \]

**Theorem 2.4** If \( u \) is a function that satisfies the wave equation in \( \Omega \), where \( \Omega \) the conical domain in \( \mathbb{R}^{n+1} \), is bounded by the backward characteristic cone with apex at \( (x^0, t^0) \in \mathbb{R}^{n+1} \) with \( t^0 > 0 \) and by the planet \( = 0 \), and \( u \) is in \( C^2(\Omega) \), such that \( u(x,0) = \frac{\partial u}{\partial t} (x,0) = 0 \) for \( x \in \tilde{B}(x^0, t^0) \) then \( u \) is vanishes in \( \tilde{\Omega} \).
Proof. For all \( x \in B(x^0, t^0) \) we have \( u_{x_1}(x,0) = u_{x_2}(x,0) = \cdots = u_{x_n}(x,0) \) because \( u(x,0) = 0 \) for all \( x \in B(x^0, t^0) \). Hence the right side of inequality (2.30) is equal to zero, and since the left side is positive, then it must be zero

\[
\left. \int_{\tilde{B}(x^0,t^0-T)} \left( u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2 \right) \right|_{t=0} dx = 0 \tag{2.38}
\]

for any \( T, \ 0 \leq T \leq t^0 \). The integrand in (2.38) is continuous and nonnegative then it must be zero for \( x \in B(x^0, t^0), \ 0 \leq T \leq t^0 \). Then, we get

\[
u_{x_1}(x,T) = \cdots = u_{x_n}(x,T) = u_t(x,T) = 0.
\]

This means that all first order partial derivatives of \( u(x,t) \) vanish in \( \Omega \), therefore \( u \) must be constant in \( \Omega \). Also, \( u \) must be constant in \( \tilde{\Omega} \) by continuity, and since \( u = 0 \) on the base of \( \tilde{\Omega} \) then it must be constant everywhere in \( \tilde{\Omega} \). From Theorem 2.4 follows uniqueness.

\[ \blacksquare \]

Corollary 2.1 If \( u_1 \) and \( u_2 \) are two functions in \( C^2(\tilde{\Omega}) \), satisfy the wave equation in \( \Omega \) and on the base of \( \Omega \), where \( \Omega \) the conical domain in \( \mathbb{R}^{n+1} \), is bounded by the backward characteristic cone with apex at \( (x^0,t^0) \in \mathbb{R}^{n+1} \) with \( t^0 > 0 \) and by the plane \( t = 0 \), and

\[
u_1 = u_2, \quad \frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial t}, \quad \text{then } u_1 = u_2 \text{ in } \tilde{\Omega}.
\]

Proof. To prove Corollary 2.1, we let \( \tilde{u} = u_1 - u_2 \) and it is not difficult to see that \( \tilde{u} \) satisfies the all hypothesis in Theorem 2.4. Then \( u_1 = u_2 \) in \( \tilde{\Omega} \).

\[ \blacksquare \]
From Corollary 2.1 and Theorem 2.4, we get the following result.

**Corollary 2.2**  \( u_1 = u_2 \) whenever \( u_1 \) and \( u_2 \) are solutions of the initial value problem (2.27)-(2.29) which are in \( C^2 \) for \( x \in \mathbb{R}^n \) and \( t \geq 0 \).

Corollary 2.2 claims the uniqueness of solution of an initial value problem and Corollary 2.1 claims the values of \( u \) and \( u_i \) on the domain of \( \Omega \) are uniquely determined by the value of \( u \) at the apex of \( \Omega \). Combining these two corollaries, we can state the following corollary.

**Corollary 2.3**  Let \( u \) be a function in \( C^2(\bar{\Omega}) \) that satisfies the wave equation in \( \Omega \) and on the base of \( \Omega \), where \( \Omega \) is the conical domain in \( \mathbb{R}^{n+1} \) bounded by the backward characteristic cone with apex at \( (x^0, t^0) \in \mathbb{R}^{n+1} \) with \( t^0 > 0 \) and by the planet \( t = 0 \). Then the value of \( u \) at \( (x^0, t^0) \) is uniquely determined by the values of \( u \) and \( u_i \) on the base of \( \Omega \).

The solution of initial value problem (2.27)-(2.29) at the point \( (x^0, t^0) \) is uniquely determined by the value of initial data \( \phi_0 \), and \( \phi_i \) on the part of the initial surface \( t = 0 \) cut off by the backward characteristic cone with apex at \( (x^0, t^0) \). Therefore, this part of the initial surface is known as the domain of dependence of the solution at the point \( (x^0, t^0) \). The values of the initial data outside of the domain of dependence do not affect the value of the solution at \( (x^0, t^0) \).
We conclude this section with a discussion of an important law of wave propagation known as conservation of energy. If $D$ is a region in the $x$-space $\mathbb{R}^n$, the integral
\[
E(u; D, T) = \frac{1}{2} \int_D (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2)_{|t=T} \, dx.
\] (2.39)

known as the energy of $u(x, t)$ contained the region $D$ at the time $t = T$.

If the solution $u(x, t)$ of the wave equation describes a wave propagation phenomenon, then the integral (2.39) is called by physicists the energy of the wave in the region $D$ at the time $t = T$. Therefore, the domain of dependence inequality is known as an energy inequality. The inequality (2.30) can be written as
\[
E(u; \tilde{B}(\mathbf{x}^0, t^0 - T), T) \leq E(u; \tilde{B}(\mathbf{x}^0, t^0), 0). \quad (2.40)
\]

We can summarize these facts as the following theorem:

**Theorem 2.5** If $u(x, t)$ is the solution of the initial value problem (2.27)-(2.29) which is in $C^2$ for $x \in \mathbb{R}^n$ and $t \geq 0$, and if initial data $\phi_0$ and $\phi_1$ vanish outside some ball $B(0, R)$ in $\mathbb{R}^n$, then the energy of $u$ contained in the whole space $\mathbb{R}^n$ remains constant for every $T \geq 0$. Specifically,
\[
\frac{1}{2} \int_{\mathbb{R}^n} (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2)_{|t=T} \, dx \quad (2.41)
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^n} (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2)_{|t=0} \, dx,
\]

Which we can write as
\[
E(u; \mathbb{R}^n, T) = E(u; \mathbb{R}^n, 0). \quad (2.42)
\]


\[\Box\]
Chapter 3

3.1 Initial-Boundary Value Problems for the Wave Equation in Two Independent Variables

Let us look at the wave equation and begin to examine initial-boundary value problems. We will show how the initial-boundary value problem for the wave equation can be solved for simple geometries by the method of separation of variables. Consider the following initial-boundary value problem given by

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < l, \quad t > 0, \quad (3.1)
\]

\[
u(x,0) = \phi_0(x)
\]

\[
\frac{\partial u}{\partial t}(x,0) = \phi_1(x) \quad 0 \leq x \leq l, \quad (3.2)
\]

\[
u(0,t) = u(l,t) = 0, \quad t \geq 0, \quad (3.3)
\]

where \(u(x,t)\) continuously assumes its initial-boundary data, \(l\) is a positive constant, and the functions \(\phi_0\) and \(\phi_1\) satisfy the conditions such as \(\phi_0(0) = \phi_0(l) = \phi_1(0) = \phi_1(l) = 0\). We need to show that there is at most one solution to the initial-boundary problem (3.1)-(3.3). If we assume that \(\phi_0(x)\) and \(\phi_1(x)\) are identically zero, then we will show that \(u(x,t)\) is identically zero in the rectangle

\[\mathcal{R} = \{(x,t): 0 < x < l, 0 < t < T\}\]

where \(T\) is an arbitrary, but is fixed, positive number. Consider energy integral and write it as the sum of the potential and kinetic energies of the physical problem modeled by (3.1)-(3.3). Assume that \(u \in C^2(\mathcal{R}) \cap C^1(\overline{\mathcal{R}})\) and
\[ E(t) = \frac{1}{2} \int_0^t \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial u}{\partial x} \right)^2 \, dx. \]

Since \( \phi_0(x) \) and \( \phi_1(x) \) are identically zero, we have \( E(0) = 0 \) and then from (3.1) and (3.3), we get

\[
\frac{\partial E(t)}{\partial t} = \int_0^t \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial u}{\partial x} \right)^2 \, dx
= \int_0^t \frac{\partial u}{\partial t} \left( \frac{\partial^2 u}{\partial t^2} \right) + \left( \frac{\partial u}{\partial x} \right)^2 \, dx + \left[ \frac{\partial u}{\partial x} \right]_{x=0}^{x=t} = 0.
\]

This implies that \( E(t) \) is independent of \( t \), and \( E(0) = E(t) = 0 \) for \( 0 \leq t \leq T \). Then, we have \( \frac{\partial}{\partial t} u(x,t) = \frac{\partial}{\partial x} u(x,t) = 0 \) and \( u(x,t) \) equals a constant in \( \Omega \). But since \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) and \( u(x,0) = 0 \), we know \( u(x,t) \) is identically zero in \( \Omega \).

Now, we want to show that the solution \( u(x,t) \) of (3.1)-(3.3) depends continuously on \( \phi_0(x) \) and \( \phi_1(x) \) with respect to the maximum norm where \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \). From all these conditions, we can conclude that \( E(t) \) is constant even if \( \phi_0(x) \) and \( \phi_1(x) \) are not identically zero. Then, if \( u(x,t) \) is a solution of (3.1)-(3.3), we have

\[
\int_0^t \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial u}{\partial x} \right)^2 \, dx = \int_0^t \left( \phi_1(x) \right)^2 + \left( \phi_0'(x) \right)^2 \, dx
\]

for \( 0 \leq t \leq T \). Therefore, if \( u^{(1)}(x,t) \) is the solution with initial data \( \phi_0^{(1)}(x) \) and \( \phi_1^{(1)}(x) \), and \( u^{(2)}(x,t) \) is the solution with initial data \( \phi_0^{(2)}(x) \) and \( \phi_1^{(2)}(x) \) such that for some positive constant \( \varepsilon \),
\[
\max_{\phi \in \mathcal{G}_l} \left| \frac{d}{dx} \phi^{(1)}(x) - \frac{d}{dx} \phi^{(2)}(x) \right| < \varepsilon ,
\]
and
\[
\max_{\phi \in \mathcal{G}_l} \left| \phi^{(1)}(x) - \phi^{(2)}(x) \right| < \varepsilon ,
\]
then for \( u(x,t) = u^{(1)}(x,t) - u^{(2)}(x,t) \), we have
\[
\int_0^1 \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial u}{\partial x} \right)^2 \right] dx \leq 2\varepsilon^2 l
\]
and
\[
u(x,t) = \int_0^x \frac{\partial u}{\partial x} dx + u(0,t)
\]
\[
= \int_0^x \frac{\partial u}{\partial x} dx .
\]

Using Schwarz inequality, see [9], we have
\[
\left| u(x,t) \right| \leq \int_0^1 \left| \frac{\partial u}{\partial x} \right| dx
\]
\[
\leq \left[ \int_0^1 \left( \frac{\partial u}{\partial x} \right)^2 dx \right]^{1/2} \int_0^1 \left( \frac{\partial u}{\partial x} \right)^2 dx \right]^{1/2}
\]
\[
\leq \sqrt{2\varepsilon l}^{1/2}
\]
for \((x,t) \in \mathcal{G}\). This implies that the continuous dependence of the solution of (3.1)-(3.3) on the initial data.

Now, we are going to use the method of separation of variables or Fourier series to construct the solution of initial-boundary value problem. We assume that the solution \( u(x,t) \) can be written as the product of \( X(x) \) and \( T(t) \), i.e.,
\[ u(x,t) = X(x)T(t) \]

where \( X(x) \) and \( T(t) \) are the functions of \( x \) and \( t \), respectively. Substituting the above relation into (3.1), we get

\[ T''X = TX'' \]

or

\[ \frac{T''}{T} = \frac{X''}{X}. \]

Since the left and right sides of this equation are functions of \( t \) and \( x \), respectively, hence

\[ \frac{T''}{T} = \frac{X''}{X} = \text{constant}. \]

Letting the constant be equal to \(-\lambda\), we get two differential equations

\[ T'' + \lambda T = 0, \quad (3.4) \]

and

\[ X'' + \lambda X = 0. \quad (3.5) \]

From the boundary condition (3.3), we get

\[ X(0) = X(l) = 0. \quad (3.6) \]

To solve equations (3.6) and (3.5), we will look at cases for \( \lambda \), because we need nontrivial solutions \( X(x) \) of the boundary value problem. From the elementary theory of ordinary differential equation it is easily seen that (3.5) and (3.6) can have a nontrivial solution only if \( \lambda > 0 \). Consider various possibilities for \( \lambda \).

If \( \lambda < 0 \), the general solution of (3.5) is

\[ X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}. \]
To satisfy the boundary conditions (3.6) the constants must be such that

\[ C_1 + C_2 = 0, \]
\[ C_1 e^{-\sqrt{-\lambda}l} + C_2 e^{-\sqrt{-\lambda}l} = 0. \]

Solving the system of equations, we get \( C_1 = C_2 = 0 \), which is the trivial solution of (3.5) and (3.6).

If \( \lambda = 0 \), the general solution of (3.5) is

\[ X(x) = C_1 + C_2 x. \]

To satisfy the boundary conditions (3.6), we must have

\[ C_1 = 0, \]
\[ C_1 + C_2 l = 0. \]

Again, the trivial solution is the only solution of (3.5) and (3.6), because we have \( C_1 = C_2 = 0 \).

Finally, if \( \lambda > 0 \) the solution of (3.5) is

\[ X(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x \]

where \( C_1 \) and \( C_2 \) are arbitrary constants. Using our boundary conditions \( X(0) = 0 \) and \( X(l) = 0 \), we have

\[ C_1 = 0, \]
\[ C_1 \cos \sqrt{\lambda} l + C_2 \sin \sqrt{\lambda} l = 0. \]

In order for \( X(x) \) not to vanish identically, we must have

\[ \lambda = \left( \frac{k\pi}{l} \right)^2, \quad \text{(} k = 1, 2, \ldots \),\]
\[ X(x) = X_k(x) = \sin \frac{k \pi x}{l} \]

Setting \( C_2 = 1 \). Then from (3.4)

\[ T(t) = T_k(t) = a_k \cos \frac{k \pi t}{l} + b_k \sin \frac{k \pi t}{l}, \]

where \( a_k \) and \( b_k \) are arbitrary constants. We have constructed a set of solutions to the wave equation (3.1) satisfying the boundary condition (3.3) given by

\[ u_k(x, t) = \sin \frac{k \pi x}{l} \left( a_k \cos \frac{k \pi t}{l} + b_k \sin \frac{k \pi t}{l} \right). \]

In order to satisfy the initial condition (3.2) we consider now the infinite series given by

\[ u(x, t) = \sum_{k=1}^{\infty} \left( a_k \cos \frac{k \pi t}{l} + b_k \sin \frac{k \pi t}{l} \right) \sin \frac{k \pi x}{l}, \tag{3.7} \]

which is the solution of (3.1) and (3.3) provided we can differentiate the series term-wise with respect to \( x \) and \( t \) twice. In order for the function \( u(x, t) \) to satisfy (3.2), we must have

\[ \phi_0(x) = u(x, 0) = \sum_{k=1}^{\infty} a_k \sin \frac{k \pi x}{l} \]

\[ \phi_1(x) = \frac{\partial u}{\partial t}(x, 0) = \frac{\pi}{l} \sum_{k=1}^{\infty} k b_k \sin \frac{k \pi x}{l}. \]

From the theory of Fourier series, using orthogonality of sine function, we get

\[ a_k = \frac{2}{l} \int_{0}^{l} \phi_0(x) \sin \frac{k \pi x}{l} \, dx \tag{3.8} \]

\[ b_k = \frac{2}{k \pi} \int_{0}^{l} \phi_1(x) \sin \frac{k \pi x}{l} \, dx. \tag{3.9} \]
The solution of problem given by (3.1), (3.2), and (3.3) is given by (3.7)-(3.9), provided that the series can be differentiated term-wise twice and continuously assumes the initial data (3.2) having the given Fourier series expansions.

Let us consider the following example.

Example 3.1 Let us solve the following problem:

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 ,
\]

\[ u(x,0) = x(l-x) \]

\[ \frac{\partial u}{\partial t}(x,0) = 0 \]

\[ u(0,t) = u(l,t) = 0 . \]

We know that \( u(x,t) \) has an expansion

\[ u(x,t) = \sum_{k=1}^{\infty} \left( a_k \cos \frac{k\pi t}{l} + b_k \sin \frac{k\pi t}{l} \right) \sin \frac{k\pi x}{l} . \]

Differentiating with respect to time yields

\[ \frac{\partial u}{\partial t}(x,t) = k\pi \sum_{k=1}^{\infty} \left( -a_k \sin \frac{k\pi t}{l} + b_k \cos \frac{k\pi t}{l} \right) \sin \frac{k\pi x}{l} . \]

Setting \( t = 0 \), we have

\[ \frac{\partial u}{\partial t}(x,0) = \sum_{k=1}^{\infty} k\pi \left( -a_k \sin \frac{k\pi 0}{l} + b_k \cos \frac{k\pi 0}{l} \right) \sin \frac{k\pi x}{l} = 0 . \]

Then

\[ 0 = \sum_{k=1}^{\infty} k\pi b_k \sin \frac{k\pi x}{l} , \text{ for all } x \in [0,l] \]

implies that all \( b_k = 0 \). Using \( u(x,0) = x \), we get

\[ x = \sum_{k=1}^{\infty} a_k \sin \frac{k\pi x}{l} . \]
For \( \phi_0(x) = x \) in the interval \((0, l)\), its Fourier sine series has the coefficients

\[
a_k = \frac{2}{l} \int_0^l \phi_0(x) \sin \frac{k\pi x}{l} \, dx
\]

\[
= \frac{2}{l} \int_0^l (x - x^2) \sin \frac{k\pi x}{l} \, dx
\]

\[
= \frac{2}{l} \left[ -\frac{l^3}{k\pi} \cos \frac{k\pi}{l} + \frac{l^2}{(k\pi)^2} \sin \frac{k\pi}{l} - \frac{l^3}{(k\pi)^3} \cos \frac{k\pi}{l} \right]_0^l
\]

\[
= (-1)^{k+1} \left( \frac{l}{k\pi} \right)^3.
\]

Therefore, the complete solution is

\[
u(x, t) = \frac{2l^3}{\pi^3} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3} \sin \frac{k\pi x}{l} \cos \frac{k\pi t}{l}.
\]

\[\square\]

**Theorem 3.1** The function \( u(x, t) \) is given by (3.7)-(3.9) has continuous second derivatives in \( \mathfrak{R} = \{ (x, t) : 0 \leq x \leq l, \ 0 \leq t \leq T \} \) and satisfies (3.1)-(3.3) whenever

\( \phi_0(0) = \phi_0(l) = 0, \ \phi_0''(0) = \phi_0''(l) = 0 \) and \( \phi_1(0) = \phi_1(l) = 0 \), where \( \phi_0 \in C^3[0,l] \), and \( \phi_1 \in C^2[0,l] \). See [3].
3.2 The Wave Equation on a Half-Line

Consider the following problem

\[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad x > 0, t > 0, \]

\[ u(x,0) = \phi_0(x), \quad x \geq 0, \]

\[ \frac{\partial u}{\partial t}(x,0) = \phi_1(x), \quad x \geq 0, \]

\[ u(0,t) = 0, \quad t \geq 0. \]

Since we are looking for the solution of this problem on the half-line, we will use d'Alembert's formula for the initial value problem on the entire line. To write a solution of this problem on the half-line, let us define odd extensions of \( \phi_0 \) and \( \phi_1 \) to the entire line.

That is,

\[ \Phi_0(x) = \begin{cases} \phi_0(x), & \text{if } x \geq 0 \\ -\phi_0(-x), & \text{if } x < 0 \end{cases} \]

and

\[ \Phi_1(x) = \begin{cases} \phi_1(x), & \text{if } x \geq 0 \\ -\phi_1(-x), & \text{if } x < 0 \end{cases} \]

\( \Phi_0 \) agrees with \( \phi_0 \), and \( \Phi_1 \) agree with \( \phi_1 \) on the half line \( x \geq 0 \). Furthermore

\[ \Phi_0(-x) = -\Phi_0(x) \]

and

\[ \Phi_1(-x) = -\Phi_1(x) \]

for all \( x \neq 0 \).
Let us recall the Cauchy problem for the wave equation in two independent variables as we have seen in Chapter 2:

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad x \in \mathbb{R}, \quad 0 \leq t < \infty,
\]

\[
u(x,0) = \phi_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \phi_1(x).
\]

It has the solution given by

\[
u(x,t) = \frac{\phi_0(x-t) + \phi_0(x+t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \phi_1(s) \, ds.
\]

Now consider, the Cauchy problem for the entire line:

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad x \in \mathbb{R},
\]

\[
u(x,0) = \Phi_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \Phi_1(x).
\]

The solution of this problem is

\[
u(x,t) = \frac{\Phi_0(x-t) + \Phi_0(x+t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \Phi_1(s) \, ds. \quad (3.10)
\]

\(u(x,t)\) satisfies the wave equation for \(x \in \mathbb{R}\) and \(t > 0\). Also for \(x > 0\) and \(t > 0\). If \(x \geq 0\) then

\[
u(x,0) = \frac{1}{2} (\Phi_0(x) + \Phi_0(x)) = \phi_0(x),
\]

and

\[
\frac{\partial u}{\partial t}(x,0) = \frac{1}{2} (-\Phi_0(x) + \Phi_0(x)) + \frac{1}{2} (\Phi_1(x) + \Phi_1(x)) = \phi_1(x).
\]

Therefore, \(u(x,t)\) satisfies the initial conditions of the problem on the half-line. We can see that \(u(0,t) = 0\) as follows. We have
\[ u(0,t) = \frac{\Phi_0(-t) + \Phi_0(t)}{2} + \frac{1}{2} \int_{-t}^{t} \Phi_1(s) ds. \]

Since we define \( \Phi_0 \) as the odd extension of \( \phi_0 \) then
\[ \Phi_0(-t) + \Phi_0(t) = -\phi_0(-t) + \phi_0(t) = 0, \]
and
\[ \int_{-t}^{t} \Phi_1(s) ds = 0 \]
because we also defined \( \Phi_1 \) to be an odd function. Thus, \( u(0,t) = 0 \).

We have shown that the solution (3.10) of the extended initial value problem on the entire line is also the solution of the initial-boundary value problem on the half-line.

We consider the following example:

**Example 3.2** Let us solve
\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad x > 0, t > 0,
\]
\[ u(x,0) = 1 - e^{-x} \]
\[ \frac{\partial u}{\partial t}(x,0) = \cos(x) \]
\[ u(0,t) = 0, \quad t \geq 0. \]

We have \( \phi_0 = 1 - e^{-x} \) and \( \phi_1 = \cos(x) \). The odd extensions of these functions to the entire line are
\[
\Phi_0(x) = \begin{cases} 
1 - e^{-x}, & \text{if } x \geq 0 \\
-1 + e^x, & \text{if } x < 0
\end{cases}
\]

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and

\[
\Phi_1(x) = \begin{cases} 
\cos(x), & \text{if } x \geq 0 \\
-\cos(x), & \text{if } x < 0.
\end{cases}
\]

Then \( \Phi_0(-x) = -\Phi_0(x) \) and \( \Phi_1(-x) = -\Phi_1(x) \). The graphs of both extensions are shown in Figure 3.1.

For \( x \geq 0 \) and \( t \geq 0 \) the solution to the initial-boundary value problem on the half-line is

\[
u(x,t) = \frac{\Phi_0(x-t) + \Phi_0(x+t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \Phi_1(s) ds.
\]

Since \( \Phi_0(x) \) and \( \Phi_1(x) \) are in the terms of \( \phi_0 \) and \( \phi_1 \), we can write the above solution in terms of the original initial position and velocity functions.

If \( x-t \geq 0 \) then

\[
u(x,t) = \frac{\Phi_0(x-t) + \Phi_0(x+t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \Phi_1(s) ds
\]

\[
= \frac{1}{2} (1 - e^{-(x-t)} + 1 - e^{-(x+t)}) + \frac{1}{2} \int_{x-t}^{x+t} \cos(s) ds
\]

\[
= 1 - \frac{1}{2} (e^{-x+t} + e^{-x-t}) + \frac{1}{2} [\sin(s)]_{x-t}^{x+t}
\]

\[
= 1 - e^{-x} \frac{1}{2} (e^t + e^{-t}) + \frac{1}{2} (\sin(x + t) - \sin(x - t))
\]

This simplifies to

\[
u(x,t) = 1 - e^{-x} \cosh(t) + \cos(x) \sin(t)
\]

If \( x-t < 0 \) then
\[
\begin{align*}
    u(x,t) &= \frac{\Phi_0(x-t) + \Phi_0(x+t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \Phi_1(s) ds \\
    &= \frac{1}{2} (-1 + e^{x-t} + 1 - e^{-(x+t)}) + \frac{1}{2} \int_{x-t}^{x+t} \cos(s) ds + \frac{1}{2} \int_{0}^{x+t} \cos(s) ds \\
    &= \frac{1}{2} (e^{x-t} - e^{-(x-t)}) - \left[ \frac{1}{2} \sin(s) \right]_{x-t}^{x+t} + \left[ \frac{1}{2} \sin(s) \right]_{0}^{x+t} \\
    &= e^{-t} \frac{1}{2} (e^x - e^{-x}) + \frac{1}{2} (\sin(x-t) + \sin(x+t)) \\
    &= e^{-t} \sinh(x) + \sin(x) \cos(t).
\end{align*}
\]

Therefore, the solution the initial-boundary value is

\[
    u(x,t) = \begin{cases} 
        1 - e^{-x} \cosh(t) + \cos(x) \sin(t), & \text{if } 0 < t < x \\
        e^{-t} \sinh(x) + \sin(x) \cos(t), & \text{if } 0 < x < t.
    \end{cases}
\]

Figure 3.1 Graphs of the odd extensions of \(\phi_0\) and \(\phi_1\) in Example 3.1
Example 3.3 Let us solve

\[ 9 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad x > 0, t > 0, \]

\[ u(x,0) = x^3 \]

\[ \frac{\partial u}{\partial t}(x,0) = e^{-x} \quad x \geq 0, \]

\[ u(0,t) = 0, \quad t \geq 0. \]

We have \( \phi_0 = x^3 \) and \( \phi_1 = e^{-x} \). Let us consider the odd extensions of these functions to the entire line:

\[ \Phi_0(x) = x^3 \quad \text{for all } x \]

and

\[ \Phi_1(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ -e^x & \text{if } x < 0. \end{cases} \]

Then \( \Phi_0(-x) = -\Phi_0(x) \) and \( \Phi_1(-x) = -\Phi_1(x) \).

For \( x \geq 0 \) and \( t \geq 0 \) the solution of the initial-boundary value problem on the half-line with \( \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \) replaced by \( \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \) is

\[ u(x,t) = \frac{\Phi_0(x-ct) + \Phi_0(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \Phi_1(s) \, ds. \]

Since \( \Phi_0(x) \) and \( \Phi_1(x) \) are in the terms of \( \phi_0 \) and \( \phi_1 \), we can write this solution in terms of the original initial position and velocity functions.

If \( x - 3t \geq 0 \) then

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\[ u(x,t) = \frac{\Phi_0(x - ct) + \Phi_0(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \Phi_1(s) ds \]

\[ = \frac{1}{2} ((x - 3t)^3 + (x + 3t)^3) + \frac{1}{6} \int_{x-3t}^{x+3t} e^{-s} ds \]

\[ = \frac{1}{2} (2x^3 + 54xt^2) + \left[ \frac{1}{6} e^{-s} \right]_{x-3t}^{x+3t} \]

\[ = x^3 + 17xt^2 + \frac{1}{6} (e^{-x}e^{-3t} - e^{-x}e^{3t}) \]

\[ = x^3 + 17xt^2 + \frac{1}{3} e^{-x} \left( e^{3t} - e^{-3t} \right). \]

Then

\[ u(x,t) = x^3 + 17xt^2 - \frac{1}{3} e^{-x} \sinh(3t). \]

If \( x - 3t < 0 \) then

\[ u(x,t) = \frac{\Phi_0(x - ct) + \Phi_0(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \Phi_1(s) ds \]

\[ = \frac{1}{2} ((x - 3t)^3 + (x + 3t)^3) + \frac{1}{6} \int_{x-3t}^{0} e^{-s} ds + \int_{0}^{x+3t} e^{-s} ds \]

\[ = \frac{1}{2} (2x^3 + 54xt^2) - \left[ \frac{1}{6} e^{s} \right]_{x-3t}^{0} - \left[ \frac{1}{6} e^{-s} \right]_{0}^{x+3t} \]

\[ = x^3 + 17xt^2 + \frac{1}{6} e^{x} - \frac{1}{6} e^{-3t} - \frac{1}{6} e^{-(x+3t)} + \frac{1}{6} \]

\[ = x^3 + 17xt^2 + \frac{1}{6} (e^{-x}e^{-3t} + e^{x}e^{-3t}). \]
\[ = x^3 + 17xt^2 + \frac{1}{6}e^{-3t}(e^x + e^{-x}) \]

\[ = x^3 + 17xt^2 + \frac{1}{3}e^{-3t} \cosh(x). \]

Therefore, the solution is

\[ u(x,t) = \begin{cases} 
  x^3 + 17xt^2 - \frac{1}{3}e^{-x} \sinh(3t), & \text{if } 0 < 3t < x \\
  x^3 + 17xt^2 + \frac{1}{3}e^{-3t} \cosh(x), & \text{if } 0 < x < 3t.
\end{cases} \]

\[ \square \]

### 3.3 Uniqueness of Solution of an Initial-Boundary Value Problem

The general initial-boundary value problem is to find a function \( u(x,t) \) defined for \( x \in \Omega \) and \( t \geq 0 \), and satisfying the wave equation

\[ \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad x \in \Omega, t > 0, \quad (3.11) \]

\[ u(x,0) = \phi_0(x) \]

\[ \frac{\partial u}{\partial t}(x,0) = \phi_1(x) \quad \text{for } x \in \Omega \quad \text{and} \quad t \geq 0, \quad (3.12) \]

\[ u(x,t) = 0 \quad \text{for } x \in \partial \Omega \quad \text{and} \quad t \geq 0, \quad (3.13) \]

or

\[ \frac{\partial u}{\partial n}(x,t) = 0 \quad \text{for } x \in \partial \Omega \quad \text{and} \quad t \geq 0. \quad (3.14) \]
The solution is required to satisfy only one of the boundary conditions (3.13) or (3.14). Differentiation in (3.14) is in the direction to the exterior normal to $\partial \Omega$. The uniqueness of the solution of the initial-boundary value problem follows from the conservation of energy.

**Theorem 3.2** If $u$ is a solution that satisfies the wave equation in $\Omega$ and one of the boundary conditions (3.13) or (3.14), suppose $u$ is in $C^2(\bar{\Omega})$, and $\Omega$ is a bounded domain in $\mathbb{R}^n$. Then

$$\int_{\Omega} \left( u_t^2 + \cdots + u_x^2 + u_i^2 \right) |_{t=T} \, dx = \int_{\Omega} \left( u_t^2 + \cdots + u_x^2 + u_i^2 \right) |_{t=0} \, dx$$

(3.15)

for every $T \geq 0$. See [11].

**Proof:** We give the proof for $n = 2$. Let $\Omega^T$ be the cylindrical domain in the $(x_1, x_2, t)$-space consisting of the points $(x_1, x_2, t)$ for which $(x_1, x_2, t) \in \Omega$ and $0 < t < T$. Since $u$ satisfies the wave equation, we have

$$\iiint_{\Omega^T} \left[ (2u, u_{x_1})_{x_1} + (2u, u_{x_2})_{x_2} - (u^2_{x_1} + u^2_{x_2} + u^2_t) \right] dx_1 dx_2 dt = 0.$$  

(3.16)

The integrand in this equation is the divergence of the vector field. Using the divergence theorem we get

$$\iiint_{\partial \Omega^T} \left[ 2u, u_{x_1} v_1 + 2u, u_{x_2} v_2 - (u^2_{x_1} + u^2_{x_2} + u^2_t) v_t \right] d\sigma = 0,$$

(3.17)

where $\partial \Omega^T$ the boundary of $\Omega^T$, and $v$ is the unit normal vector on $\partial \Omega^T$ in the direction of exterior to $\Omega^T$. Here, we split our integral into three integrals because our boundary,
∂Ω^T consists of three parts. On the top and bottom of the cylinder, \((x_1, x_2) \in \Omega, \; t = T,\)
\(v = (0,0,1)\) and \((x_1, x_2) \in \Omega, \; t = 0, \; v = (0,0,-1),\) respectively. The integrals are

\[- \iint_\Omega (u^2 + u_x^2 + u_t^2)_{t=T}^t dx_1 dx_2 = 0.\] (3.18)
\[\iint_\Omega (u^2 + u_x^2 + u_t^2)_{t=0}^t dx_1 dx_2 = 0.\] (3.19)

On the cylindrical surface, \((x_1, x_2) \in \partial \Omega, \; 0 \leq t \leq T, \; \nu = (n_1, n_2, 0),\) where \(n = (n_1, n_2)\) is the exterior normal to \(\partial \Omega,\) and the integral is

\[\int_0^T \int_{\partial \Omega} (2u_x n_1 + 2u_t n_2) ds dt = \int_0^T 2u_t \frac{\partial u}{\partial n} ds dt,\] (3.20)

where \(ds\) is the element of length on the boundary curve \(\partial \Omega.\) Now, we can write equation (3.17) as the sum of equations (3.18), (3.19) and (3.20)

\[0 = -\iint_\Omega (u^2 + u_x^2 + u_t^2)_{t=T}^t dx_1 dx_2 + \iint_\Omega (u^2 + u_x^2 + u_t^2)_{t=0}^t dx_1 dx_2 + \int_0^T 2u_t \frac{\partial u}{\partial n} ds dt.\] (3.21)

In both cases, where \(u\) satisfies the boundary condition (3.13) or (3.14), the last integral in (3.21) is zero. Hence, in either case, (3.20) yields the conclusion (3.15) of the theorem for \(n = 2.\) If \(u(x,t)\) satisfies the initial condition (3.12), then the energy of \(u\) in \(\Omega\) can be computed in terms of the initial data. Then our equation (3.15) can be written as

\[\iint_\Omega (u^2 + \cdots + u^2_{x_n} + u_t^2)_{t=T}^t dx = \iint_\Omega (\phi_0^2 + \cdots + \phi_{x_n}^2 + \phi_t^2) dx.\]

From the above theorem, we can conclude the following.
Corollary 3.1 There can be at most one solution of the initial-boundary value problem (3.11), (3.12) and (3.13) or (3.11), (3.12) and (3.14).

3.4 Fourier’s Method for the Wave Equation in Three Independent Variables

Consider the problem of finding the solution of the following initial-boundary value problem for the wave equation in a rectangular cylinder. We want to find a function $u = (x_1, x_2, t)$ such that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial t^2} = 0 \quad ((x_1, x_2) \in \mathbb{R}, \ t > 0) \quad (3.22)$$

$$u(x_1, x_2, 0) = \phi_0(x_1, x_2) \quad ((x_1, x_2) \in \mathcal{G}) \quad (3.23)$$

$$u(0, x_2, t) = u(a, x_2, t) = 0 \quad 0 \leq x_2 \leq b, t \geq 0$$

$$u(x_1, 0, t) = u(x_1, b, t) = 0 \quad 0 \leq x_1 \leq a, t \geq 0 \quad (3.24)$$

where $u(x_1, x_2, t)$ continuously assumes its initial-boundary data, $\mathcal{G} = \{(x, t) : 0 < x_1 < a, 0 < x_2 < b\}$, $a$ and $b$ are positive constants. Assuming that functions $\phi_0(x_1, x_2)$ and $\phi_1(x_1, x_2)$ are smooth enough, we shall formally construct a solution to (3.22)-(3.24) by Fourier’s method. See [8]. We assume that the solution $u(x_1, x_2, t)$ can be written as the product of $v(x_1, x_2)$ and $T(t)$, i.e.
\[ u(x_1, x_2, t) = v(x_1, x_2)T(t) \]

where \( v(x_1, x_2) \) and \( T(t) \) are functions of \( x_1, x_2 \) and \( t \), respectively, such that (3.24) is satisfied. Substituting the above relation into (3.22), we get

\[
\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \frac{T''}{v} = \text{constant.}
\]

Letting the constant be equal \(-k^2\), we get

\[ T'' + k^2 T = 0 \]  

(3.25)

\[ \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + k^2 v = 0 \]

\[ ((x_1, x_2) \in \Re) \]  

(3.26)

\[ v(0, x_2) = v(a, x_2) = 0 \]
\[ 0 \leq x_2 \leq b \]

\[ v(x_1, 0) = v(x_1, b) = 0 \]
\[ 0 \leq x_1 \leq a. \]

(3.27)

We have an eigenvalue problem (3.26) and (3.27) for the partial differential equation. We solve it by using the method of separation of variables. Let

\[ v(x_1, x_2) = X(x_1)Y(x_2). \]

Then, substituting into (3.26), we have

\[
\frac{Y''}{Y} + k^2 = \frac{X''}{X} = k_i^2
\]

where \( k_i \) is a constant, or

\[ X'' + k_i^2 X = 0 \]

\[ Y'' + k_2^2 Y = 0 \]

where \( k^2 = k_1^2 + k_2^2 \). Solving these differential equations gives

\[ X(x_1) = c_1 \cos k_1 x_1 + c_2 \sin k_1 x_1 \]

\[ Y(x_2) = c_3 \cos k_2 x_2 + c_4 \sin k_2 x_2, \]
where \( c_1, c_2, c_3 \) and \( c_4 \) are arbitrary constants. From (3.27), we have

\[
X(0) = X(a) = 0 \\
Y(0) = Y(b) = 0.
\]

Letting \( c_2 = c_4 = 1 \), we get

\[
X(x_1) = \sin k_1 x_1 \\
Y(x_2) = \sin k_2 x_2
\]

where

\[
\sin k_1 a = 0 \\
\sin k_2 b = 0.
\]

Hence

\[
k_1 = k_{1m} = \frac{m\pi}{a} \quad (m = 1, 2, \ldots)
\]

\[
k_2 = k_{2n} = \frac{n\pi}{b} \quad (n = 1, 2, \ldots)
\]

Thus, the eigenvalues of (3.26) and (3.27) are given by

\[
k_{mn}^2 = k_{1m}^2 + k_{2n}^2 = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \quad (m, n = 1, 2, 3, \ldots)
\]

With the corresponding eigenfunctions

\[
v_{mn}(x_1, x_2) = \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b}.
\]

The solution of (3.25) is now given by

\[
T_{mn}(t) = a_{mn} \cos k_{mn} t + b_{mn} \sin k_{mn} t
\]

and hence formally
\[ u(x_1, x_2, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( a_{mn} \cos k_{mn} t + b_{mn} \sin k_{mn} t \right) \sin \frac{m \pi x_1}{a} \sin \frac{n \pi x_2}{b} \]  \hspace{1cm} (3.28)

will be a solution of (3.22) and (3.24). In order to satisfy (3.23), we must have

\[ u(x_1, x_2, 0) = \phi_0(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m \pi x_1}{a} \sin \frac{n \pi x_2}{b} \]

\[ \frac{\partial u}{\partial t}(x_1, x_2, 0) = \phi_1(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} k_{mn} \sin \frac{m \pi x_1}{a} \sin \frac{n \pi x_2}{b} \]

From Fourier's theorem, we should choose the coefficients \( a_{mn} \), and \( b_{mn} \) to be

\[ a_{mn} = \frac{4}{ab} \int_0^b \int_0^a \phi_0(x_1, x_2) \sin \frac{m \pi x_1}{a} \sin \frac{n \pi x_2}{b} \, dx_1 dx_2 \]  \hspace{1cm} (3.29)

\[ b_{mn} = \frac{4}{abk_{mn}} \int_0^b \int_0^a \phi_1(x_1, x_2) \sin \frac{m \pi x_1}{a} \sin \frac{n \pi x_2}{b} \, dx_1 dx_2 \]  \hspace{1cm} (3.30)

Equations (3.28)-(3.30) define the formal solution of (3.22)-(3.24). In order to verify that it is a solution, we must make smoothness assumptions on \( \phi_0(x_1, x_2) \) and \( \phi_1(x_1, x_2) \), and argue as in the Section 3.1. For details see [3].
4.1 Solution of Initial Value Problems for the Nonhomogeneous Wave Equation in Two Independent Variables

Consider the initial-value problem

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t),
\]

(4.1)

\[
u(x,0) = \phi_0(x)\]

(4.2)

\[
\frac{\partial u}{\partial t}(x,0) = \phi_1(x).
\]

In considering the wave equation as modeling vibrations along a string, \( f(x,t) \) could be interpreted as an external driving or damping force. We will use the characteristic triangle to write the solution of this initial value problem.

Suppose we want to obtain the solution \( u(x_0,t_0) \). Let \( \Omega \) denote the characteristic triangle having vertices \((x_0,t_0), (x_0 - ct_0,0)\) and \((x_0 + ct_0,0)\), as shown in Figure 4.1. \( \Omega \) includes the sides \( L, M, \) and \( I \) of the triangle.

We integrate \( f = u_t - c^2 u_{xx} \) over the triangle \( \Omega \) and apply the Green's theorem. See [6]. Thus

\[
\iiint_{\Omega} f \, dxdt = \iint_{\Omega} (u_t - c^2 u_{xx}) \, dxdt.
\]

Green's theorem says that

\[
\iiint_{\Omega} (P_x - Q_t) \, dt \, dx = \oint_C P \, dt + Q \, dx
\]

for any functions \( P \) and \( Q \), where the line integral on the boundary is taken counterclockwise. \( C \) is the boundary of \( \Omega \). Thus, we get
\[
\int \int \alpha \, f(x, t) \, dx \, dt = \int \left( -c^2 u_x \, dt - u_t \, dx \right) .
\]

Now, evaluate the line integral over each of the line segments comprising \( \Omega \). On \( I \),
\[ dt = 0, \quad u_t(x,0) = \phi(x) \quad \text{and} \quad x \text{ varies from } (x_0 - ct_0) \text{ to } (x_0 + ct_0). \]
Then
\[
\int \frac{c^2 u_x \, dt + u_t \, dx}{I} = \int_{x_0 - ct_0}^{x_0 + ct_0} u_t(x,0) \, dx = \int_{x_0 - ct_0}^{x_0 + ct_0} \phi(s) \, ds.
\]
On \( M \), \( x - ct = x_0 - ct_0 \) so that \( dx = c \, dt \) and
\[
\int \frac{c^2 u_x \, dt + u_t \, dx}{L} = \int_{L} c \, dt + c^2 u_x \frac{1}{c} \, dx = c \int_{L} u_t \, dt + u_x \, dx = c(u(x_0 - ct_0,0) - u(x_0, t_0))
\]
because in the counterclockwise orientation \( M \) extends from \( (x_0, t_0) \) to \( (x_0 - ct_0,0) \).
On $L$, $x + ct = x_0 + ct_0$, so $dx = -cdt$ and

$$
\int_{M} c^2 u_x dt + u_x dx = \int_{M} u_x (-c) dt + c^2 u_x (-\frac{1}{c}) dx
$$

$$
= -c(u(x_0, t_0) - u(x_0 + ct_0, 0))
$$

Adding these three results, we get

$$
\int_{\Omega} \phi dxdt = \int_{x_0 - c t_0}^{x_0 + c t_0} \phi(s) ds - c(u(x_0 - ct_0, 0)
$$

$$
- u(x_0, t_0)) + c(u(x_0 + ct_0, t_0) - u(x_0 - ct_0, 0).
$$

Now $u(x_0 - ct_0, 0) = \phi_0(x_0 - ct_0)$ and $u(x_0 + ct_0, 0) = \phi_0(x_0 + ct_0)$. We can therefore solve for $u(x_0, t_0)$ in the last equation to obtain

$$
u(x_0, t_0) = \frac{1}{2} \left( \phi_0 (x_0 + ct_0) + \phi_0 (x_0 - ct_0) \right)
$$

$$
+ \frac{1}{2c} \int_{x_0 - c t_0}^{x_0 + c t_0} \phi_1 (s) ds + \frac{1}{2c} \iint_{\Omega} f dxdt.
$$

We have used $x_0$ and $t_0$ in this derivation to be able to use $(x, t)$ for a point of $\Omega$. Then we can write its solution as

$$
u(x, t) = \frac{1}{2} \left( \phi_0 (x + ct) + \phi_0 (x - ct) \right)
$$

$$
+ \frac{1}{2c} \int_{x - c t}^{x + c t} \phi_1 (s) ds + \frac{1}{2c} \iint_{\Omega} f(\xi, \eta) d\xi d\eta.
$$

**Example 4.1** Let us consider the following initial-value problem

$$
4 \frac{\partial^2 u}{\partial x^2} + x \cos(t) = \frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty, \quad t > 0,
$$

$$
\frac{\partial u}{\partial t} = 0, \quad x = \pm \infty.
$$

An example of this kind is the wave equation

$$
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},
$$

with initial conditions

$$
u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x), \quad -\infty < x < \infty.
$$

The solution can be written

$$
u(x, t) = \frac{1}{2} \left( f(x + ct) + f(x - ct) \right) + \frac{1}{2c} \int_{x - ct}^{x + ct} g(s) ds.
$$

This is the d'Alembert solution for the wave equation.
\[ u(x,0) = e^{-x} \]
\[ \frac{\partial u}{\partial t}(x,0) = \sin(x) \quad -\infty \leq x \leq \infty. \]

Evaluate

\[ \int_\Omega \xi \cos(\eta) d\xi d\eta = \int_0^t \left( \int_{x-2t+2\eta}^{x+2t-2\eta} \xi d\xi \right) \cos(\eta) d\eta \]
\[ = \frac{1}{2} \int_0^t \left( \int_{x-2t+2\eta}^{x+2t-2\eta} \xi \cos(\eta) d\eta \right) \cos(\eta) d\eta \]
\[ = \int_0^t 4x(t-\eta) \cos(\eta) d\eta \]
\[ = 4tx \sin t - 4x \int_0^t \eta \cos(\eta) d\eta \]
\[ = 4tx \sin t - 4x \left( \sin t \left|_0^t \right. - \int_0^t \sin \eta d\eta \right) \]
\[ = 4tx \sin t - 4tx \sin t - 4x \cos t + 4x \]
\[ = 4x(1 - \cos t) \]

The solution of the initial value problem is

\[ u(x,t) = \frac{1}{2} (\phi_0(x + ct) + \phi_0(x - ct)) \]
\[ + \frac{1}{2c} \int_{x-ct}^{x+ct} \phi_1(s) ds + \frac{1}{2c} \int_\Omega f(\xi, \eta) d\xi d\eta \]
\[ = \frac{1}{2} (e^{-x-2t} + e^{-x+2t}) + \frac{1}{4} \int_{x-2t}^{x+2t} \sin(s) ds + x(1 - \cos t) \]
\[
\frac{1}{2} e^{-x}(e^{-2t} + e^{2t}) - \frac{1}{4} \cos(x + 2t) - \cos(x - 2t) + x(1 - \cos t).
\]

We can rewrite it as

\[
u(x,t) = e^{-x} \cosh(2t) + \frac{1}{2} \sin(x) \sin(2t) + x(1 - \cos t).
\]

4.2 A Nonhomogeneous Problem on a Bounded Interval

Consider the following initial-boundary value problem:

\[
\frac{\partial^2 u}{\partial x^2} + Ax = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < l, t > 0,
\]

(4.4)

\[
u(x,0) = \frac{\partial u}{\partial t} (x,0) = 0, \quad 0 \leq x \leq l,
\]

(4.5)

\[
u(0,t) = \nu(l,t) = 0, \quad t \geq 0.
\]

(4.6)

This initial-boundary value problem models the motion of a string pegged at its ends, with initial position and velocity given by \( \phi_0 \) and \( \phi_1 \), respectively, and \( A \) is a positive constant.

We attempt to find a solution of the form

\[
u(x,t) = X(x)T(t),
\]

the product of a function of \( x \) and a function of \( t \). Substitute this into the wave equation to obtain

\[
T^n X = TX'' + Ax.
\]
We cannot isolate all terms involving \( x \) on one side of the equation, and all the terms involving \( t \) on the other side. Therefore, we instead let

\[
u(x,t) = U(x,t) + f(x)\,.
\]

We must choose \( f \) so that \( U \) satisfies an initial-boundary value problem we can solve. Substituting \( u \) into the wave equation, we get

\[
\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + f''(x) + Ax.
\]

This is the homogeneous wave equation for \( U \), if we choose \( f \) so that

\[
f''(x) + Ax = 0.
\]

Consider the boundary conditions. First, we note that

\[
U(0,t) = u(0,t) - f(0) = -f(0)
\]

and

\[
U(l,t) = u(l,t) - f(l) = -f(l)
\]

for \( t \geq 0 \). If we choose \( f \) such that

\[
f(l) = f(0) = 0,
\]

then \( U(0,t) = 0 \) and \( U(l,t) = 0 \).

Therefore, we want to choose \( f \) so that

\[
f''(x) + Ax = 0; f(l) = f(0) = 0.
\]

First integrate \( f''(x) = -Ax \) twice to obtain

\[
f(x) = -\frac{A}{6} x^3 + Bx + C.
\]

Then

\[
f(0) = C = 0,
\]
and

\[ f(l) = -\frac{A}{6} l^3 + Bl = 0 \]

implies that \( B = (A/6)l^2 \). Using the boundary conditions, we obtain

\[ f(x) = -\frac{1}{6} Ax^3 + \frac{1}{6} Al^2 x \]

\[ = \frac{A}{6} x(l^2 - x^2). \]

Now, consider the initial conditions in the problem for \( U \). We need

\[ U(x,0) = u(x,0) - f(x) = -f(x) \]

and

\[ \frac{\partial U}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,0) = 0. \]

The initial-boundary value problem for \( U \) is

\[ \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial t^2} = 0 < x < l, t > 0, \]

\[ U(x,0) = -\frac{A}{6} x(l^2 - x^2) \]

\[ \frac{\partial U}{\partial t}(x,0) = 0 \]

\[ 0 \leq x \leq l, \]

\[ U(0,t) = U(l,t) = 0 \quad t \geq 0. \]

We know the solution of this initial-boundary value problem from Section 3.1, except here we have \( U \) in place of \( u \), \( \phi_0(x) = -\frac{A}{6} x(l^2 - x^2) \) and \( \phi_1(x) = 0 \). From equation (3.7), the solution can be written as

\[ U(x,t) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{l}\right) \cos\left(\frac{k\pi t}{l}\right), \]
in which

\[ a_k = \frac{2}{l} \int_0^l \phi_0(x) \sin \frac{k \pi x}{l} \, dx \]

\[ = \frac{2}{l} \int_0^l -\frac{A}{6} x(l^2 - x^2) \sin \frac{k \pi x}{l} \, dx \]

\[ = -\frac{Al}{3} \int_0^l x \sin \frac{k \pi x}{l} \, dx + \frac{Al}{3} \int_0^l x^3 \sin \frac{k \pi x}{l} \, dx \]

\[ = -\frac{Al}{3} \left( -\frac{x l}{\pi k} \cos \frac{k \pi x}{l} \right)_0^l + \frac{Al}{3} \int_0^l x^3 \cos \frac{k \pi x}{l} \, dx \]

\[ = \frac{Al^3}{3 \pi k} \cos k \pi - \frac{Al^3}{3 \pi k} \sin k \pi + \frac{Al^3}{3 \pi k} \int_0^l x^3 \cos \frac{k \pi x}{l} \, dx \]

\[ = \frac{2Al^3}{3 \pi k} \cos k \pi + \frac{Al^3}{(k \pi)^2} \sin k \pi - \frac{2Al}{(k \pi)^2} \left( -\frac{x l}{\pi k} \cos \frac{k \pi x}{l} \right)_0^l + \frac{Al}{\pi k} \int_0^l x^3 \cos \frac{k \pi x}{l} \, dx \]

\[ = (-1)^k \frac{2Al^3}{(k \pi)^3}. \]

That is,

\[ a_k = \frac{2Al^3 (-1)^k}{\pi^3 k^3}. \]

Therefore, the solutions is given by

\[ u(x, t) = \frac{2Al^3}{\pi^3} \sum_{k=1}^{\infty} (-1)^k \sin \left( \frac{k \pi x}{l} \right) \cos \left( \frac{k \pi t}{l} \right) + \frac{A}{6} x(l^2 - x^2). \]
Bibliography


