Remarks on a Semilinear Elliptic Equation on $\mathbb{R}^n$

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1. INTRODUCTION

Existence and symmetry properties of solutions are among the major questions in the study of partial differential equations. In this paper we consider the following semilinear elliptic equation

$$\Delta u - u + Q|u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^n,$$

$$u \geq 0, u \neq 0 \quad \text{in } \mathbb{R}^n, \quad \text{and} \quad u \to 0 \quad \text{at } \infty,$$

where $p$ is a constant satisfying

$$1 < p < \frac{n+2}{n-2}. \quad (1.2)$$

(For $n = 2$, the number $(n + 2)/(n - 2)$ is considered to be $\infty$.)

We study the existence of solutions of Eq. (1.1) with some restrictions on $Q$ in Section 2 below. In Sections 3 and 4 below, the symmetry properties and nonexistence of solutions are investigated. We find a class of radially symmetric potentials $Q$ for which positive nonradial solutions of (1.1) exist, and a nonexistence theorem is proved for certain radial potentials.

Equation (1.1) is referred to as a (nonautonomous) scalar field equation. It arises in various branches of applied mathematics, for example, in the study of standing wave solutions of nonlinear Klein-Gordon equations and of nonlinear Schrödinger equations. For existence theory of Eq. (1.1), a major difficulty is that in $\mathbb{R}^n$ we no longer have Sobolev Compact Embedding Theorems. Nevertheless, the important special case $Q \equiv 1$ and its generalizations have been investigated extensively by various authors, which in particular include Nehari [10], Synge [13], Berger [2], Coffman [3], Strauss [12], Berestycki and Lions [1], and McLeod and Serrin [9]. In 1963 Nehari [10] showed that in $\mathbb{R}^3$ Eq. (1.1) with $Q \equiv 1$ has a positive radial solution provided that $1 < p \leq 4$, and that in case $p = 5$, such a
solution does not exist. Nehari’s results may be extended to general \( p \) and \( n \) with \( 1 < p < (n+2)/(n-2) \) for existence and \( p \geq (n+2)/(n-2) \) for nonexistence (see, e.g., [2,11]). This nonexistence result together with the fact that any solution of (1.1) with \( Q \equiv 1 \) must be radial (see [5,6]) gives us a nonexistence result of positive solution for (1.1) in case \( p \geq (n+2)/(n-2) \).

For general potentials \( Q \), existence theorems have been established recently under various kinds of hypotheses on \( Q \) by Lions [8] and by Ding and Ni [4]. However, very simple examples (see Ni [11] for more details) show that in general one cannot hope to solve (1.1) even for bounded \( Q \)'s. On the other hand, a result recently given by Ding and Ni [4] shows that the solvability is guaranteed for any nonnegative bounded \( Q \) provided that it is radial. The results presented in Section 2 below improve some of Lions in [8]. Symmetry properties of solutions have been studied by Gidas, Ni, and Nirenberg in a series of elegant papers [5,6]. Our results here (Section 3 below) indicate that the radial symmetry of solutions of (1.1) is, in general, very sensitive to perturbations of the potential \( Q \). In [4], the existence of positive radial solutions of (1.1) has been studied by Ding and Ni for a radial potential \( Q \). It seems that our nonexistence result in Section 4 below shows that Corollary 4.8 in Ding and Ni [4] is optimal and thus completes the theory in some sense for radial cases.

We would like to point out that the results in Sections 2 and 3 can be extended to more general second order elliptic operators than \( \Delta \).

2. EXISTENCE RESULTS

2.1. Preliminaries. In this section we study the existence of solutions of Eq. (1.1). We shall use a variational approach, namely the so-called “Concentration-Compactness Principle” developed by Lions (see [8]) in solving Eq. (1.1).

First, for convenience, some notations need to be introduced. Let

\[
J(Q)[u] = \int_{\mathbb{R}^n} Q(x) |u|^{p+1}(x) \, dx,
\]

(2.1)

where \( Q(x) \) is continuous and bounded in \( \mathbb{R}^n \) with \( Q^+(x) \neq \in \mathbb{R}^n \) and \( u \in L^{p+1}(\mathbb{R}^n) \), where \( Q^+(x) = \max \{ Q(x), 0 \} \). Next for \( \lambda > 0 \), we define

\[
I(\lambda) = \inf \{ \| u \|^2 : u \in H^1(\mathbb{R}^n) \text{ and } J(Q)[u] = \lambda \}.
\]

(2.2)

Recall that \( H^1(\mathbb{R}^n) \) is the space of the closure of \( C_0^\infty(\mathbb{R}^n) \) under the following norm

\[
\| u \|^2 = \| u \|^2_{L^2} = \int_{\mathbb{R}^n} (|\nabla u|^2 + u^2) \, dx.
\]

Finally, let us denote \( Q^* \) as \( \limsup_{x \to \infty} Q(x) \) and \( Q_* \) as \( \inf_{x \in \mathbb{R}^n} Q(x) \).
Remarks 2.1. By (1.2), it is clear that $H^1(\mathbb{R}^n) \subset L^{p+1}(\mathbb{R}^n)$, so that $J(Q)[\cdot]$ is well defined in $H^1(\mathbb{R}^n)$.

2.2. Since it is assumed that $Q^+ \neq 0$ in $\mathbb{R}^n$, we have that \( \{ u \in H^1(\mathbb{R}^n) : J(Q)[u] = \lambda \} \) is not empty for every $\lambda > 0$.

Now, one may ask the following

**Question.** Is there a $u \in H^1(\mathbb{R}^n)$ such that $J(Q)[u] = \lambda$ and $I_\lambda(Q) = \|u\|^2$?

This is a problem of existence of minimizers, and we will denote it by $(P_\lambda(Q))$ or simply by $(P)$. It is clear that to establish the existence of solutions of Eq. (1.1), one can instead prove the existence for $(P_\lambda(Q))$ for some $\lambda > 0$ because such a minimizer will be a solution of Eq. (1.1) after a scaling.

It is known (see [4, 8]) that

(A) $(P)$ always possesses a positive minimizer if $Q^* = Q_*$, or $Q^* \leq 0$.

(B) $(P)$ has no minimizers if $\lim_{x \to \infty} Q(x) = \sup_{x \in \mathbb{R}^n} Q(x)$ and $Q$ is not a constant.

In view of (A), we shall assume that $Q^* > 0$ for the rest of this paper.

For $(P_\lambda(Q))$ we can always choose minimizing sequences and it is known (see [8]) that if $\{u_m\}$ is a minimizing sequence of $(P_\lambda(Q))$, there exists a subsequence, say, without loss of generality $\{u_m\}$ itself, and a sequence $\{y_m\}$ in $\mathbb{R}^n$, such that for any $\varepsilon > 0$, there is a $R_\varepsilon < +\infty$ so that

$$\int_{B_{R_\varepsilon}(y_m)} |u_m|^{p+1}(x) \, dx > \lambda_m - \varepsilon \quad \text{for all } m \geq 1, \quad (2.3)$$

where

$$\lambda_m' = \int_{\mathbb{R}^n} |u_m|^{p+1}(x) \, dx \quad \text{and} \quad \lambda_m' \to \lambda' > 0 \quad \text{as } m \to \infty \quad (2.4)$$

and $\{u_m\}$ converges to some function $u$ in $H^1(\mathbb{R}^n)$ weakly. By the lower semicontinuity of norm, we have $\|u\|^2 \leq I_\lambda(Q)$. Therefore, to show that $u$ is indeed a minimizer of $(P_\lambda(Q))$, what we have to show is that $J(Q)[u] \geq \lambda$!

**Remark 2.3.** It is clear that we may assume that $\{u_m\}$ and $u$ are non-negative in this section, because if we replace $u$ by $|u|$, $J(Q)[\cdot]$ remains the same and $\|\cdot\|$ can be only reduced (see [7, pp. 152]).

For the sake of convenience we state the following fact as a lemma and give a simple proof.
LEMMA 2.1. Suppose that \( \{u_m\} \) is a minimizing sequence of \( (P_\lambda(Q)) \) which converges to some function \( u \) in \( H^1(\mathbb{R}^n) \) weakly. If there exists a bounded sequence \( \{y_m\} \) in \( \mathbb{R}^n \) such that the inequality (2.3) holds for all \( \varepsilon > 0 \), then \( J(Q)[u] = \lambda \).

Proof: Since \( \{y_m\} \) is bounded, (2.3) becomes \( \forall \varepsilon > 0, \exists R_\varepsilon < +\infty \), so that

\[
\int_{B_{R_\varepsilon}(0)} u_m^{\rho+1}(x) \, dx > \lambda_m^* - \varepsilon
\]

(2.3)'

and correspondingly, since

\[
\lambda = J(Q)[u_m] = \int_{B_{R_\varepsilon}(0)} Q(x) \, u_m^{\rho+1}(x) \, dx + \int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(0)} Q(x) \, u_m^{\rho+1}(x) \, dx
\]

i.e.,

\[
\int_{B_{R_\varepsilon}(0)} Q(x) \, u_m^{\rho+1}(x) \, dx = \lambda - \int_{\mathbb{R}^n \setminus B_{R_\varepsilon}(0)} Q(x) \, u_m^{\rho+1}(x) \, dx
\]

by (2.3)' and (2.4), we have then

\[
\lambda - \|Q\|_{L^\infty} \varepsilon \leq \int_{B_{R_\varepsilon}(0)} Q(x) \, u_m^{\rho+1}(x) \, dx \leq \lambda + \|Q\|_{L^\infty} \varepsilon.
\]

(2.5)

But on \( B_{R_\varepsilon}(0) \), we have the Sobolev Compactness Theorem. Hence let \( m \to \infty \) in (2.5), it becomes

\[
\lambda - \|Q\|_{L^\infty} \varepsilon \leq \int_{B_{R_\varepsilon}(0)} Q(x) \, u_m^{\rho+1}(x) \, dx \leq \lambda + \|Q\|_{L^\infty} \varepsilon
\]

which of course implies that

\[
\int_{\mathbb{R}^n} Q(x) \, u^{\rho+1}(x) \, dx = \lambda.
\]

Q.E.D.

2.2. THEOREM 2.1. If \( \forall \varepsilon > 0 \), we have

\[
|\{x \in \mathbb{R}^n, \, Q(x) \geq Q_* + \varepsilon\}| < \infty.
\]

Then \( (P_\lambda(Q)) \) has at least one positive minimizer for every \( \lambda > 0 \).

Proof: Let \( \{u_m\} \), \( \{y_m\} \), and \( u \) be as in (2.3) and (2.4). There are two possible cases:
First, there exists a bounded subsequence of \( \{ y_m \} \), then the conclusion follows from the argument in Lemma 2.1.

Second, if \( y_m \to \infty \) as \( m \to \infty \). Let

\[
\tilde{u}_m(x) = u_m(x + y_m).
\]

Hence for \( \tilde{u}_m(x) \), we have

\[
\| \tilde{u}_m \|^2 = \| u_m \|^2 \to I_\lambda(Q) \quad \text{as} \quad m \to \infty,
\]

and \( \forall \epsilon > 0, \exists R_\epsilon < +\infty \), so that

\[
\int_{B_{R_\epsilon}(0)} \tilde{u}_m^{p+1}(x) \, dx > \lambda_m' - \epsilon,
\]

where again we have

\[
\lambda_m' = \int_{\mathbb{R}^n} \tilde{u}_m^{p+1}(x) \, dx = \int_{\mathbb{R}^n} u_m^{p+1}(x) \, dx.
\]

Now, since \( \{ \tilde{u}_m \} \) is bounded in \( H^1(\mathbb{R}^n) \), there is a subsequence of \( \{ \tilde{u}_m \} \), say \( \{ \tilde{u}_m \} \) itself again converges to a function \( \tilde{u} \) in \( H^1(\mathbb{R}^n) \) weakly, with

\[
\| \tilde{u} \|^2 \leq I_\lambda(Q).
\]

Next we want to show that

\[
J(Q)[\tilde{u}] \geq \lambda
\]

which will end the proof.

From our assumption on \( Q \), it is clear that for any \( \epsilon > 0 \), there is a \( \bar{R}_\epsilon < +\infty \), such that

\[
|\{ Q > Q_* + \epsilon \} \setminus B_{\bar{R}_\epsilon}(0) | < \epsilon.
\]

Since \( y_m \to m \infty \), there exists \( N_\epsilon \), such that \( | y_m | > R_\epsilon + \bar{R}_\epsilon \) for \( m \geq N_\epsilon \) (\( R_\epsilon \) as in (2.3)). Therefore for \( m \geq N_\epsilon \)

\[
\int_{B_{R_\epsilon}(0)} Q(x) \tilde{u}_m^{p+1}(x) \, dx - \int_{B_{R_\epsilon}(y_m)} Q(x) u_m^{p+1}(x) \, dx
\]

\[
= \int_{B_{R_\epsilon}(y_m)} [Q(x - y_m) - Q(x)] u_m^{p+1}(x) \, dx
\]

\[
= \int_{B_{R_\epsilon}(y_m) \cap \{ Q > Q_* + \epsilon \}} [Q(x - y_m) - Q(x)] u_m^{p+1}(x) \, dx
\]

\[
+ \int_{B_{R_\epsilon}(y_m) \cap \{ Q < Q_* + \epsilon \}} [Q(x - y_m) - Q(x)] u_m^{p+1}(x) \, dx
\]
\[
\geq - (\|Q\|_{L^\infty} - Q_*) \int_{\{Q > Q_+ + \varepsilon\} \setminus B_R(0)} u_m^{p+1}(x) \, dx \\
- \varepsilon \int_{B_{R_\varepsilon}(y_m) \cap \{Q < Q_+ + \varepsilon\}} u_m^{p+1}(x) \, dx \\
\geq - (\|Q\|_{L^\infty} - Q_*) \left( \int_{\{Q > Q_+ + \varepsilon\} \setminus B_R(0)} \, dx \right)^{1/q'} \left( \int_{\mathbb{R}^n} u_m^{2n/(n-2)}(x) \, dx \right)^{1/p'} \\
- \varepsilon \lambda_m' \geq - C(n, Q, p, \lambda) \varepsilon^{1/q'} - \varepsilon \lambda_m',
\]

where \( p'(p + 1) = 2n/(n-2) \), and \( 1/p' + 1/q' = 1 \) and \( C \) is a constant depending only on \( n, Q, p, \) and \( \lambda \). And because \( \int_{B_R(0)} Q(x) u_m^{p+1}(x) \, dx > \lambda - \|Q\|_{L^\infty} \varepsilon \) by (2.3) and (2.4), we have

\[
\int_{B_{R_\varepsilon}(0)} Q(x) \tilde{u}_m^{p+1}(x) \, dx > \lambda - \|Q\|_{L^\infty} \varepsilon - C \varepsilon^{1/q'} - \varepsilon \lambda_m',
\]

and applying the Compactness Theorem on \( B_R(0) \) for \( \{\tilde{u}_m\} \), we have

\[
\int_{B_{R_\varepsilon}(0)} Q(x) \tilde{u}^{p+1}(x) \, dx > \lambda - \|Q\|_{L^\infty} \varepsilon - C \varepsilon^{1/q'} - \varepsilon \lambda'
\]

for any \( \varepsilon > 0 \). Therefore, we finally get

\[
\int_{\mathbb{R}^n} Q(x) \tilde{u}^{p+1}(x) \, dx \geq \lambda.
\]

This completes the proof. Q.E.D.

**Theorem 2.2.** Let \( D(r) = \int_{|x| \leq r} [Q(x) - Q^*] \, dx \), for \( r \geq 0 \). If \( D(r) \geq 0 \) in \( \mathbb{R}^+ \) and is not identically 0 then \((P_\lambda(Q))\) has at least one minimizer in \( H^1(\mathbb{R}^n) \) for every \( \lambda > 0 \).

**Proof.** First we want to show that under the above assumption on \( Q \), we have for every \( \lambda > 0 \) the following:

\[
I_\lambda(Q) < I_\lambda(Q^*). \tag{2.6}
\]

Let \( u_0 \) be a positive minimizer of \((P_1(1))\) in \( \mathbb{R}^n \). Then \( u_0 \) must be radially symmetric, \( u_0'(r) < 0 \) for \( r > 0 \) and more

\[
\lim_{r \to \infty} r^{(n-1)/2} e^r u_0(r) = \mu > 0 \quad \text{(see [6])}. \tag{2.7}
\]

And \( u_0 \) is also a minimizer of \((P_{Q^*}(Q^*))\), but since

\[
I_\lambda = \lambda^{2(p+1)} I_1 = \left( \frac{\lambda}{Q^*} \right)^{1/(p+1)} I_{Q^*},
\]

where \( p'(p + 1) = 2n/(n-2) \), and \( 1/p' + 1/q' = 1 \) and \( C \) is a constant depending only on \( n, Q, p, \) and \( \lambda \). And because

\[
\int_{B_R(0)} Q(x) u_m^{p+1}(x) \, dx > \lambda - \|Q\|_{L^\infty} \varepsilon \] by (2.3) and (2.4), we have

\[
\int_{B_{R_\varepsilon}(0)} Q(x) \tilde{u}_m^{p+1}(x) \, dx > \lambda - \|Q\|_{L^\infty} \varepsilon - C \varepsilon^{1/q'} - \varepsilon \lambda_m',
\]

and applying the Compactness Theorem on \( B_R(0) \) for \( \{\tilde{u}_m\} \), we have

\[
\int_{B_{R_\varepsilon}(0)} Q(x) \tilde{u}^{p+1}(x) \, dx > \lambda - \|Q\|_{L^\infty} \varepsilon - C \varepsilon^{1/q'} - \varepsilon \lambda'
\]

for any \( \varepsilon > 0 \). Therefore, we finally get

\[
\int_{\mathbb{R}^n} Q(x) \tilde{u}^{p+1}(x) \, dx \geq \lambda.
\]

This completes the proof. Q.E.D.

**Theorem 2.2.** Let \( D(r) = \int_{|x| \leq r} [Q(x) - Q^*] \, dx \), for \( r \geq 0 \). If \( D(r) \geq 0 \) in \( \mathbb{R}^+ \) and is not identically 0 then \((P_\lambda(Q))\) has at least one minimizer in \( H^1(\mathbb{R}^n) \) for every \( \lambda > 0 \).

**Proof.** First we want to show that under the above assumption on \( Q \), we have for every \( \lambda > 0 \) the following:

\[
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\[
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\]

And \( u_0 \) is also a minimizer of \((P_{Q^*}(Q^*))\), but since

\[
I_\lambda = \lambda^{2(p+1)} I_1 = \left( \frac{\lambda}{Q^*} \right)^{1/(p+1)} I_{Q^*},
\]
for any $Q$ with $Q^* > 0$, we get that

$$I_\lambda(Q) < I_\lambda(Q^*), \forall \lambda > 0 \iff I_{Q^*}(Q) < I_{Q^*}(Q^*).$$

Now, by the hypotheses on $Q$, we have

$$\int_{\mathbb{R}^n} Q(x) u^{p+1}_0(x) \, dx - \int_{\mathbb{R}^n} Q^* u^{p+1}_0(x) \, dx$$

$$= \int_{\mathbb{R}^n} \left[ Q(x) - Q^* \right] u^{p+1}_0(x) \, dx = \lim_{r \to +\infty} \int_{|x| = r} \left[ Q(x) - Q^* \right] u^{p+1}_0(x) \, dx$$

$$= \lim_{r \to +\infty} \int_0^r \left[ u^{p+1}_0(t) D'(t) \, dt \right]$$

$$= \lim_{r \to +\infty} \left[ u^{p+1}_0(r) D(r) - (p+1) \int_0^r D(t) u^{p}_0(t) u'_0(t) \, dt \right]$$

$$= -(p+1) \int_0^\infty D(t) u^{p}_0(t) u'_0(t) \, dt > 0$$

by (2.7) which implies that $I_{Q^*}(Q) < I_{Q^*}(Q^*)$.

Second, we want to show that $\{y_m\}$ has a bounded subsequence. If not, suppose $y_m \to \infty$. But we have from (2.3) and (2.4)

$$\int_{B_{R\epsilon}(y_m)} Q(x) u^{p+1}_m(x) \, dx > \lambda - \|Q\|_{L^\infty} \epsilon, \quad \forall \epsilon > 0.$$ 

Since

$$\int_{\mathbb{R}^n} Q^* u^{p+1}_m(x) \, dx \geq \int_{B_{R\epsilon}(y_m)} Q^* u^{p+1}_m(x) \, dx$$

$$= \int_{B_{R\epsilon}(y_m)} \limsup_{y \to \infty} Q(y) u^{p+1}_m(x) \, dx \quad (2.8)$$

and $y_m \to \infty$ as $m \to \infty$, we will have the following: let

$$\lambda_m^* = \int_{\mathbb{R}^n} Q^* u^{p+1}_m(x) \, dx,$$

which converges to $Q^* \lambda' > 0$ by (2.4), and with (2.8), we have that $Q^* \lambda' \geq \lambda$. But

$$I_{Q^*\lambda'}(Q^*) \leq \liminf_{m \to \infty} \|u_m\|^2,$$

giving us that $I_{Q^*\lambda'}(Q^*) \leq I_\lambda(Q)$, in particular

$$I_\lambda(Q^*) \leq I_\lambda(Q) \quad \text{since} \quad \lambda \leq Q^* \lambda'.$$
contradicting the above result. Therefore, \( \{y_m\} \) does have a bounded subsequence. This ends the proof by Lemma 2.1. Q.E.D.

Remark 2.4. From the argument of Theorem 2.2 it follows that Eq. (1.1) possesses a solution if

\[
\int_{\mathbb{R}^n} Q(x) u_0^{r+1} (x) \, dx > \limsup_{x \to \infty} Q(x).
\]

2.3. Examples. (1) For any \( f(x) \in C^0(\mathbb{R}^n) \), \( f(x) \geq 0 \), and \( \text{supp} f \subseteq B_1(0) \), let \( Q_f(x) = 1 + \sum_{k=1}^{\infty} f(k^2 (x - k v)) \) for some \( v \in S^{n-1} \). Then \( Q_* = 1 \), \( Q^* = 1 + \|f\|_{L^\infty} = \sup_{x \in \mathbb{R}^n} Q(x) \), but the hypothesis of Theorem 2.1 is satisfied. Note that \( D(r) < 0 \) for \( r \) large enough if \( f \neq 0 \) in \( \mathbb{R}^n \).

(2) Letting \( Q_0(x) = 1 + e^{-|x|^n} \cos(|x|^n) \), we see that \( Q_0^* = 1 > Q_{0*} \). Theorem 2.2 can be applied for \( Q_0 \):

\[
D_0(r) = \omega_n \int_0^r t^{n-1} e^{-t^n} \cos(t^n) \, dt
\]

\[
= \frac{\omega_n}{2n} \left[ 1 + \sqrt{2} e^{-r^n} \sin \left( r^n - \frac{\pi}{4} \right) \right] > 0
\]

for all \( r > 0 \).

3. An Example of Positive Nonradial Solutions

3.1. Preliminaries. We shall consider only radial potentials in this section. It is well known that every positive solution of Eq. (1.1) must be radially symmetric if \( Q \) is a positive constant, and it is also the case for the Dirichlet Problem with boundary value zero on \( \partial B_R(0) \) (see \([5, 6]\)). But nevertheless, for certain type \( Q(r) \), the existence of positive nonradial solutions in \( B_R(0) \) has been proved in \([4]\) for large \( R \). Here we will give an example showing the existence of positive nonradial solutions in \( \mathbb{R}^n \) with \( n \geq 3 \).

Let \( H^1_0(\mathbb{R}^n) \) be the closure of compactly supported smooth radial functions on \( \mathbb{R}^n \) in \( H^1(\mathbb{R}^n) \). For a given \( Q(r) \in C^0(\mathbb{R}^+ \) ), which is bounded and \( Q^+(r) \neq 0 \) in \( \mathbb{R}^+ \), define

\[
M_*(Q) = \sup \{ J(Q)[u] : u \in H^1_0(\mathbb{R}^n) \text{ and } \|u\| = 1 \} \quad (3.1)
\]

and

\[
M(Q) = \sup \{ J(Q)[u] : u \in H^1(\mathbb{R}^n) \text{ and } \|u\| = 1 \}. \quad (3.2)
\]
Remarks. 3.1. It is obvious that $M_r(Q) \leq M(Q)$ and we may assume that the maximizers (if they exist) are nonnegative. And for bounded $Q$ in $\mathbb{R}^n$ we always have $M(Q) < \infty$.

By the Sobolev inequality we know that for all $u$ on the unit sphere in $H^1(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |u|^{2n/(n-2)}(x) \, dx \leq c(n),$$

where $c(n)$ is some positive constant depending only on $n$.

Now because of condition (1.2), we have for some positive constant $\tilde{C}(p)$ that

$$t^{p+1} \leq \tilde{C}(p)(t^2 + t^{2n/(n-2)}) \text{ in } \mathbb{R}^+.$$

Hence for any $u$ in $H^1(\mathbb{R}^n)$ with $\|u\| = 1$, we have the following

$$J(Q)[u] \leq \sup_{\mathbb{R}^n} Q \cdot \int_{\mathbb{R}^n} |u|^{p+1}(x) \, dx$$

$$\leq \tilde{C}(p) \sup_{\mathbb{R}^n} Q \cdot \int_{\mathbb{R}^n} (|u|^2 + |u|^{2n/(n-2)})(x) \, dx$$

$$\leq \tilde{C}(p)(1 + C(n)) \cdot \sup_{\mathbb{R}^n} Q$$

$$< \infty.$$

Q.E.D.

3.2. It is well known (see [12]) that every $u \in H^1(\mathbb{R}^n)$ is almost everywhere equal to a function $\tilde{u}(x)$, which is continuous for $x \neq 0$ and such that

$$|\tilde{u}(x)| \leq C_n |x|^{(1-n)/2} \|u\|, \quad \text{for } \|x\| \geq \alpha_n,$$

where $C_n$ and $\alpha_n$ are positive constants depending only on $n$.

3.3. By a compactness result (see [1, 12]) for $H^1(\mathbb{R}^n)$, $M_r(Q)$ is realized for every such $Q$.

3.2. Example. We are now ready to construct an example for which a positive nonradial solution of (1.1) exists. Let

$$\varphi_0(t) = 1 + \cos \left( t - \frac{\pi}{2} \right) \chi_{[0,1]}(t), \quad t \in \mathbb{R},$$

where $\chi_E$ denotes the characteristic function of set $E \subset \mathbb{R}$. Set

$$\varphi_x(t) = \varphi_0(t - x);$$

(3.4)
we know that $M_\alpha(\phi_\alpha)$ (and $M(\phi_\alpha)$, resp.) is achieved by some positive function in $H^1_\alpha(\mathbb{R}^n)$ (in $H^1(\mathbb{R}^n)$, resp.) [4, 8]. Denote $u_\alpha$ as a maximizer which assumes $M_\alpha(\phi_\alpha)$ in $H^1_\alpha(\mathbb{R}^n)$. But by (3.3), if $\alpha \geq \alpha_\alpha + \pi/2$, we have

$$J(1)[u_\alpha] - J(\phi_\alpha)[u_\alpha]$$

$$= \int_{\alpha \leq |x| \leq \alpha + \pi} \left[ 1 - \left( 1 + \cos \left( \frac{|x| - \pi/2}{2} - \alpha \right) \right) \right] u_\alpha^{p+1}(|x|) \, dx$$

$$\geq - \int_{\alpha \leq |x| \leq \alpha + \pi} u_\alpha^{p+1}(|x|) \, dx$$

$$\geq - \left( \int_{\alpha \leq |x| \leq \alpha + \pi} u_\alpha^2(|x|) \, dx \right)^{(p+1)/(p+1)} \left( \int_{\alpha \leq |x| \leq \alpha + \pi} u_\alpha^q(|x|) \, dx \right)^{((p+1)/q)(1-\lambda)}$$

where $1 = ((p+1)/2) + ((p+1)/q)(1-\lambda)$, $2 < p+1 < q$ by the Holder Inequality. Since $\|u_\alpha\| = 1$, we then have by choosing $q = 2n/(n-2) > p+1$

$$J(1)[u_\alpha] - J(\phi_\alpha)[u_\alpha]$$

$$\geq - \left( \int_{\alpha \leq |x| \leq \alpha + \pi} u_\alpha^q(|x|) \, dx \right)^{(p+1)/(p+1)} \left( \int_{\alpha \leq |x| \leq \alpha + \pi} u_\alpha^q(|x|) \, dx \right)^{((p+1)/q)(1-\lambda)}$$

$$\geq - C(p, n) \alpha^{-(p-1)(n-1)/2},$$

where $C(p, n)$ is a positive constant depending only on $p$ and $n$.

Combining with the fact that $M_\alpha(\phi_\alpha) \geq M_\alpha(1)$, we finally obtain that

$$M_\alpha(\phi_\alpha) \to M_\alpha(1) \quad \text{as} \quad \alpha \to +\infty.$$

Now observing that $M_\alpha(1) = M(1)$ (see [6]), and that $M(\phi_\alpha) \geq M(\varphi_{-\pi/2}) > M(1)$ for all $\alpha \geq -\pi/2$, because by the maximum principle each maximizer must be strictly positive, we conclude that $M_\alpha(\phi_\alpha)$ is strictly less than $M(\varphi_\alpha)$ for $\alpha$ sufficiently large, which in turn implies that maximizers attaining $M(\varphi_\alpha)$ must be nonradial for large $\alpha$. Because otherwise, if one of the positive maximizers, say $u_\alpha$, is radially symmetric about some point $x_0$ in $\mathbb{R}^n$, then we know that $u_\alpha$ is a solution of Eq. (1.1) with the potential $\varphi_\alpha$ and more we have that $(u_\alpha - \Delta u_\alpha)/u_\alpha^p = \varphi_\alpha$ is also radially symmetric about $x_0$, which implies that $x_0$ must be the origin because $\varphi_\alpha$ is radially symmetric only about the origin.

Therefore $u_\alpha$ is in $H^1_\alpha(\mathbb{R}^n)$ and $M(\varphi_\alpha)$ must thus be equal to $M_\alpha(\phi_\alpha)$ which contradicts our argument for large $\alpha$.

**Remark 3.4.** It is clear from this proof that nonradial solutions exist for various positive perturbations of constant $Q \equiv 1$. 


4. A NONEXISTENCE RESULT

While one could expect the existence of solutions of Eq. (1.1) for bounded $Q$ under mild restrictions, it was not known in general, for the situations where $Q$ is unbounded near infinity. Recently, in a paper by Ding and Ni [4], it was proved that Eq. (1.1) always possesses a positive radial solution in $\mathbb{R}^n$ provided that $Q(x)$ is radially symmetric and

$$0 \leq Q(x) \leq (\text{positive constant})(1 + |x|)^l,$$

where $0 \leq l < (n-1)(p-1)/2$.

Our concern here is the nonexistence of solutions of (1.1) for radial potentials; for this we have obtained

**Theorem 4.1.** There is no positive radial solution of (1.1), if $Q(r) \geq 0$ and $Q(r) r^{-(n-1)(p-1)/2}$ is nondecreasing, where $Q(r) \in C^{0,1}(\mathbb{R}^+)$ and $n \geq 3$.

4.1. Preliminaries. Suppose that $u(x) \in C^2(\mathbb{R}^n)$ is a solution of (1.1). Set $V(x) = K(|x|) u(x)$ where $0 < K \in C^2(\mathbb{R}^+)$. Then

$$\Delta V(x) - \frac{2K'(r)}{rK(r)} x \cdot \nabla V(x) - \left\{ 1 + \frac{K''(r)}{K(r)} + \frac{(n-1)K'(r)}{rK(r)} \right\} V(x) + Q(r) K^{1-p(r)} V^p(x) = 0 \quad (4.1)$$

in $\mathbb{R}^n \setminus \{0\}$, where $r = |x|$.

Now, for a special $K(r) = r^{(n-1)/2}$, we have

$$\Delta V(x) - \frac{n-1}{r^2} x \cdot \nabla V(x) - \left\{ 1 + \frac{(n-1)(n-3)}{4r^2} \right\} V(x) + Q(r) r^{-(n-1)(p-1)/2} V^p(x) = 0 \quad (4.2)$$

in $\mathbb{R}^n \setminus \{0\}$.

If we assume further that $u$ is radial (and so is $V$), we then have

$$V''(r) - \left\{ 1 + \frac{(n-1)(n-3)}{4r^2} \right\} V(r) + Q(r) r^{-(n-1)(p-1)/2} V^p(r) = 0 \quad (4.3)$$

in $\mathbb{R}^+$.

4.2. Some lemmas. Here we will first prove some lemmas for Theorem 4.1. The case $n = 3$ appears to be easier in (4.3) and we have a better understanding about it.
Lemma 4.1. The initial value problem

\[
V''(t) - V(t) + V^p(t) = 0, \quad p > 1, \\
V(0) = 0, \quad V'(0) = a > 0
\]  

(4.4)

does not possess any positive solution in \((0, \infty)\).

Proof. Suppose \(V\) is a positive solution of Eq. (4.4). We then have

\[
V''(t) = V(t)\left[1 - V^{p-1}(t)\right] = \begin{cases} 
> 0 & \text{if } 0 < V < 1 \\
< 0 & \text{if } V > 1.
\end{cases}
\]  

(4.5)

Claim. \(\exists t_0 > 0\), so that \(V(t_0) = 0\).

Suppose not; i.e., we suppose that \(V(t) > 0\) for \(t > 0\).

(i) If \(V < 1\) for all \(t > 0\), then \(V'' > 0\) by (4.5) together with \(V'(t) > 0\) for \(t > 0\) will imply that \(V(t) \rightarrow +\infty\) as \(t \rightarrow +\infty\), a contradiction;

(ii) If for some \(t\), \(V(t) \geq 1\). Now, let \(t_1\) be the point where \(V = 1\). Therefore \(V(t) \geq V(t_1 + 1) > 1\) for \(t \geq t_1 + 1\). Since \(V'' > 0\) for \(t > 0\), then, \(V''\) is decreasing in \((t_1 + 1, \infty)\) since \(V''(t) < 0\) there, which in turn implies that

\[
V'(t) = V'(t_1 + 1) + \int_{t_1 + 1}^{t} V''(s) \, ds \\
\leq V'(t_1 + 1) + V''(t_1 + 1)(t - t_1 - 1) \rightarrow -\infty
\]

as \(t \rightarrow +\infty\), a contradiction.

Therefore, let \(t_0\) be the first point where \(V'(t_0) = 0\). By the uniqueness and time-invertible property of Eq. (4.4), we conclude that \(V(2t_0) = 0\), a contradiction again. This completes the proof. Q.E.D.

Lemma 4.2. Every positive solution for the following initial value problem

\[
V''(t) - \left(1 + \frac{c}{t^2}\right) V(t) + W(t) V^p(t) = 0 \quad \text{for } t > 0 \\
V(0) = 0, \quad V'(0) = a > 0
\]  

(4.6)

where \(c > 0, p > 1\), \(W\) is a nondecreasing function in \(C^{0,1}(\mathbb{R}^+)\) with \(W(0) \geq 0\), \(W \not< 0\), and \((1 - c) a^2 \geq 0\), satisfies \(V(t) = 0(e^{-2t}) at \infty for some \alpha > 0\).

Proof. Let \(V\) be such a solution. We first claim that \(\exists t_0 > 0\), so that \(V(t_0) = 0\). For otherwise, since \(V(0) = 0\), \(V'(0) \geq 0\), and \(V(t) > 0\) for \(t > 0\), we must have that \(V(t) > 0\) for \(t > 0\). Now let \(\beta = \lim_{t \rightarrow +\infty} W(t)\) (\(\beta\) could be \(\infty\)). We discuss three possible cases.
(i) $\beta < \infty$ and $\beta V^{p-1}(t) < 1$ for all $t > 0$. Then $V''(t) > 0$ for all $t > 0$. It gives a contradiction as in Lemma 4.1 (i);

(ii) $\beta < \infty$ and there exists a point $t_1$ at which $\beta V^{p-1}(t_1) = 1$. Then $V(t) > V(t_1 + 1) > (1/\beta)^{1/(p-1)}$, for all $t \geq t_1 + 1$ since $V'' > 0$ and

$$V''(t) = V(t) \left[ \left( 1 + \frac{c}{t^2} \right) - W(t) V^{p-1}(t) \right] \leq -d(c, \beta) < 0$$

for all $t$ large enough, where $d$ is a constant depending only on $c$ and $\beta$ for all large $t$. But as in Lemma 4.1 (ii), this again gives a contradiction;

(iii) $\beta = \infty$. This case can be handled as in case (ii) above. Therefore, our assertion follows.

Multiplying Eq. (4.6) by $V'(t)$ and integrating over $[\epsilon, t]$ for any $\epsilon > 0$, we have

$$\frac{1}{2} V'^2(t) - \frac{1}{2} V'^2(\epsilon) - \frac{1}{2} \int_\epsilon^t \left( 1 + \frac{c}{s^2} \right) (V^2)'(s) \, ds$$

$$+ \frac{1}{p+1} \int_\epsilon^t W(s)(V^{p+1})'(s) \, ds = 0.$$ 

Integration by parts gives

$$\frac{1}{2} V'^2(t) - \frac{1}{2} V'^2(\epsilon) - \frac{1}{2} \left( 1 + \frac{c}{\epsilon^2} \right) V^2(t) + \frac{1}{2} \left( 1 + \frac{c}{\epsilon^2} \right) V^2(\epsilon)$$

$$- \int_\epsilon^t \frac{c}{s^3} V^2(s) \, ds + \frac{1}{p+1} W(t) V^{p+1}(t) - \frac{1}{p+1} W(\epsilon) V^{p+1}(\epsilon)$$

$$- \frac{1}{p+1} \int_\epsilon^t W'(s) V^{p+1}(s) \, ds = 0.$$ 

Now, letting $\epsilon \to 0$ and noting that $V(\epsilon)/\epsilon \to V'(0)$, we obtain

$$V'^2(t) = \left( 1 + \frac{c}{t^2} \right) V^2(t) + 2c \int_0^t \frac{V^2(s)}{s^3} \, ds - \frac{2}{p+1} W(t) V^{p+1}(t)$$

$$+ \frac{2}{p+1} \int_0^t W'(s) V^{p+1}(s) \, ds + (1-c) a^2. \quad (4.7)$$

Next, we multiply (4.6) by $V(t)$ and subtract (4.7) from

$$V(t) V''(t) - V'^2(t) = -2c \int_0^t \frac{V^2(s)}{s^3} \, ds - \frac{2}{p+1} W(t) V^{p+1}(t)$$

$$- \frac{2}{p+1} \int_0^t W'(s) V^{p+1}(s) \, ds - (1-c) a^2,
therefore

\[
\left[ \frac{V'(t)}{V(t)} \right]' = - \frac{2c}{V^2(t)} \int_0^t V^2(s) \, ds - \frac{p-1}{p+1} W(t) V^p - 1(t) W(t) V^{p-1}(t)
- \frac{2}{(p+1) V^2(t)} \int_0^t W'(s) V^{p+1}(s) \, ds - (1-c) a^2 V^2(t).
\]

Integrating over \([t_0, t]\) for \(r > t_0\), we have

\[
\frac{V'(t)}{V(t)} = -2c \int_{t_0}^t \frac{dr}{V^2(r)} \int_0^r \frac{V^2(s)}{s^3} \, ds - \frac{p-1}{p+1} \int_{t_0}^t W(r) V^{p-1}(r) \, dr
- \frac{2}{p+1} \int_{t_0}^t \frac{dr}{V^2(r)} \int_0^r W'(s) V^{p+1}(s) \, ds - (1-c) a^2 \int_{t_0}^t \frac{dr}{V^2(r)}.
\]

Thus \(V'(t) < 0\) for all \(t > t_0\); i.e., \(V(t)\) decreases in \((t_0, \infty)\) and

\[
\frac{V'(t)}{V(t)} \leq -2c \int_{t_0}^t \frac{dr}{V^2(r)} \int_0^r \frac{V^2(s)}{s^3} \, ds
\leq -2c \int_{t_0}^t \frac{dr}{V^2(t_0)} \int_0^r \frac{V^2(s)}{s^3} \, ds
= -2\alpha (t - t_0),
\]

where \(\alpha = c [V(t_0)]^{-2} \int_0^{t_0} V^2(s) s^{-3} \, ds\), and which gives us the desired result.

**Q.E.D.**

**Lemma 4.3.** Suppose \(V\) is a positive solution for the following initial value problem

\[
V''(t) - V(t) W(t) V^p(t) = 0 \quad \text{for} \quad t > 0
V(0) = 0, \quad V'(0) = a \geq 0,
\]

where \(p > 1\) and \(W\) is as in Lemma 4.2. Then \(V(t) \cdot e^{\alpha t^2}\) is bounded in \(\mathbb{R}^+\) for some \(\alpha > 0\), provided that \(W\) is not a constant.

The proof follows from Eq. (4.8), where we have

\[
\frac{V'(t)}{V(t)} = -\frac{p-1}{p+1} \int_{t_0}^t W(r) V^{p-1}(r) \, dr - \frac{2}{p+1} \int_{t_0}^t \frac{dr}{V^2(r)} \int_0^r W'(s) V^{p+1}(s) \, ds
- a^2 \int_{t_0}^t \frac{dr}{V^2(r)}
\]

(4.10)
and we make use of the following

\[ \frac{V'(t)}{V(t)} \leq -\frac{2}{p+1} \int_0^t \frac{dr}{V^2(r)} \int_0^r W''(s) V^{p+1}(s) \, ds. \]

Then the very same arguments above work provided that \( W \) is not a constant, i.e., \( \exists \tilde{t} > 0 \), so that

\[ \int_0^\tilde{t} W'(s) V^{p+1}(s) \, ds > 0. \]

4.3. Proof of Theorem 4.1. Suppose \( u \) is a positive radial solution of Eq. (1.1) with \( Q \) satisfying the hypotheses of Theorem 4.1. Since \( Q(r) \geq 0 \), we have

\[ \Delta u - u \leq 0 \]

which implies that (see [6]) for some \( \mu > 0 \):

\[ u(x) \geq \mu e^{-\frac{|x|}{|x|^{(n-1)/2}}} \text{ near } \infty. \]

Hence, if we set \( V(r) = r^{(n-1)/2} u(r) \), we get

\[ V(r) \geq \mu e^{-r} \text{ for large } r \quad (4.11) \]

and by (4.3) we see that \( V \) satisfies Eq. (4.6). Now (4.11) contradicts the conclusions of Lemmas 4.1, 4.2, and 4.3. Q.E.D.

Remark 4.1. Theorem 4.1 shows that the existence result obtained by Ding and Ni in [4] (namely, Corollary 4.8 in [4]) is optimal in a certain sense.

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